Homework set 5 — MATH 393C — Spring 2019

Due in class on Thursday May 9, 2019. Hand in solutions to 3 problems of your choice.

Problem 5.1: Consider a sparse matrix $\mathbf{A}$ resulting from a finite difference discretization of an elliptic PDE on a regular $n \times n$ grid. In this exercise, you will test the performance of Matlab sparse solvers. (You could alternatively use any other environment that can call UMFPACK or a similar library.) You will time two commands

```
x = A \ b;
```

and

```
[L,U,P,Q] = lu(A);
```

where $\mathbf{A}$ is of course generated and stored in a sparse format.

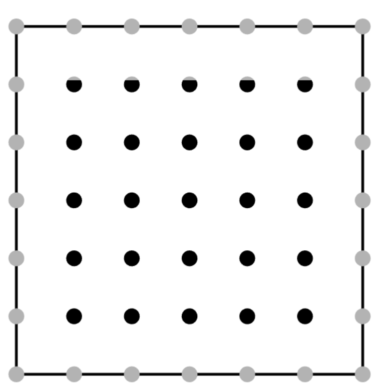
(a) Let $\mathbf{A}$ denote a matrix resulting from the plain 5-point stencil for the Laplace equation. Plot timings for solving a linear system, and for doing an LU factorization, respectively. Also plot the times for solving the linear system once you have the LU factorization, and for the memory requirements.

(b) Repeat problem (a), but now implement higher order stencils, and see how this changes the timings and the memory requirements. For instance, try the 9 and 13 point stencils obtained by using centered 5 and 7 point finite difference operators of orders $O(h^4)$ and $O(h^6)$, respectively. Discuss your findings. Note: For a higher order method, one has to think a little about how to handle nodes close to the boundary. The full stencil would in principle "stick out" of the grid for these nodes. In this problem, you can ignore this complication. Just restrict the "interior" nodes in I_{i} to be only the ones that are sufficiently far removed from the boundary.

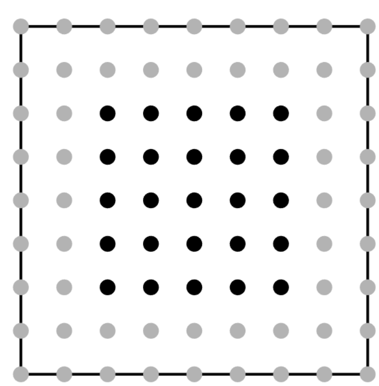
(c) Repeat problem (a) for a problem in 3D set on a regular grid with $N = n \times n \times n$ points. Do just the plain 7 point stencil of order $O(h^2)$ and compare with your results for two dimensions.

(d) Say you are working on a machine with 32GB of RAM. Estimate the largest problem size that you could solve for the four different matrices (5, 9, and 13 point stencil in 2D, 7-point stencil in 3D). A rough estimate based only on how much memory is required to store the L and U factors is enough — you do not need to worry about additional temporary storage.

Hint: The file `HW5p1.m` may be helpful.

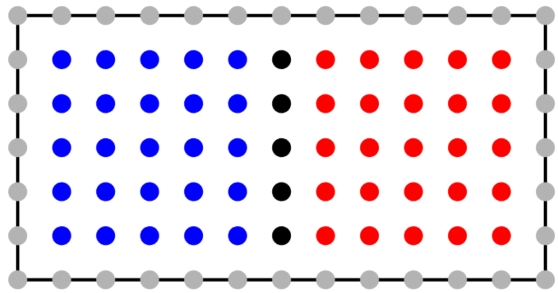


The grid in Problem 5.1, shown for $n = 5$. Dirichlet data is specified on the grey nodes, and the block nodes involve hold the $N = 25$ unknowns.



For the 9 point stencil in problem (c), you are allowed to just add more nodes at the boundary to get in this case the 5×5 grid shown in black.

Problem 5.2: In this exercise, you will numerically investigate the ranks of the off-diagonal blocks of the Schur complements that arise in the LU factorization of a finite difference matrix $\mathbf{A}$. Use a computational domain $\Omega = [-1, 1] \times [0, 1]$ discretized on a uniform grid of $(2n + 3) \times (n + 2)$ grid points, as shown (for $n = 5$):



Specify Dirichlet boundary data on the grey nodes, and partition the remaining nodes into I_1 (black), I_2 (blue), and I_3 (red), corresponding to the center, left, and right blocks, respectively. Then form the Schur complement

$$\mathbf{S} = \mathbf{A}(I_1, I_1) - \mathbf{A}(I_1, I_2)\big(\mathbf{A}(I_2, I_2)\big)^{-1}\mathbf{A}(I_2, I_1) - \mathbf{A}(I_1, I_3)\big(\mathbf{A}(I_3, I_3)\big)^{-1}\mathbf{A}(I_3, I_1),$$

and let $\mathbf{S}_{2,3}$ denote the top right quarter of $\mathbf{S}$, as usual. Let σ_j denote the j'th singular values of $\mathbf{S}_{2,3}$.

(a) Let $\mathbf{A}$ be our standard example of the 5-point stencil discretizing the Poisson equation. Provide graphs of σ_j/σ_1 versus j for some different choices of n. The larger n the better, but eventually things will get too slow, or you will run out of memory. Scale the x-axis so that you only show "interesting" values of j. Comment on what you see, and formulate a hypothesis on how the numerical rank depends on n.

(b) Repeat problem (a), but now consider the matrix $\mathbf{A}$ associated with a mesh conductivity problem with random conductivities. Investigate how the ranks depend on the contrast ratio between the largest and the smallest conductivities.

(c) Let κ be a real positive number, and set $\mathbf{A} = \mathbf{L} - \kappa^2\mathbf{I}$, where $\mathbf{L}$ represents the standard 5-point stencil for the Laplace operator $-\Delta$. Fix an n (the largest you can comfortably handle), then investigate how the numerical rank of $\mathbf{S}_{2,3}$ depends on κ. Discuss your results, and formulate a hypothesis.

(d) Pick some other problem involving an elliptic equation on the same grid that you may find interesting. Investigate the ranks numerically, and describe your findings. For instance, you may want to consider some other elliptic operator of the form

$$-\Delta u(\boldsymbol{x}) + b(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_1} + c(\boldsymbol{x})\frac{\partial u(\boldsymbol{x})}{\partial x_2} + d(\boldsymbol{x})\, u(\boldsymbol{x}) = f(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega.$$

Hint: The file `HW5p2.m` may be helpful.

Problem 5.3: In the class notes you will find the following result:

Lemma 1 (Variation of Woodbury)**.** *Suppose that* $\mathbf{A}$ *is an invertible* $N \times N$ *matrix, that* K *is a positive integer smaller than* N*, and that* $\mathbf{A}$ *admits the factorization*

$$\underset{N\times N}{\mathbf{A}} = \underset{N\times K}{\mathbf{U}} \; \underset{K\times K}{\tilde{\mathbf{A}}} \; \underset{K\times N}{\mathbf{V}^*} + \underset{N\times N}{\mathbf{D}}. \tag{1}$$

Then

$$\underset{N\times N}{\mathbf{A}^{-1}} = \underset{N\times K}{\mathbf{E}} \; \underset{K\times K}{\big(\tilde{\mathbf{A}} + \hat{\mathbf{D}}\big)^{-1}} \; \underset{K\times N}{\mathbf{F}^*} + \underset{N\times N}{\mathbf{G}}, \tag{2}$$

where

$$\hat{\mathbf{D}} = \big(\mathbf{V}^*\,\mathbf{D}^{-1}\,\mathbf{U}\big)^{-1}, \tag{3}$$

$$\mathbf{E} = \mathbf{D}^{-1}\,\mathbf{U}\,\hat{\mathbf{D}}, \tag{4}$$

$$\mathbf{F} = (\hat{\mathbf{D}}\,\mathbf{V}^*\,\mathbf{D}^{-1})^*, \tag{5}$$

$$\mathbf{G} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\,\mathbf{U}\,\hat{\mathbf{D}}\,\mathbf{V}^*\,\mathbf{D}^{-1}, \tag{6}$$

provided all inverses that appear in the formulas exist.

Prove that the matrix $\mathbf{G}$ has rank at most $N - K$.

Problem 5.4: In this problem, you will investigate the properties of Chebyshev spectral collocation methods.

(a) Consider the following five two-point boundary value problems on $I = [-1, 1]$:

$$\begin{aligned} -u''(x) - 4u(x) &= 0, & u(-1) &= 0, & u(1) &= 1, \\ -u''(x) - 100u'(x) &= 0, & u(-1) &= 0, & u(1) &= 1, \\ -u''(x) - e^{10x}\,u(x) &= 0, & u(-1) &= 0, & u(1) &= 1, \\ -u''(x) - 10|x|\,u(x) &= 0, & u(-1) &= 0, & u(1) &= 1, \\ -u''(x) - u(x) &= |x|, & u(-1) &= 0, & u(1) &= 1. \end{aligned}$$

Solve each problem using a spectral collocation method, using an n-point Chebyshev grid. Provide plots of the error versus n. For the first equation, you can easily find the exact solution. For the others, you may need to estimate the error at each step. Plot all five convergence lines in the same plot. Plot the condition numbers of the coefficient matrices in the linear systems against n. Estimate how the condition number depends on n. Discuss your results.

(b) Let $\Omega = [-1, 1]^2$, let $\Gamma = \partial\Omega$, and consider the Laplace equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ \quad u(\boldsymbol{x}) = f(\boldsymbol{x}), & \boldsymbol{x} \in \Gamma, \end{cases}$$

Solve the equation using a Chebyshev collocation method on an $n \times n$ grid for the right hand sides

$$\begin{aligned} f(\boldsymbol{x}) &= \log\big((x_1 + 1.5)^2 + (x_2 - 0.5)^2\big), \\ f(\boldsymbol{x}) &= \cos(11x_2)e^{x_1}, \\ f(\boldsymbol{x}) &= \sqrt{|\boldsymbol{x} - (0.5, 1)|} \end{aligned}$$

Provide plots of the estimated errors against $N = n^2$. Plot the condition number of the coefficient matrix against N and estimate the growth rate. Discuss your results.

(c) Let $\Omega = [-1, 1]^2$, let $\Gamma = \partial\Omega$, and consider the Helmholtz equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) - 1000u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ \qquad\qquad\quad u(\boldsymbol{x}) = 1, & \boldsymbol{x} \in \Gamma. \end{cases}$$

Solve the equation using a Chebyshev collocation method on an $n \times n$ grid, and provide a plot of the estimated errors against $N = n^2$ for each case. Discuss your results.

(d) [Optional:] Solve the BVP

$$\begin{cases} -\Delta u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ \quad u(\boldsymbol{x}) = \log\big((x_1 + 1.5)^2 + (x_2 - 0.5)^2\big), & \boldsymbol{x} \in \Gamma, \end{cases}$$

using both a spectral collocation method on an $n \times n$ grid, and a finite difference method using the standard 5-point stencil on a uniform grid. Plot the errors for both methods versus $N = n^2$. Observe that the analytical solution is obvious in this case. Then plot the errors versus the solution time for both methods. Theoretically estimate the asymptotic behavior of the flop count as a function of requested accuracy for the two methods, and compare with your observed results. Repeat the problem for the Helmholtz equation

$$\begin{cases} -\Delta u(\boldsymbol{x}) - 1000u(\boldsymbol{x}) = 0, & \boldsymbol{x} \in \Omega, \\ \qquad\qquad u(\boldsymbol{x}) = Y_0(\sqrt{1000}|\boldsymbol{x} - (1.5, 0.5)|), & \boldsymbol{x} \in \Gamma, \end{cases}$$

where Y_0 is the zeroth order Bessel function of the second kind. (You can compute it in Matlab using the command `bessely(0,·)`.)

Hint: The file `HW5p4.m` may be helpful.