MATH 393C: Fast Methods in Scientific Computing

Lecture on March 13, 2019.

Supplementary material on adaptive FMM – new material on page 25 onwards.

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The outgoing expansion

Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

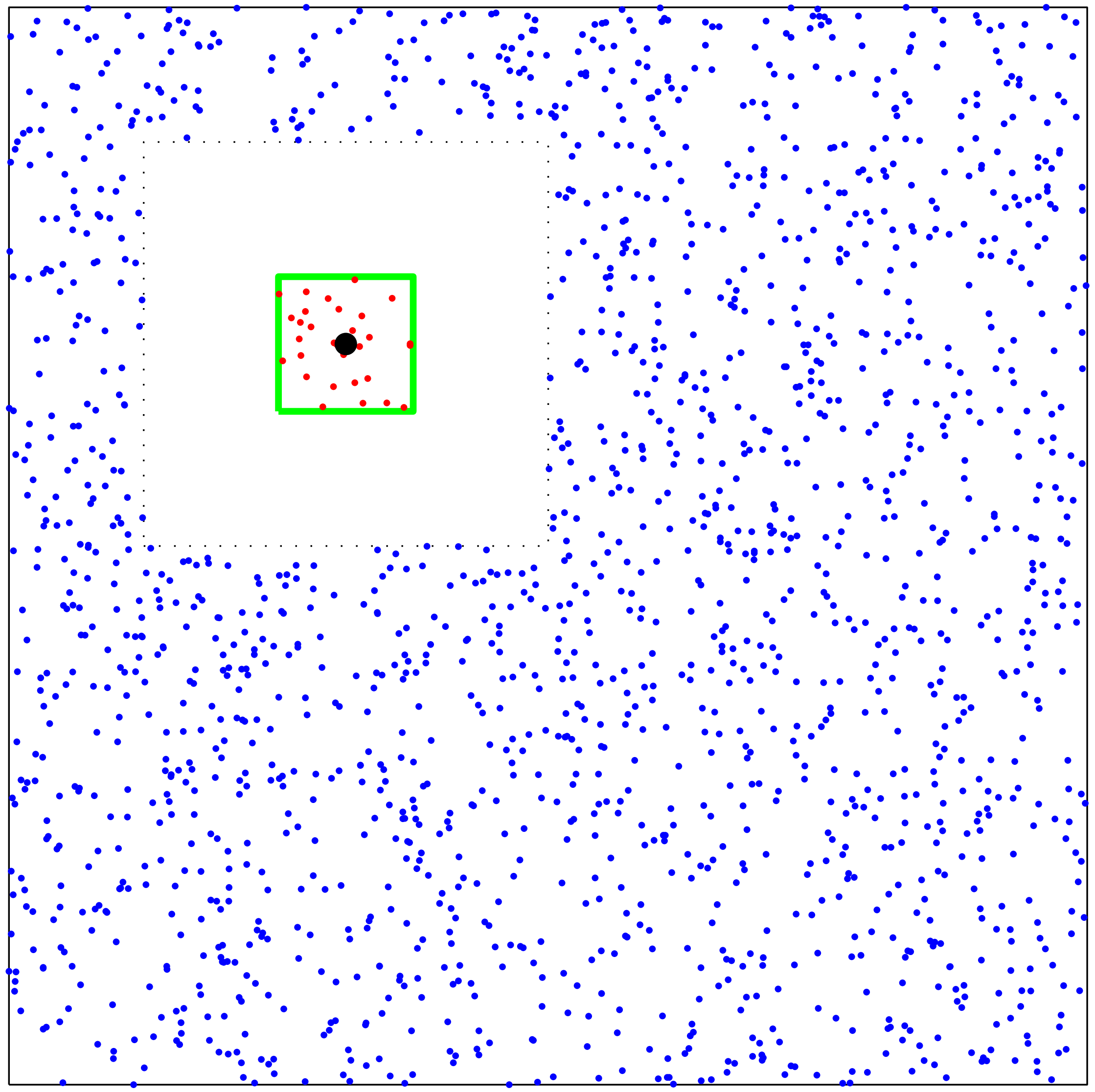
Let $\mathbf{y}_j$ be source locations in τ (red).

Let q_j be the strength of source j.

Let $\mathbf{x}_i$ be targets well separated from τ (blue).

Let u denote the potential

$$u(\mathbf{x}_i) = \sum_j q_j \log(\mathbf{x}_i - \mathbf{y}_j).$$



The *outgoing expansion* of τ is a vector $\hat{\mathbf{q}} = [\hat{q}_p]_{p=0}^{P}$ of complex numbers such that

(1) $$u(\mathbf{x}) \approx \hat{q}_0 \log|\mathbf{x} - \mathbf{c}_\tau| + \sum_{p=1}^{P} \hat{q}_p \frac{1}{(\mathbf{x} - \mathbf{c}_\tau)^p}, \qquad \mathbf{x} \in \Omega_\tau^{\mathrm{far}}.$$

The outgoing expansion is a compact representation of the sources inside τ (it encodes both the source locations and the magnitudes).

The incoming expansion

Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

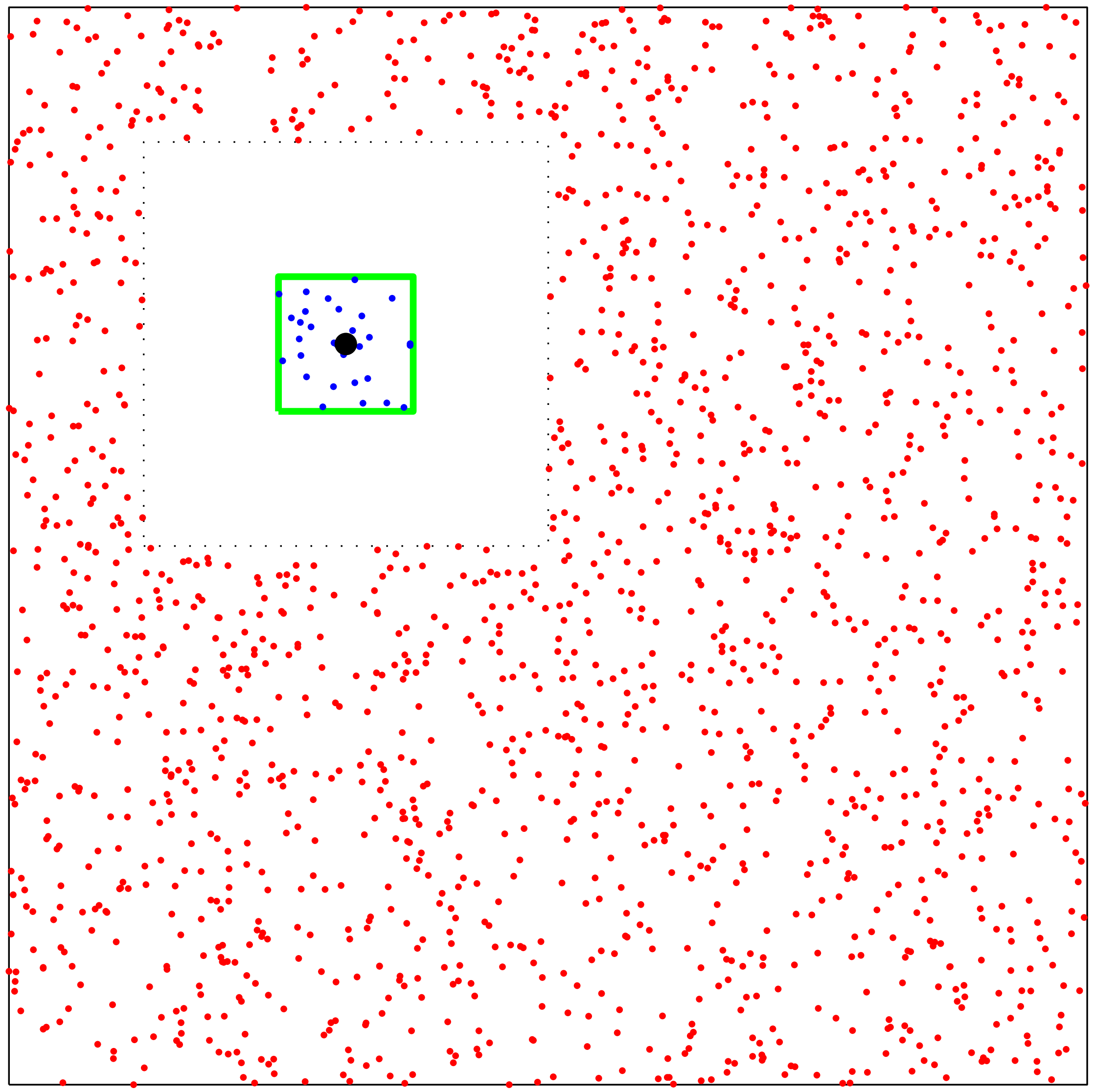
Let $\mathbf{y}_j$ be sources well-separated from τ (red).

Let q_j be strength of source j.

Let $\mathbf{x}_i$ be targets inside τ (blue).

Let u denote the potential

$$u(\mathbf{x}_i) = \sum_j q_j \log(\mathbf{x}_i - \mathbf{y}_j).$$



The *incoming expansion* of τ is a vector $\hat{\mathbf{u}} = [\hat{u}_p]_{p=0}^P$ of complex numbers such that

(2) $$u(\mathbf{x}) \approx \sum_{p=0}^{P} \hat{u}_p (\mathbf{x} - \mathbf{c}_\tau)^p, \qquad \mathbf{x} \in \Omega_\tau.$$

The incoming expansion is a compact representation of the sources well-separated from τ

(it encodes both the source locations and the magnitudes).

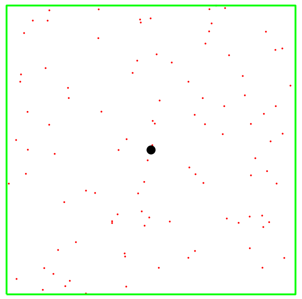
The *outgoing-from-sources* translation operator $\mathbf{T}_\tau^{(\mathrm{ofs})}$

Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

Let $\{\mathbf{y}_j\}_j^{N_\tau}$ be source locations in τ (red).

Let q_j be strength of source j.



The operator $\mathbf{T}_\tau^{(\mathrm{ofs})}$ constructs the outgoing expansion directly from the vector of charges.

$$\begin{array}{ccccc} \hat{\mathbf{q}}_\tau & = & \mathbf{T}_\tau^{(\mathrm{ofs})} & & \mathbf{q} \\ (P+1)\times 1 & & (P+1)\times N_\tau & & N_\tau \times 1 \end{array}$$

$$\mathbf{T}_{\tau,0,j}^{(\mathrm{ofs})} = 1 \qquad 1 \leq j \leq N_\tau$$

$$\mathbf{T}_{\tau,p,j}^{(\mathrm{ofs})} = -\frac{1}{p}(\mathbf{y}_j - \mathbf{c}_\tau)^p \qquad 1 \leq p \leq P \qquad 1 \leq j \leq N_\tau.$$

The *outgoing-from-outgoing* translation operator $\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma}$

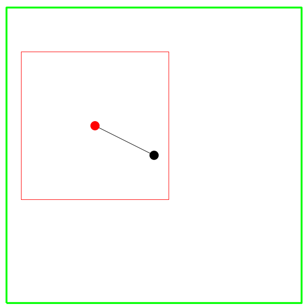
Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of $\mathbf{c}_\tau$ (black).

Let σ denote a box contained in τ.

Let $\mathbf{c}_\sigma$ denote the center of σ (red).

Let $\hat{\mathbf{q}}_\sigma$ be outgoing expansion of σ.



$\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma}$ constructs the outgoing expansion of τ from the outgoing expansion of σ

$$\underset{(P+1)\times 1}{\hat{\mathbf{q}}_\tau} = \underset{(P+1)\times(P+1)}{\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma}} \; \underset{(P+1)\times 1}{\hat{\mathbf{q}}_\sigma}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma}$ is a lower tridiagonal matrix with entries

$$\begin{aligned}
\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma,0,0} &= 1 \\
\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma,p,0} &= -\frac{1}{p}\mathbf{d} && 1 \leq p \leq P \\
\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma,p,q} &= \binom{p}{q}\mathbf{d}^{p-q} && 1 \leq q \leq p \leq P.
\end{aligned}$$

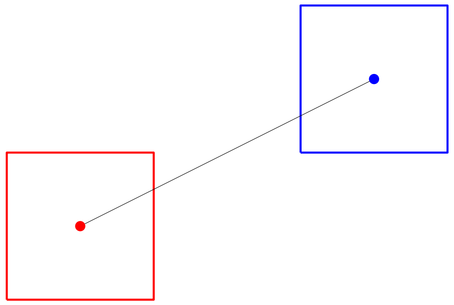
The *incoming-from-outgoing* translation operator $\mathbf{T}^{(\mathrm{ifo})}_{\tau,\sigma}$

Let σ be a source box (red) with center $\mathbf{c}_\sigma$.

Let τ be a target box (blue) with center $\mathbf{c}_\tau$.

Let $\hat{\mathbf{q}}_\sigma$ be the outgoing expansion of σ.

Let $\hat{\mathbf{u}}_\tau$ represent the potential in τ caused by sources in σ.



$\mathbf{T}^{(\mathrm{ifo})}_{\tau,\sigma}$ constructs the incoming expansion of τ from the outgoing expansions of σ:

$$\underset{(P+1)\times 1}{\hat{\mathbf{u}}_\tau} = \underset{(P+1)\times(P+1)}{\mathbf{T}^{(\mathrm{ifo})}_{\tau,\sigma}} \; \underset{(P+1)\times 1}{\hat{\mathbf{q}}_\sigma}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathbf{T}^{(\mathrm{ifo})}_{\tau,\sigma}$ is a matrix with entries

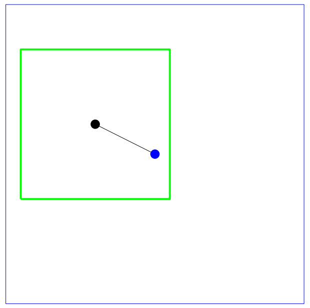
$$\mathbf{T}^{(\mathrm{ofo})}_{\tau,\sigma,p,q} = ?$$

The *incoming-from-incoming* translation operator $\mathbf{T}^{(\mathrm{ifi})}_{\tau,\sigma}$

Let τ be a box (green) with center $\mathbf{c}_\tau$ (black).

Let σ be a box (blue) containing τ with center $\mathbf{c}_\sigma$.

Let $\hat{\mathbf{u}}_\sigma$ be an incoming expansion for σ.



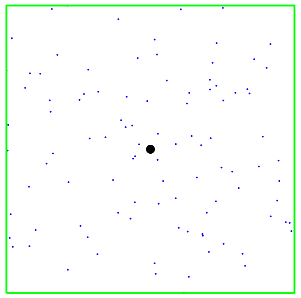
$\mathbf{T}^{(\mathrm{ifi})}_{\tau,\sigma}$ constructs the incoming expansion of τ from the incoming expansion of σ

$$\begin{array}{cccc} \hat{\mathbf{u}}_\tau & = & \mathbf{T}^{(\mathrm{ifi})}_{\tau,\sigma} & \hat{\mathbf{u}}_\sigma \\ (P+1)\times 1 & & (P+1)\times(P+1) & (P+1)\times 1 \end{array}$$

With $\mathbf{d} = \mathbf{c}_\sigma - \mathbf{c}_\tau$, $\mathbf{T}^{(\mathrm{ifi})}_{\tau,\sigma}$ is a matrix with entries

$$\mathbf{T}^{(\mathrm{ifi})}_{\tau,\sigma,p,q} = ?$$

The *targets-from-incoming* translation operator $\mathbf{T}_{\tau}^{(\mathrm{tfi})}$



Let τ be a box (green).

Let $\mathbf{c}_{\tau}$ be the center of τ (black).

Let $\{\mathbf{x}_i\}_{i}^{N_{\tau}}$ be target locations in τ (blue).

Let $\hat{\mathbf{u}}_{\tau}$ be the incoming expansion of τ.

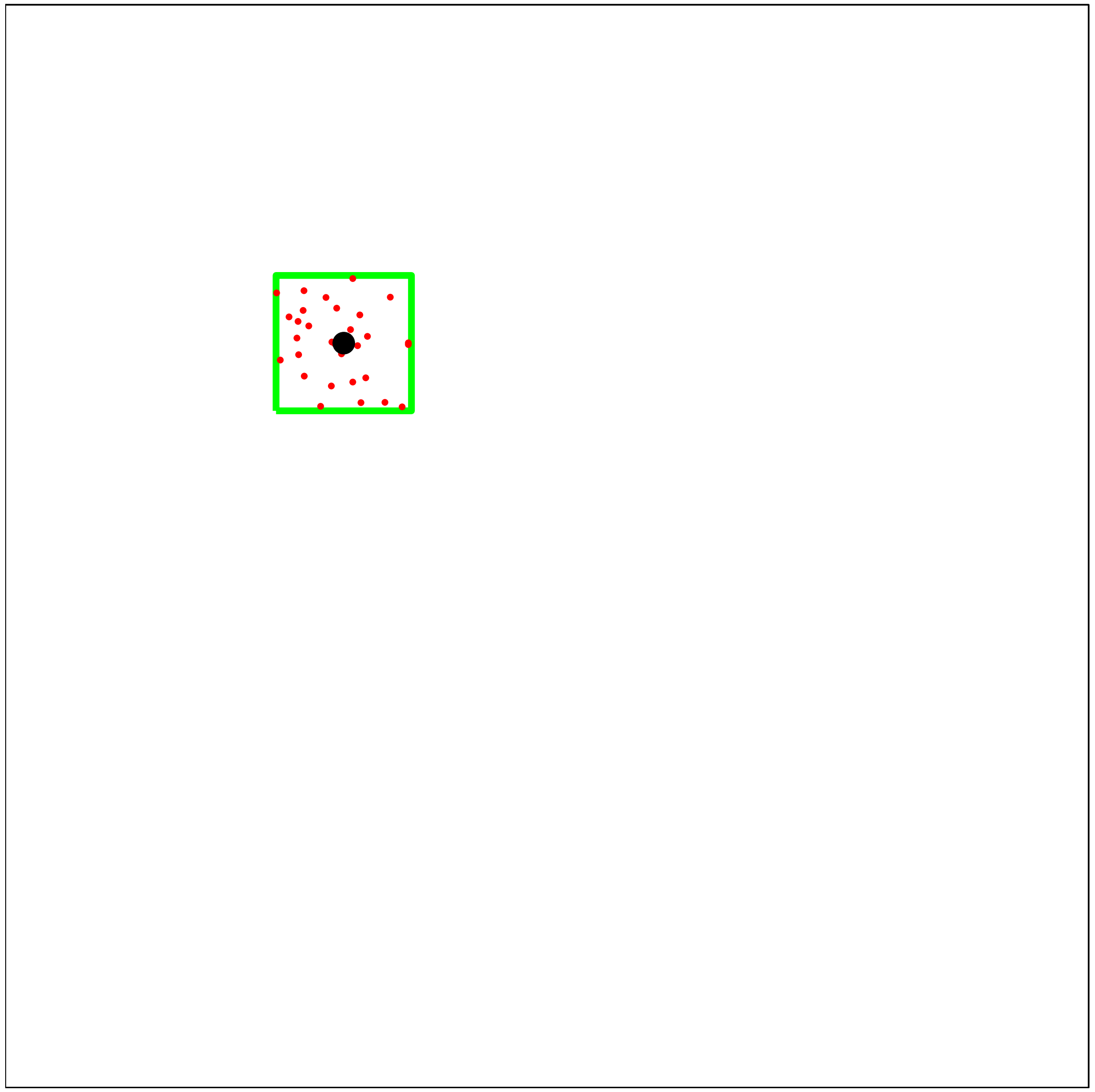
$\mathbf{T}_{\tau}^{(\mathrm{tfi})}$ constructs the potentials in τ from the incoming expansion

$$\begin{array}{ccccc} \mathbf{u}_{\tau} & = & \mathbf{T}_{\tau}^{(\mathrm{tfi})} & & \hat{\mathbf{u}}_{\tau} \\ N_{\tau}\times 1 & & N_{\tau}\times(P+1) & & (P+1)\times 1 \end{array}$$

$$\mathbf{T}_{\tau,i,p}^{(\mathrm{tfi})} = (\mathbf{x}_i - \mathbf{c}_{\tau})^{p} \qquad 1 \le i \le N_{\tau} \quad 0 \le p \le P.$$

How do you compute the expansions of a box?

Computing the outgoing expansion of a leaf



Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

Let $\{\mathbf{y}_j\}_j^{N_\tau}$ be source locations in τ (red).

Let q_j be strength of source j.

There is an analytic formula:

$$\hat{q}_0 = \sum_{j=1}^{N_\tau} q_j \qquad \hat{q}_p = -\frac{1}{p}\sum_{j=1}^{N_\tau} q_j\,(\mathbf{y}_j - \mathbf{c}_\tau)^p, \qquad p = 1,\,2,\,\dots,\,P.$$

We write the formula compactly as

$$\hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\,\mathbf{q}_\tau.$$

Computing the outgoing expansion of a parent

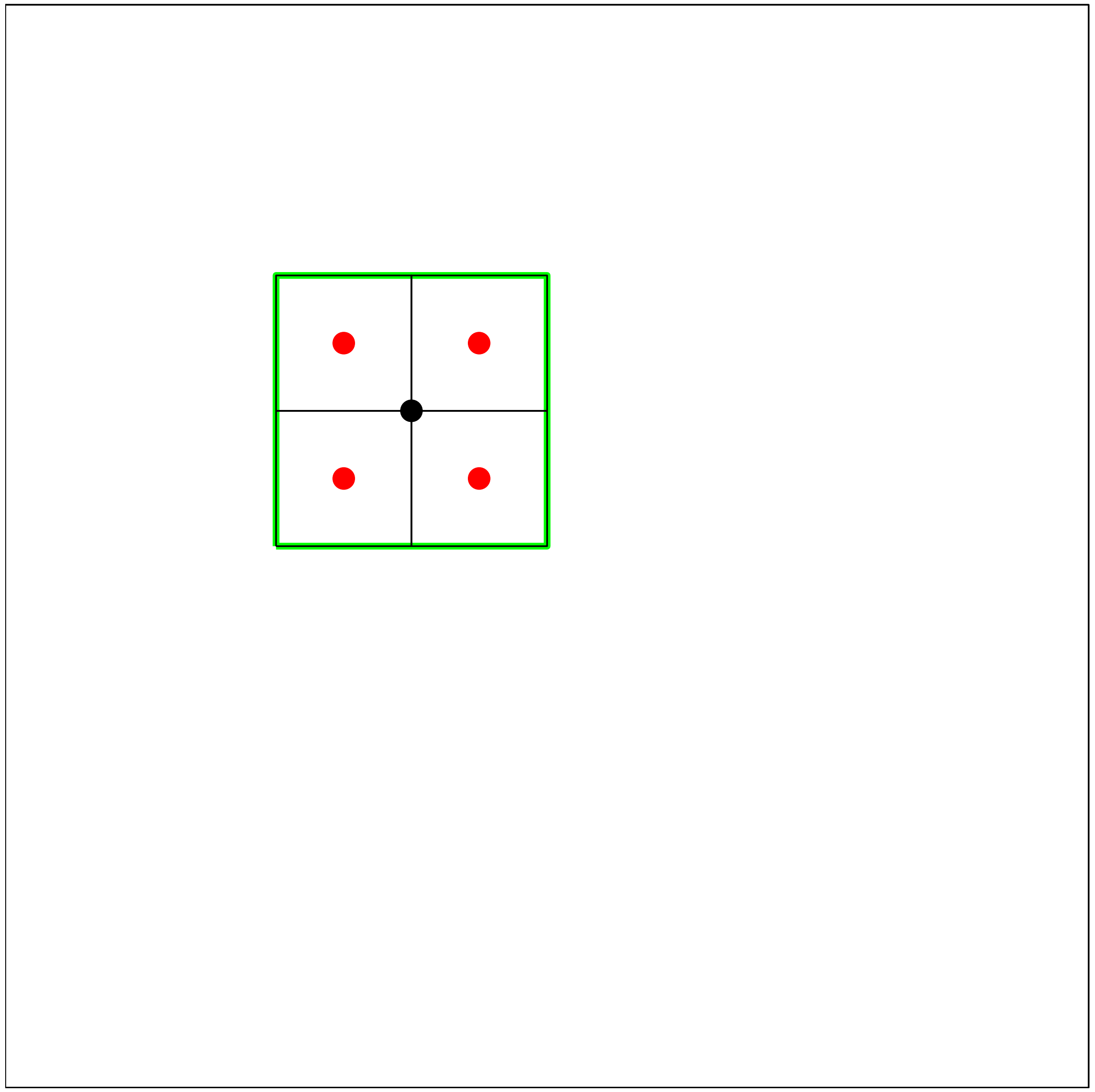
Let τ be a box (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

Let $\mathcal{L}_\tau^{(\text{child})}$ denote the children of τ.

Let $\mathbf{c}_\sigma$ be the center of child σ.

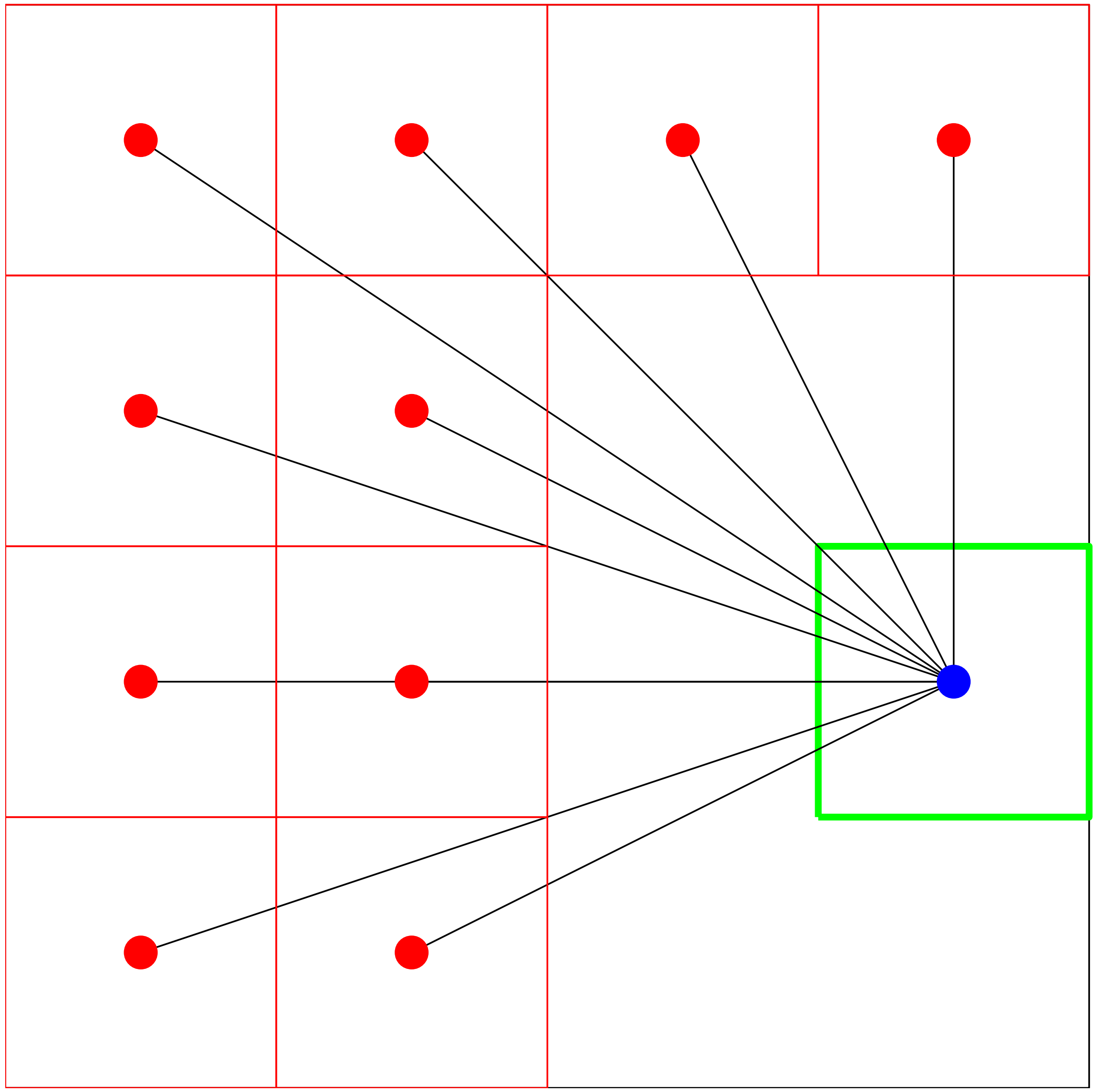
Let $\hat{\mathbf{q}}_\sigma$ be the outgoing expansion of child σ.



The outgoing expansion of τ can be computed from the outgoing expansions of its children:

$$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathbf{T}_{\tau,\sigma}^{(\text{ofo})} \hat{\mathbf{q}}_\sigma.$$

Computing the incoming expansions on level 2



Let τ be a box on level 2 (green).

Let $\mathbf{c}_\tau$ be the center of τ (black).

The well-separated boxes on level 2 are red.

The incoming expansion of τ is computed from the outgoing expansions of boxes in its interaction list

$$\hat{\mathbf{u}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{int})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifo})} \, \hat{\mathbf{q}}_\sigma.$$

Computing the incoming expansions on level ℓ when $\ell > 2$

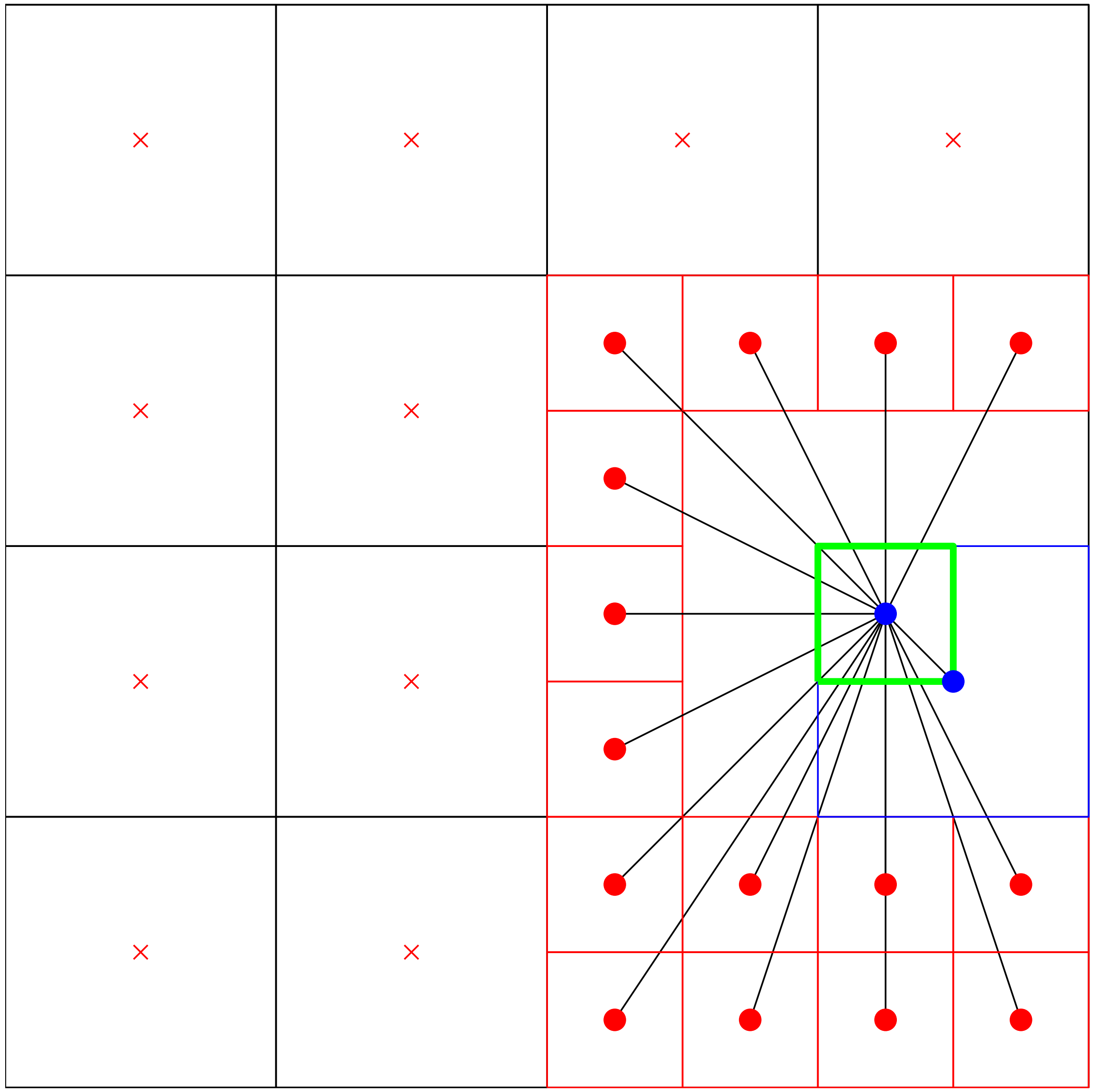
Let τ be a box on level $\ell = 3$ (green).

Let ν be the parent of τ (blue).

Let $u_{\rm in}^{\tau}$ denote the potential caused by charges that are well-separated from τ — these are charges in the boxes marked with red dots and crosses. We have

$$u_{\rm in}^{\tau} = u_{\rm in}^{\nu} + v,$$

where $u_{\rm in}^{\nu}$ is the incoming field for τ's parent (caused by the boxes with red crosses), and v is the field caused by boxes in the interaction list of τ (boxes with a red dot).



The field $u_{\rm in}^{\nu}$ was computed on the previous level and is represented by $\hat{\mathbf{u}}_{\nu}$.

The field v is computed by transferring the outgoing expansions $\hat{\mathbf{q}}_{\sigma}$ for $\sigma \in \mathcal{L}_{\tau}^{(\rm int)}$.

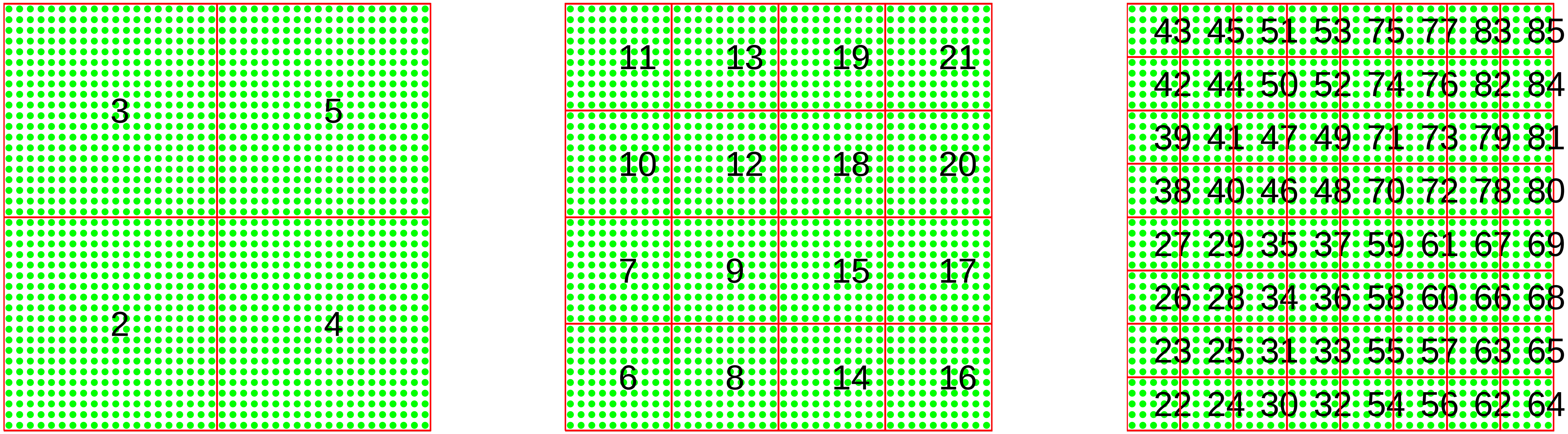
$$\hat{\mathbf{u}}_{\tau} = \mathbf{T}_{\tau,\nu}^{(\rm ifi)}\,\hat{\mathbf{u}}_{\nu} + \sum_{\sigma \in \mathcal{L}_{\tau}^{(\rm int)}} \mathbf{T}_{\tau,\sigma}^{(\rm ifo)}\,\hat{\mathbf{q}}_{\sigma}$$

$$\sim u_{\rm in}^{\tau} \qquad \sim u_{\rm in}^{\nu} \qquad \sim v$$

The classical Fast Multipole Method in $\mathbb{R}^2$

1. Construct the tree and all "interaction lists."
2. For each leaf node, compute its outgoing expansion directly from the charges in the box via the *outgoing-from-sources operator.*
3. For each parent node, compute its outgoing expansion by merging the expansions of its children via the *outgoing-from-outgoing operator.*
4. For each node, compute its incoming expansion by transferring the incoming expansion of its parent (via the *incoming-from-incoming operator*), and then add the contributions from all charges in its interaction list (via the *incoming-from-outgoing operator*).
5. For each leaf node, evaluate the incoming expansion at the targets (via the *targets-from-incoming operator*), and compute near-field interactions directly.

Construct the tree and all interaction lists.



Let L denote the number of levels in the tree.

Set all potentials to zero:

For all boxes τ

$$\begin{aligned}\hat{\mathbf{u}}_\tau &= \mathbf{0} \\ \hat{\mathbf{q}}_\tau &= \mathbf{0}.\end{aligned}$$

Set the potential to zero:

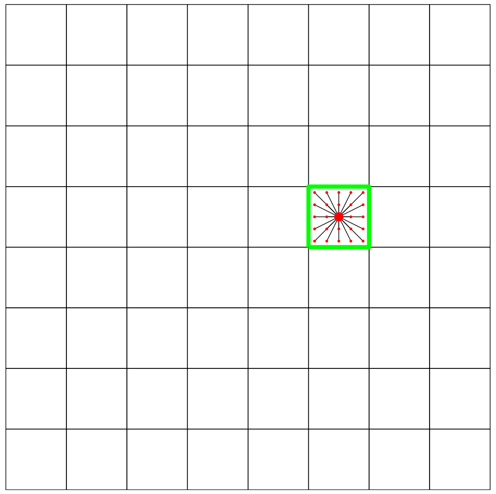
$$\mathbf{u} = \mathbf{0}.$$

Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators:*

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\,\mathbf{q}(J_\tau)$$

end loop



Compute the outgoing expansion of each parent by merging the expansions of its children via application of the *outgoing-from-outgoing operators:*

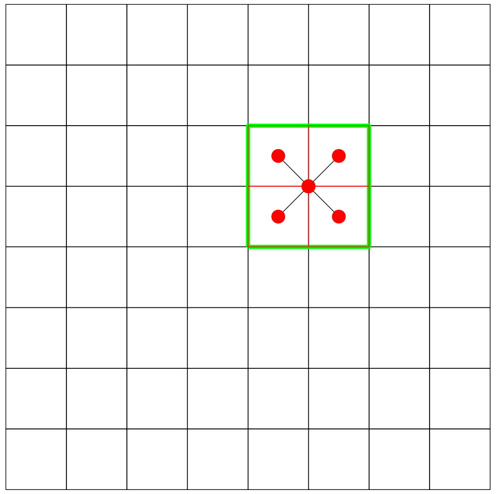
loop over levels $\ell = L-1, L-2, \ldots, 2$

loop over all nodes τ on level ℓ

$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{child})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ofo})} \hat{\mathbf{q}}_\sigma$

end loop

end loop

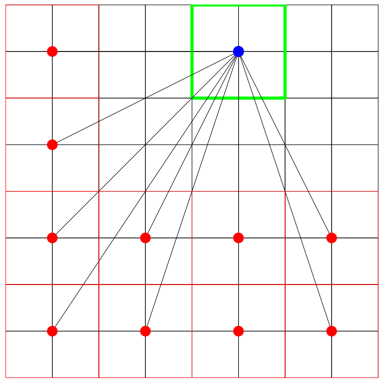


Add contributions from boxes in the interaction list of each box via the *incoming-from-outgoing operators:*

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{int})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifo})}\, \hat{\mathbf{q}}_\sigma.$$

end loop



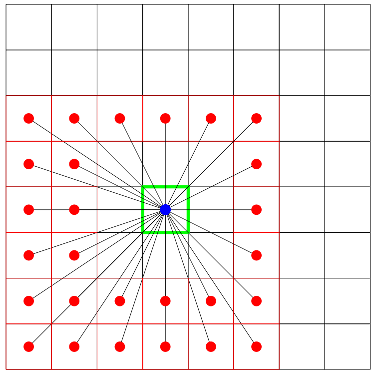
Add contributions from boxes in the interaction list of each box via the

incoming-from-outgoing operators:

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{int})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifo})} \, \hat{\mathbf{q}}_\sigma.$$

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators:*

loop over levels $\ell = 2, 3, 4, \dots, L-1$

  loop over all nodes τ on level ℓ

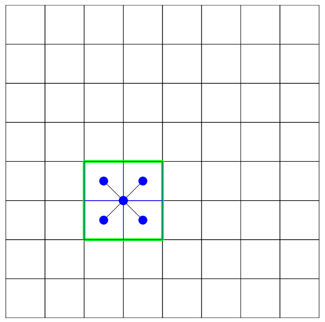
    loop over all children σ of τ

      $\hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathbf{T}^{(\mathrm{ifi})}_{\sigma,\tau}\,\hat{\mathbf{u}}_\tau.$

    end loop

  end loop

end loop

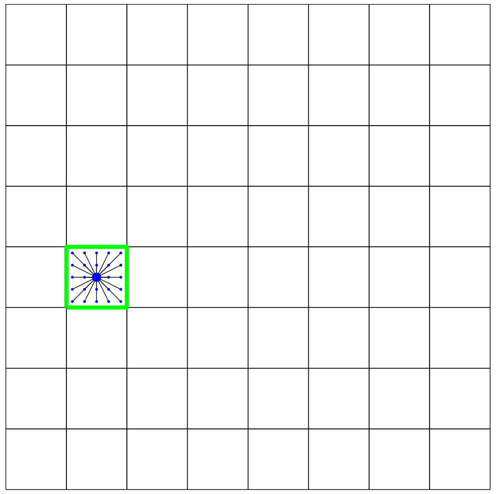


Compute the potential on every leaf by expanding its incoming potential via the *targets-from-incoming operators:*

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{T}_\tau^{(\mathrm{tfi})}\,\hat{\mathbf{u}}_\tau$$

end loop

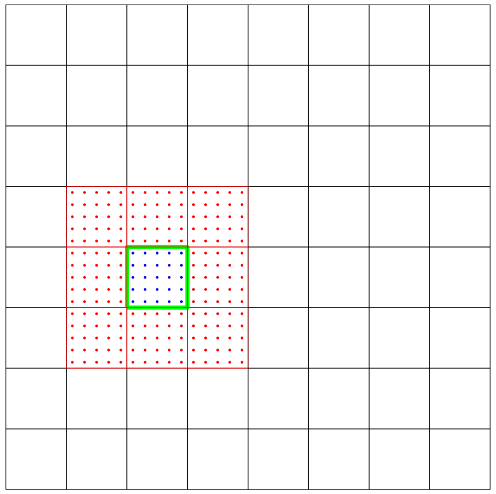


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau)\,\mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{nei})}} \mathbf{A}(J_\tau, J_\sigma)\,\mathbf{q}(J_\sigma)$$

end loop



Set $\hat{\mathbf{u}}_\tau = \mathbf{0}$ and $\hat{\mathbf{q}}_\tau = \mathbf{0}$ for all τ.

loop over all leaf nodes τ

$\quad \hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\, \mathbf{q}(J_\tau)$

end loop

loop over levels $\ell = L, L-1, \dots, 2$

$\quad$ **loop** over all nodes τ on level ℓ

$\quad\quad \hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{child})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ofo})}\, \hat{\mathbf{q}}_\sigma$

$\quad$ **end loop**

end loop

loop over all nodes τ

$\quad \hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{int})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifo})}\, \hat{\mathbf{q}}_\sigma.$

end loop

loop over levels $\ell = 2, 3, 4, \dots, L-1$

$\quad$ **loop** over all nodes τ on level ℓ

$\quad\quad$ **loop** over all children σ of τ

$\quad\quad\quad \hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathbf{T}_{\sigma,\tau}^{(\mathrm{ifi})}\, \hat{\mathbf{u}}_\tau.$

$\quad\quad$ **end loop**

$\quad$ **end loop**

end loop

loop over all leaf nodes τ

$\quad \mathbf{u}(J_\tau) = \mathbf{T}_\tau^{(\mathrm{tfi})}\, \hat{\mathbf{u}}_\tau$

end loop

loop over all leaf nodes τ

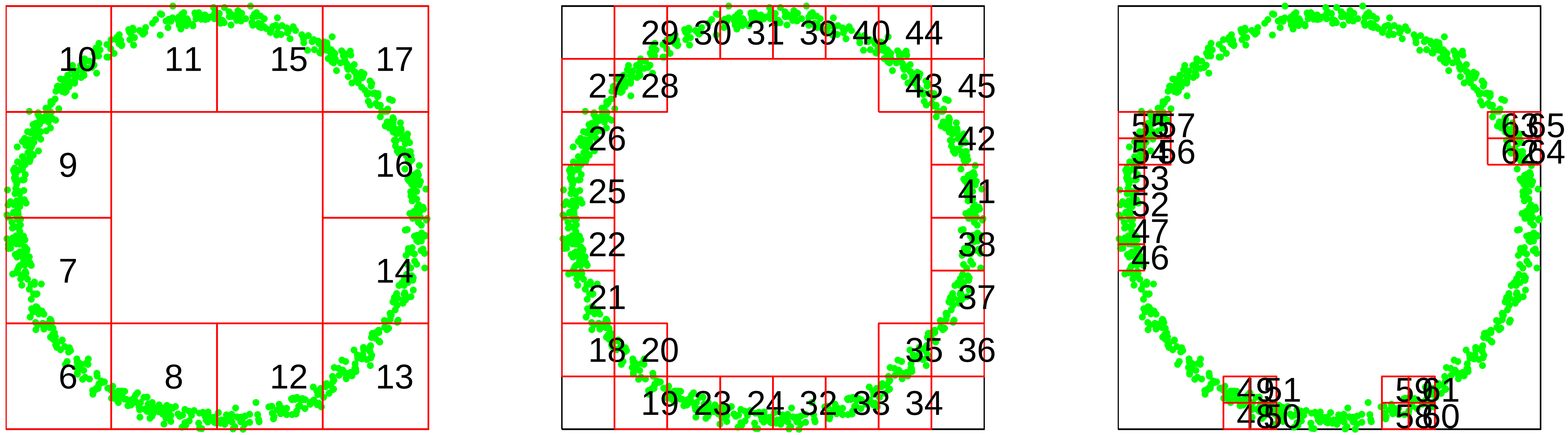
$\quad \mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau)\, \mathbf{q}(J_\tau)$

$\quad\quad + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{nei})}} \mathbf{A}(J_\tau, J_\sigma)\, \mathbf{q}(J_\sigma)$

end loop

Now let us consider a non-uniform tree.

Construct the tree and all interaction lists.



Let L denote the number of levels in the tree.

Set all potentials to zero:

For all boxes τ

$$\begin{aligned}\hat{\mathbf{u}}_\tau &= \mathbf{0} \\ \hat{\mathbf{q}}_\tau &= \mathbf{0}.\end{aligned}$$

Set the potential to zero:

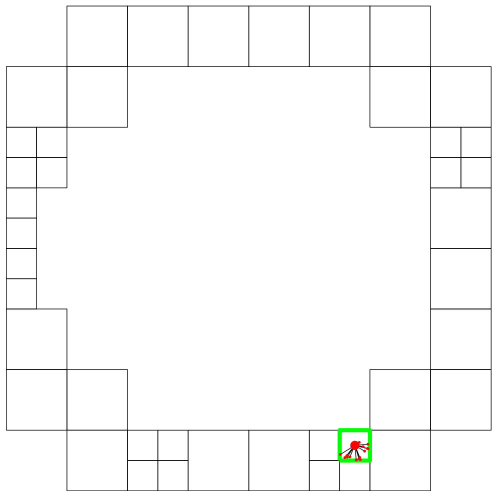
$$\mathbf{u} = \mathbf{0}.$$

Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators:*

loop over all leaf nodes τ

$\hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\,\mathbf{q}(J_\tau)$

end loop

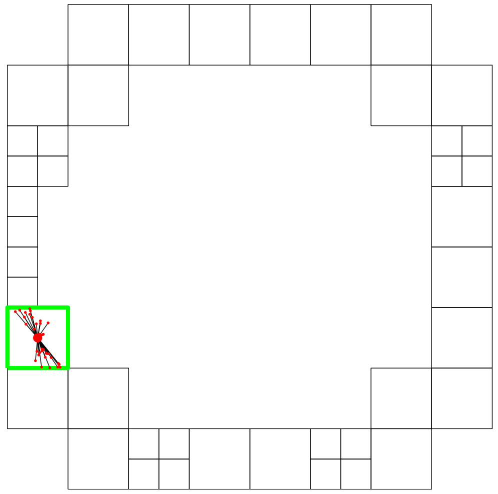


Compute the outgoing expansion on each leaf via application of the *outgoing-from-source operators:*

loop over all leaf nodes τ

$$\hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\,\mathbf{q}(J_\tau)$$

end loop



Compute the outgoing expansion of each parent by merging the expansions of its children via application of the *outgoing-from-outgoing operators:*

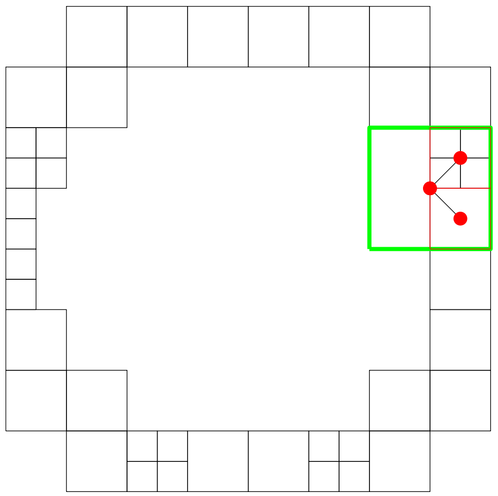
loop over levels $\ell = L-1, L-2, \ldots, 2$

loop over all nodes τ on level ℓ

$$\hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\text{child})}} \mathbf{T}_{\tau,\sigma}^{(\text{ofo})} \, \hat{\mathbf{q}}_\sigma$$

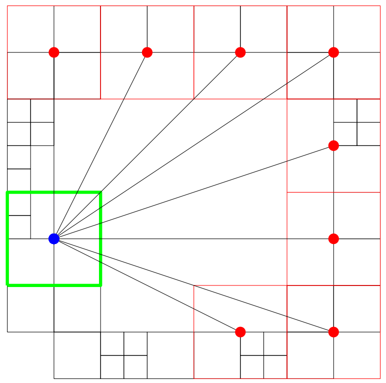
end loop

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators:*

loop over levels $\ell = 2, 3, 4, \ldots, L-1$
  loop over all nodes τ on level ℓ
    loop over all children σ of τ
      $\hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathbf{T}^{(\mathrm{ifi})}_{\sigma,\tau}\,\hat{\mathbf{u}}_\tau$.
    end loop
  end loop
end loop

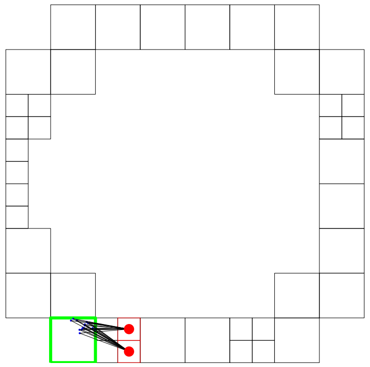


New: Some leaves τ (e.g. the green one below) must collect contributions to their potentials from outgoing expansions on boxes σ (red) on finer levels via the *targets-from-outgoing operator:*

loop over all nodes leaf τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{tfo})}\, \hat{\mathbf{q}}_\sigma.$$

end loop

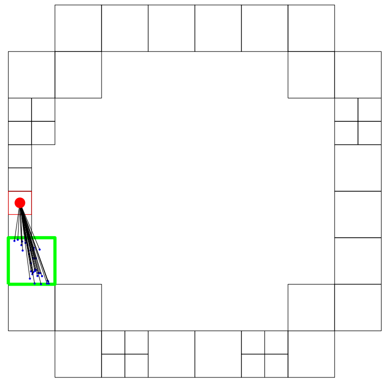


New: Some leaves τ (e.g. the green one below) must collect contributions to their potentials from outgoing expansions on boxes σ (red) on finer levels via the *targets-from-outgoing operator:*

loop over all nodes leaf τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{tfo})}\, \hat{\mathbf{q}}_\sigma.$$

end loop

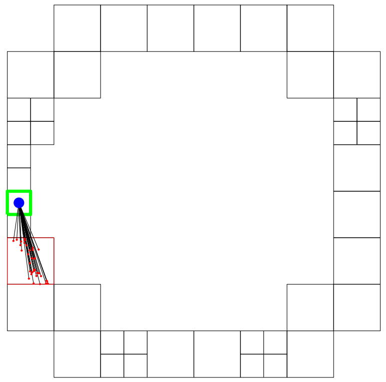


New: Some boxes τ (e.g. the green one) must collect contributions to their incoming expansions directly from the sources in some leaves σ (red) via the *incoming-from-sources operator:*

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(4)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifs})} \mathbf{q}(J_\sigma).$$

end loop

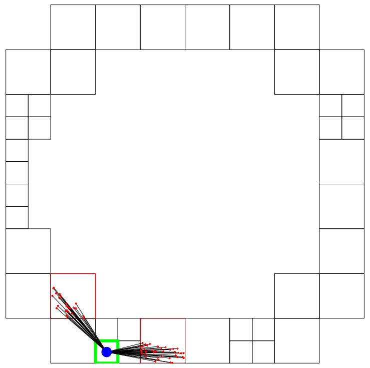


New: Some boxes τ (e.g. the green one) must collect contributions to their incoming expansions directly from the sources in some leaves σ (red) via the *incoming-from-sources operator:*

loop over all nodes τ

$$\hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(4)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifs})} \, \mathbf{q}(J_\sigma).$$

end loop



Add contributions from the parent of each box via via the *incoming-from-incoming operators:*

loop over levels $\ell = 2, 3, 4, \dots, L-1$

  loop over all nodes τ on level ℓ

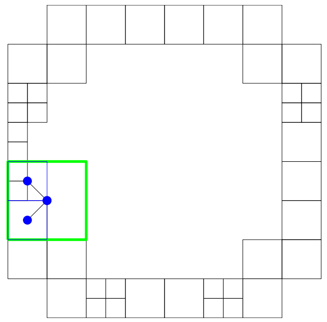
    loop over all children σ of τ

      $\hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathbf{T}^{(\mathrm{ifi})}_{\sigma,\tau}\,\hat{\mathbf{u}}_\tau.$

    end loop

  end loop

end loop

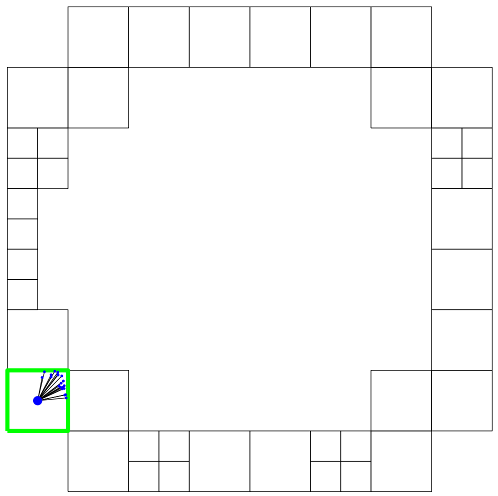


Compute the potential on every leaf by expanding its incoming potential via the *targets-from-incoming operators:*

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{T}_\tau^{(\mathrm{tfi})}\,\hat{\mathbf{u}}_\tau$$

end loop

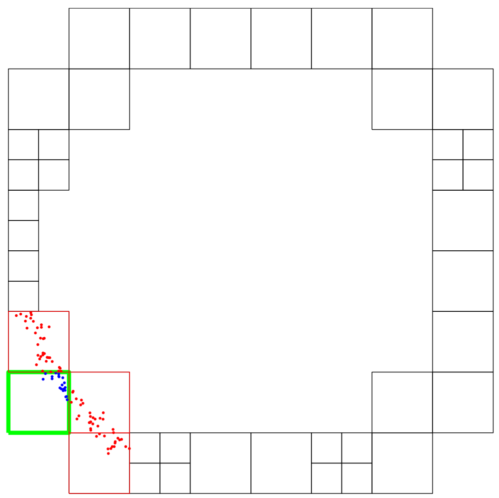


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau)\,\mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{nei})}} \mathbf{A}(J_\tau, J_\sigma)\,\mathbf{q}(J_\sigma)$$

end loop

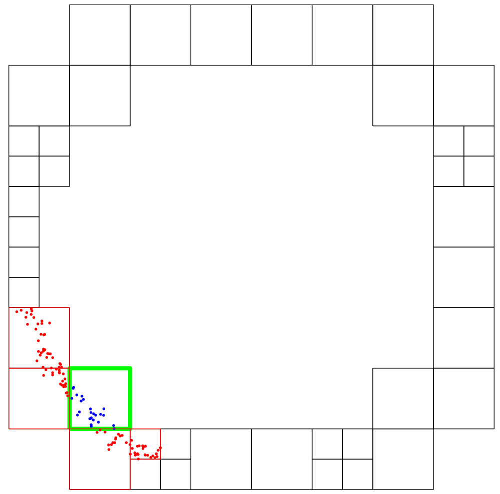


Add to the leaf potentials the interactions from direct neighbors:

loop over all leaf nodes τ

$$\mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau)\,\mathbf{q}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{nei})}} \mathbf{A}(J_\tau, J_\sigma)\,\mathbf{q}(J_\sigma)$$

end loop



Set $\hat{\mathbf{u}}_\tau = \mathbf{0}$ and $\hat{\mathbf{q}}_\tau = \mathbf{0}$ for all τ.

loop over all leaf nodes τ
$\quad \hat{\mathbf{q}}_\tau = \mathbf{T}_\tau^{(\mathrm{ofs})}\, \mathbf{q}(J_\tau)$
end loop

loop over levels $\ell = L,\, L-1,\, \dots,\, 2$
$\quad$ **loop** over all nodes τ on level ℓ
$\quad\quad \hat{\mathbf{q}}_\tau = \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{child})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ofo})}\, \hat{\mathbf{q}}_\sigma$
$\quad$ **end loop**
end loop

loop over all nodes τ
$\quad \hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{int})}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifo})}\, \hat{\mathbf{q}}_\sigma.$
end loop

loop over all nodes τ
$\quad \hat{\mathbf{u}}_\tau = \hat{\mathbf{u}}_\tau + \sum_{\sigma \in \mathcal{L}_\tau^{(4)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{ifs})}\, \mathbf{q}(J_\sigma).$
end loop

loop over levels $\ell = 2,\, 3,\, 4,\, \dots,\, L-1$
$\quad$ **loop** over all nodes τ on level ℓ
$\quad\quad$ **loop** over all children σ of τ
$\quad\quad\quad \hat{\mathbf{u}}_\sigma = \hat{\mathbf{u}}_\sigma + \mathbf{T}_{\sigma,\tau}^{(\mathrm{ifi})}\, \hat{\mathbf{u}}_\tau.$
$\quad\quad$ **end loop**
$\quad$ **end loop**
end loop

loop over all leaf nodes τ
$\quad \mathbf{u}(J_\tau) = \mathbf{T}_\tau^{(\mathrm{tfi})}\, \hat{\mathbf{u}}_\tau$
end loop

loop over all nodes τ
$\quad \mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \sum_{\sigma \in \mathcal{L}_\tau^{(3)}} \mathbf{T}_{\tau,\sigma}^{(\mathrm{tfo})}\, \hat{\mathbf{q}}_\sigma.$
end loop

loop over all leaf nodes τ
$\quad \mathbf{u}(J_\tau) = \mathbf{u}(J_\tau) + \mathbf{A}(J_\tau, J_\tau)\, \mathbf{q}(J_\tau)$
$\quad\quad + \sum_{\sigma \in \mathcal{L}_\tau^{(\mathrm{nei})}} \mathbf{A}(J_\tau, J_\sigma)\, \mathbf{q}(J_\sigma)$
end loop

A summary of the lists needed:

$\mathcal{L}_\tau^{(\text{child})}$ The children of τ.

$\mathcal{L}_\tau^{(\text{parent})}$ The parent of τ.

$\mathcal{L}_\tau^{(\text{nei})}$ For a leaf box τ, this is a list of the leaf boxes that directly border τ.
For a non-leaf box, $\mathcal{L}_\tau^{(\text{nei})}$ is empty.

$\mathcal{L}_\tau^{(\text{int})}$ A box $\sigma \in \mathcal{L}_\tau^{(\text{int})}$ iff σ and τ are on the same level,
σ and τ are well-separated,
but the parents of σ and τ are not well-separated.

$\mathcal{L}_\tau^{(3)}$ For a *leaf* box τ, a box $\sigma \in \mathcal{L}_\tau^{(3)}$ iff σ lives on a finer level than τ,
τ is well-separated from σ, but τ is not well-separated from the parent of σ.
For a non-leaf box τ, $\mathcal{L}_\tau^{(3)}$ is empty.

$\mathcal{L}_\tau^{(4)}$ The dual of $\mathcal{L}_\tau^{(3)}$. In other words, $\sigma \in \mathcal{L}_\tau^{(4)}$ if and only if $\tau \in \mathcal{L}_\sigma^{(3)}$.

A summary of the translation operators: