Sections 4.2 and 4.3: Higher order equations

Recall that we seek a solution of the form $x(t) = e^{rt}$ to the ODE

(1) $$\ddot{x} + b\,\ddot{x} + c\,x = 0.$$

Inserting $x = e^{rt}$ into (1) we find

$$r^2\,e^{rt} + br\,e^{rt} + c\,e^{rt} = 0.$$

Since $e^{rt} \neq 0$, we find that (1) is satisfied if and only if

(2) $$r^2 + br + c = 0.$$

The roots of (2) are $r_1 = -\frac{b}{2} + \sqrt{\frac{b^2}{4} - c}$, and $r_2 = -\frac{b}{2} - \sqrt{\frac{b^2}{4} - c}$.

Good news: Since (2) has always at least one root, there is at least one solution $x = e^{rt}$!

Less good news: The roots could be *complex.*

There are three different cases, depending on the sign of $b^2/4 - c$.

Case 1: $b^2/4 - c > 0 \quad \Rightarrow \quad r_{1,2} = -b/2 \pm \alpha$ where $\alpha = \sqrt{b^2/4 - c}$.

Case 2: $b^2/4 - c < 0 \quad \Rightarrow \quad r_{1,2} = -b/2 \pm i\beta$ where $\beta = \sqrt{c - b^2/4}$.

Case 3: $b^2/4 - c = 0 \quad \Rightarrow \quad r = r_2 = r_2 = -b/2.$

Recall: Solution of $\ddot{x} + b\ddot{x} + cx = 0$ depends on the roots $r_{1,2} = -\frac{b}{2} + \sqrt{b^2/4 - c}$ of $r^2 + br + c = 0$.

Case 1: $b^2/4 - c > 0$ Set $\alpha = \sqrt{b^2/4 - c}$ so $r_{1,2} = -b/2 \pm \alpha$. Then the solution is

$$x(t) = A\,e^{r_1 t} + B\,e^{r_2 t}.$$

Case 2: $b^2/4 - c < 0$ Set $\beta = \sqrt{c - b^2/4}$ so $r_{1,2} = -b/2 \pm i\beta$. Then the solution is

$$x(t) = A\,e^{r_1 t} + B\,e^{r_2 t} = A\,e^{-bt/2 + i\beta t} + B\,e^{-bt/2 - i\beta t}$$

$$= e^{-bt/2}\left(A\,e^{i\beta t} + B\,e^{-i\beta t}\right) = e^{-bt/2}\left(A\cos(\beta t) + i\,A\sin(\beta t) + B\cos(\beta t) - i\,B\sin(\beta t)\right)$$

$$= (A{+}B)\,e^{-bt/2} A\cos(\beta t) + (iA{-}iB)\,e^{-bt/2}\sin(\beta t) = C\,e^{-bt/2} A\cos(\beta t) + D\,e^{-bt/2}\sin(\beta t),$$

where we defined new constants $C = A + B$ and $D = iA - iB$.

Case 3: $b^2/4 - c = 0$ Now we have a double-root $r = r_1 = r_2 = -b/2$. In this case, one solution is given by

$$x(t) = e^{rt} = e^{-bt/2}.$$

But what is the other solution? ... One can show (and we will!) that the other solution is $x(t) = t\,e^{-bt/2}$, so the final general solution is

$$x(t) = A\,e^{-bt/2} + B\,t\,e^{-bt/2}.$$

Example: Solve the initial value problem

$$\ddot{x} + b\,\ddot{x} + 4\,x = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

for the values $b = 0$, $b = 2$, $b = 5$, and $b = 4$.

Suppose $b = 0$: We seek to solve

$$\ddot{x} + 4\,x = 0.$$

The characteristic equation is

$$r^2 + 4 = 0,$$

with roots

$$r_1 = 2i \qquad r_2 = -2i.$$

The general solution is

$$x(t) = A\,\cos(2t) + B\,\sin(2t).$$

Using the initial conditions, we find

$$A\,\cos(0) + B\,\sin(0) = 1 \qquad -2A\,\sin(0) + 2B\,\cos(0) = 0,$$

with solution $A = 1$ and $B = 0$. So the final solution is

$$x(t) = \cos(2t).$$

Example: Solve the initial value problem

$$\ddot{x} + b\,\ddot{x} + 4\,x = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

for the values $b = 0$, $b = 2$, $b = 5$, and $b = 4$.

Suppose $b = 2$: We seek to solve

$$\ddot{x} + 2\dot{x} + 4\,x = 0.$$

The characteristic equation is

$$r^2 + 2r + 4 = 0,$$

with roots

$$r_1 = -1 + i\sqrt{3} \qquad r_2 = -1 - i\sqrt{3}.$$

The general solution is

$$x(t) = A\,e^{-t}\cos(\sqrt{3}t) + B\,e^{-t}\sin(\sqrt{3}t).$$

Using the initial conditions, we find

$$A = 1 \qquad -A + \sqrt{3}B = 0,$$

with solution $A = 1$ and $B = 1/\sqrt{3}$. So the final solution is

$$x(t) = e^{-t}\left(\cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right).$$

Example: Solve the initial value problem

$$\ddot{x} + b\,\ddot{x} + 4\,x = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

for the values $b = 0$, $b = 2$, $b = 5$, and $b = 4$.

Suppose $b = 5$: We seek to solve

$$\ddot{x} + 5\dot{x} + 4\,x = 0.$$

The characteristic equation is

$$r^2 + 5r + 4 = 0,$$

with roots

$$r_1 = -1 \qquad r_2 = -4.$$

The general solution is

$$x(t) = A\,e^{-t} + B\,e^{-4t}.$$

Using the initial conditions, we find

$$A + B = 1 \qquad -A - 4B = 0,$$

with solution $A = 4/3$ and $B = -1/3$. So the final solution is

$$x(t) = \frac{4}{3}\,e^{-t} - \frac{1}{3}\,e^{-4t}.$$

Example: Solve the initial value problem

$$\ddot{x} + b\,\ddot{x} + 4\,x = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 0$$

for the values $b = 0$, $b = 2$, $b = 5$, and $b = 4$.

Suppose $b = 4$: We seek to solve

$$\ddot{x} + 4\dot{x} + 4\,x = 0.$$

The characteristic equation is

$$r^2 + 4r + 4 = 0,$$

with *double root*

$$r = -2.$$

The general solution is

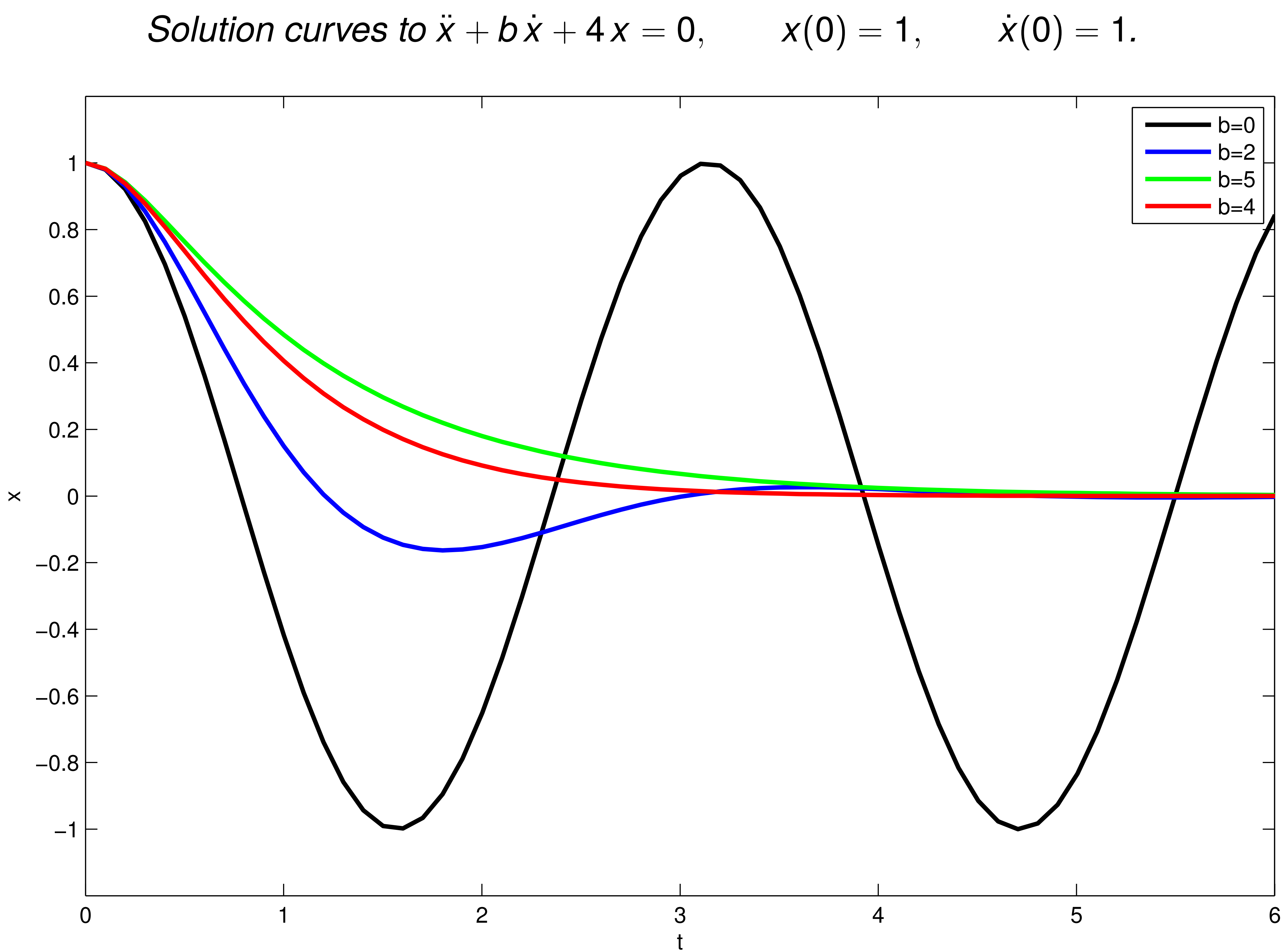
$$x(t) = A\,e^{-2t} + B\,t\,e^{-2t}.$$

Using the initial conditions, we find

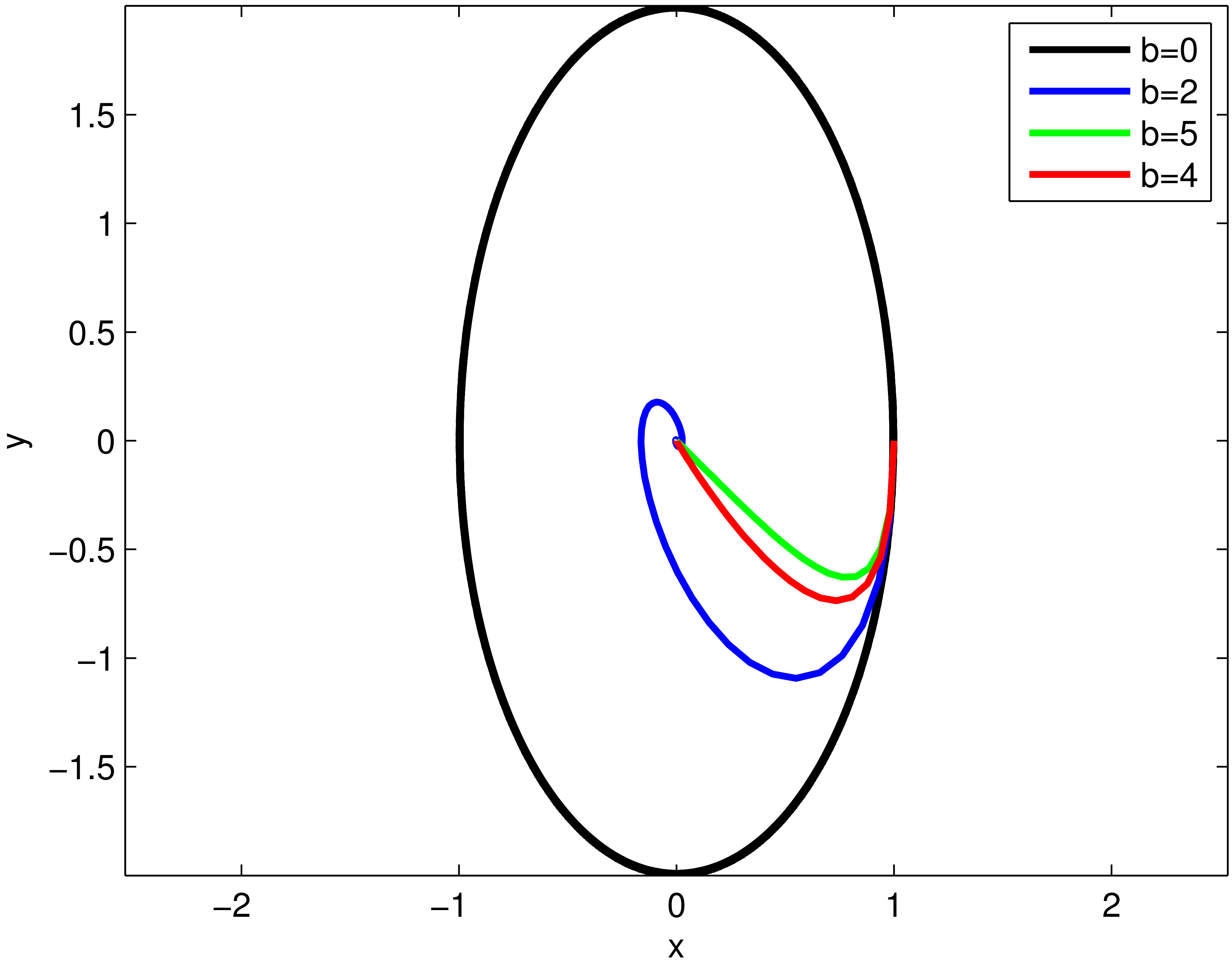
$$A = 1 \qquad -2A + B = 0,$$

with solution $A = 1$ and $B = 2$. So the final solution is

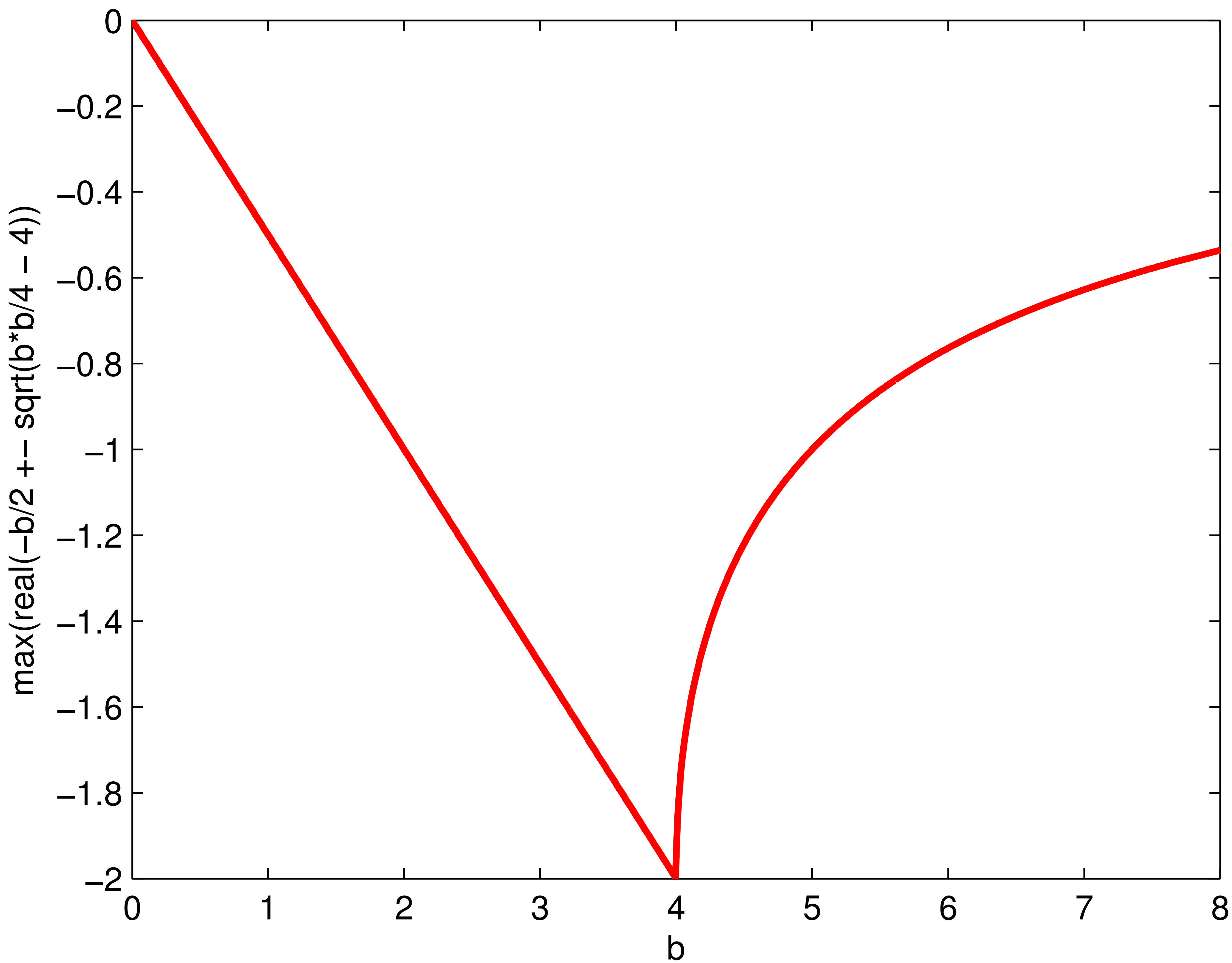
$$x(t) = (1 + 2t)\,e^{-2t}.$$


Solution curves to $\ddot{x} + b\,\dot{x} + 4\,x = 0, \quad x(0) = 1, \quad \dot{x}(0) = 1.$
b=0
b=2
b=5
b=4
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
x
0
1
2
3
4
5
6
t

Phase plots to $\ddot{x} + b\,\dot{x} + 4\,x = 0, \qquad x(0) = 1, \qquad \dot{x}(0) = 1.$



Plot of $\max(\mathit{real}(-b/2 \pm \sqrt{b^2/4 - c}))$.



The min at $b = 4$ corresponds to *critical damping* — the fastest return to equilibrium.

Theorem: Let (a, b) be an interval on the real line, and let $t_0 \in (a, b)$.

Let p and q be *continuous* functions on (a, b). Then:

1. For any real numbers x_0 and y_0, there is a unique solution on (a, b) to the equation

$$(3) \qquad \begin{cases} \ddot{x} + p\,\dot{x} + q\,x = & 0 \\ x(t_0) = & x_0 \\ \dot{x}(t_0) = & y_0. \end{cases}$$

2. If x_1 and x_2 are two linearly independent solutions of $\ddot{x} + p\,\dot{x} + q\,x = 0$, then any solution of (3) takes the form $x = c_1\,x_1 + c_2\,x_2$, for some constants c_1 and c_2.
3. The set $V = \{x \in C^2(I) : \ \ddot{x} + p\,\dot{x} + q\,x = 0\}$ is a two-dimensional vector space.

Proof: Rewrite as a *system* of first order equations, and apply Picard's theorem.

See book for details.

Theorem: Let (a,b) be an interval on the real line, and let $t_0 \in (a,b)$.
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2. If x_1 and x_2 are two linearly independent solutions of $\ddot{x} + p\,\dot{x} + q\,x = 0$, then any solution of (3) takes the form $x = c_1\,x_1 + c_2\,x_2$, for some constants c_1 and c_2.
3. The set $V = \{x \in C^2(I) : \ \ddot{x} + p\,\dot{x} + q\,x = 0\}$ is a two-dimensional vector space.

Let b be a real number, and let us apply the theorem to the equation $\ddot{x} - 2b\,\dot{x} + b^2\,x = 0$.
The characteristic equation is $r^2 - 2b\,r + b^2 = 0$, which has the double root $r = b$.
So we know that $x_1(t) = e^{bt}$ is one solution.
Now consider the function $x_2(t) = t\,e^{bt}$.
We find that $\dot{x}_2 = e^{bt} + b\,t\,e^{bt}$ and $\ddot{x}_2 = 2b\,e^{bt} + b^2\,t\,e^{bt}$, so

$$\ddot{x} - 2b\,\dot{x} + b^2\,x = 2b\,e^{bt} + b^2\,t\,e^{bt} - 2b\,e^{bt} - 2b^2\,t\,e^{bt} + b^2\,t\,e^{bt} = 0.$$

We have found two solutions to the ODE. The set $\{x_1, x_2\}$ is linearly independent. The theorem shows that *any* solution to $\ddot{x} - 2b\,\dot{x} + b^2\,x = 0$ is of the form $x = A\,e^{bt} + B\,t\,e^{bt}$.

Theorem: Let (a, b) be an interval on the real line, and let $t_0 \in (a, b)$.

Let p and q be *continuous* functions on (a, b). Then:

1. For any real numbers x_0 and y_0, there is a unique solution on (a, b) to the equation

(3)
$$\begin{cases} \ddot{x} + p\,\dot{x} + q\,x = & 0 \\ x(t_0) = & x_0 \\ \dot{x}(t_0) = & y_0. \end{cases}$$

2. If x_1 and x_2 are two linearly independent solutions of $\ddot{x} + p\,\dot{x} + q\,x = 0$, then any solution of (3) takes the form $x = c_1\,x_1 + c_2\,x_2$, for some constants c_1 and c_2.
3. The set $V = \{x \in C^2(I) : \ \ddot{x} + p\,\dot{x} + q\,x = 0\}$ is a two-dimensional vector space.

Corollary: Suppose that b is a real number, and consider the ODE

(4)
$$\ddot{x} - 2b\,\dot{x} + b^2\,x = 0.$$

The characteristic equation is $r^2 - 2b\,r + b^2$ which has the double root $r = b$. Then

1. The functions $x_1(t) = e^{bt}$ and $x_2(t) = t\,e^{bt}$ both solve (4).
2. Any solution to (4) takes the form $x = c_1\,x_1 + c_2\,x_2$ for some constants c_1 and c_2.

The existence theorem can be generalized to higher order equations:

Theorem: Let (a, b) be an interval on the real line, and let $t_0 \in (a, b)$.
Let $a_0, a_1, a_2, \ldots, a_{n-1}$ be continuous functions on (a, b). Then the equation

$$\frac{dx^n}{dt^n} + a_{n-1}(t)\,\frac{dx^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\,\frac{dx}{dt} + a_0(t)\,x(t) = 0 \tag{5}$$

has a solution space of dimension precisely n. Moreover, for any $t_0 \in (a, b)$, and for any real numbers $b_0, b_1, b_2, \ldots, b_{n-1}$, there is precisely one solution of (5) that satisfies

$$x(t_0) = b_0, \qquad x'(t_0) = b_1, \qquad x''(t_0) = b_2, \qquad x^{(n-1)}(t_0) = b_{n-1}.$$

Now suppose that we *somehow* (it doesn't matter how!) find a set of solutions $\{x_1, x_2, \ldots, x_m\}$ to (5). Do these form a basis for the solution space?

- If $m < n$, then no — they cannot possibly span an n-dimensional space.
- If $m > n$, then no — they cannot possibly be linearly independent.
- If $m = n$, you need to check if they are *linearly indep.* If they are, then yes!

Wronskians: Suppose that we are given a set $\{f_1, f_2, \dots, f_n\}$ of functions on an interval I, and want to know if the set is linearly independent. One technique that is conceptually straight-forward, but can take some work to execute if n is larger than 3, is to form the so called *Wronskian*,

$$W(t) = \det \begin{bmatrix} f_1(t) & f_2(t) & f_3(t) & \cdots & f_n(t) \\ f_1'(t) & f_2'(t) & f_3'(t) & \cdots & f_n'(t) \\ f_1''(t) & f_2''(t) & f_3''(t) & \cdots & f_n''(t) \\ \vdots & \vdots & \vdots & & \vdots \\ f_1^{(n)}(t) & f_2^{(n)}(t) & f_3^{(n)}(t) & \cdots & f_n^{(n)}(t) \end{bmatrix}.$$

The Wronskian can be used to detect linear independence:

$W(t) \neq 0$ for some $t \in I$ $\quad\Rightarrow\quad$ the set is linearly independent.

In general, it might be that $W(t) = 0$ for every t even for a linearly independent set.

However: If $\{f_1, f_2, \dots, f_n\}$ all solve a n'th order linear DE, then we have

$W(t) = 0$ for some $t \in I$ $\quad\Rightarrow\quad$ the set is linearly dependent.

Example: Let $I = [1, 3]$, and consider for t in this interval, the equation

$$t^3 y''' - t^2 y'' + 2t\, y' - 2y = 0. \tag{6}$$

Set $y_1(t) = t, \quad y_2(t) = t^2, \quad y_3(t) = t\ \log(t)$.

Prove that any solution of (6) takes the form $x = c_1 y_1 + c_2 y_2 + c_3 y_3$!

Solution: For $t \in I$, we have $t^3 \neq 0$, so (6) $\Leftrightarrow y''' - \frac{1}{t} y'' + \frac{2}{t^2} y' - \frac{2}{t^3} y = 0$.
Existence theorem applies, and the solution space has dimension three. So the claim follows if we can prove (a) that every y_j is a solution and (b) that $\{y_1, y_2, y_3\}$ is lin. indep.

Verify that y_3 is a solution: We find $y_3' = \log t + 1$, $y_3'' = 1/t$, and $y_3''' = -1/t^2$. Inserting into (6) we find $t^3 (-1/t^2) - t^2 (1/t) + 2t (\log t + 1) - 2t \log t = 0$.

... you show that y_1 and y_2 are solutions analogously ...

Show that $\{y_1, y_2, y_3\}$ is linearly indep: We form the Wronskian

$$W(t) = \det \begin{bmatrix} t & t^2 & t \log t \\ 1 & 2t & 1 + \log t \\ 0 & 2 & 1/t \end{bmatrix} = 2t + 0 + 2t \log t - 0 - 2t(1 + \log t) - t = -t.$$

Since $W(t) \neq 0$ on I, we knot that $\{y_1, y_2, y_3\}$ is linearly independent!

Now consider a *constant coefficient* ODE

$$x^{(n)} + a_{n-1}\, x^{(n-1)} + a_{n-2}\, x^{(n-2)} + \cdots + a_1\, x' + a_0\, x = 0. \tag{7}$$

Inserting the test solution $x(t) = e^{rt}$ into (7), we find

$$\left(r^n + a_{n-1}\, r^{n-1} + a_{n-2}\, r^{n-2} + \cdots + a_1\, r + a_0\right) e^{rt} = 0.$$

The characteristic polynomial always has at least one root, so there is *always* at least one solution of the form $x(t) = e^{rt}$.

Suppose that the characteristic polynomial has *n distinct roots* $r_1, r_2, \ldots, r_n$. Then

$$\left\{e^{r_1 t}, e^{r_2 t}, \ldots, e^{r_n t}\right\}$$

is a basis for the solution space of (7), and so *any* solution can be written

$$x(t) = A_1\, e^{r_1 t} + A_2\, e^{r_2 t} + \cdots + A_n\, e^{r_n t}.$$

Example: Find the general solution to

$$\frac{dx^4}{dt^4} - 16\,x = 0.$$

Solution: First observe that the characteristic equation is

$$r^4 - 16 = 0.$$

The roots of this equation are

$$r_1 = 2, \qquad r_2 = -2, \qquad r_3 = 2i, \qquad r_4 = -2i,$$

and so the general solution is

$$x(t) = c_1\,e^{2t} + c_2\,e^{-2t} + c_3\,e^{i2t} + c_4\,e^{-i2t}.$$

If we want a purely *real* formulation, then observe that

$$x(t) = c_1\,e^{2t} + c_2\,e^{-2t} + (c_3 + c_4)\,\cos(2t) + (ic_3 - ic_4)\,\sin(2t).$$

Now set $d_1 = c_3 + c_4$ and $d_2 = ic_3 - c_4$ to obtain

$$x(t) = c_1\,e^{2t} + c_2\,e^{-2t} + d_1\,\cos(2t) + d_2\,\sin(2t).$$

Example: Find the general solution to

$$x''' + 5\,x'' + 3\,x' - 9 = 0.$$

Hint: One solution is given by $x(t) = e^t$!

Solution: First observe that the characteristic equation is

$$r^3 + 5r^2 + 3r - 9 = 0.$$

The hint tells us that $r_1 = 1$ is one solution, which allows us to factor

$$r^3 + 5r^2 + 3r - 9 = (r - 1)\,(r^2 + 6r + 9).$$

The remaining two roots are then

$$r_{2,3} = -3 \pm \sqrt{3^2 - 9} = -3.$$

Since -3 is a double root, we find that the general solution is

$$x(t) = A\,e^t + B\,e^{-3t} + C\,t\,e^{-3t}.$$

	Exam 1	Exam 2	Exam 3	Final
Mean	64	72		
Median	65	73		
SD		15		
C- cutoff	38	49		
High score	99 ($\times$ 3)	100 ($\times$ 5)		