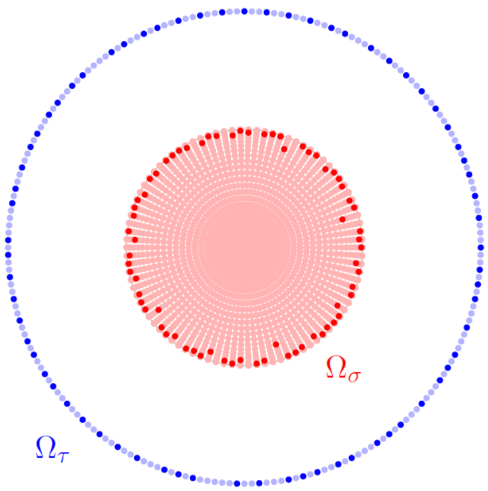
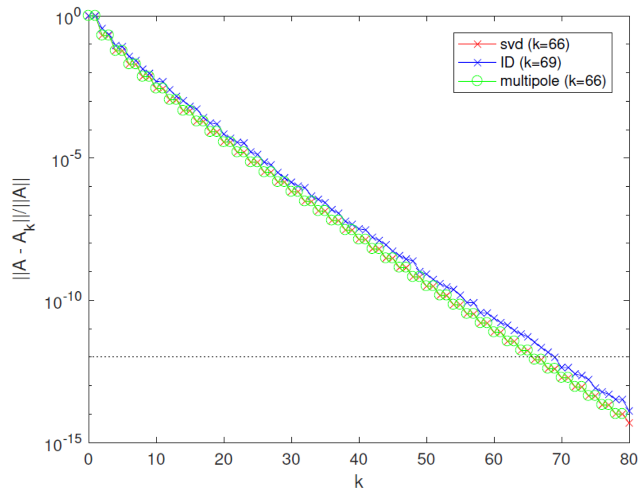
Homework set 4 — MATH 393C — Spring 2019

Due in class on Thursday April 18, 2019. Hand in solutions to all problems.

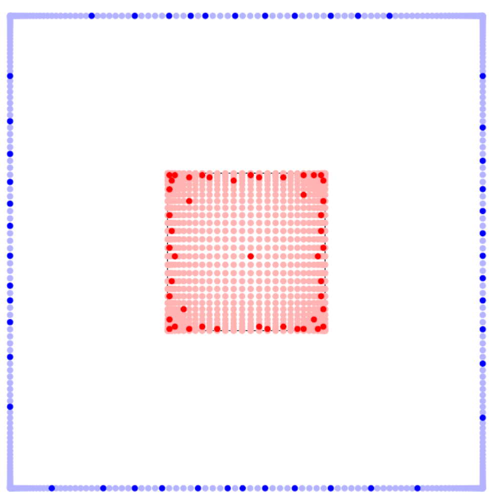
Problem 4.1: In the course notes posted on the class website on the FMM, Figure 4.3 shows in part the following:



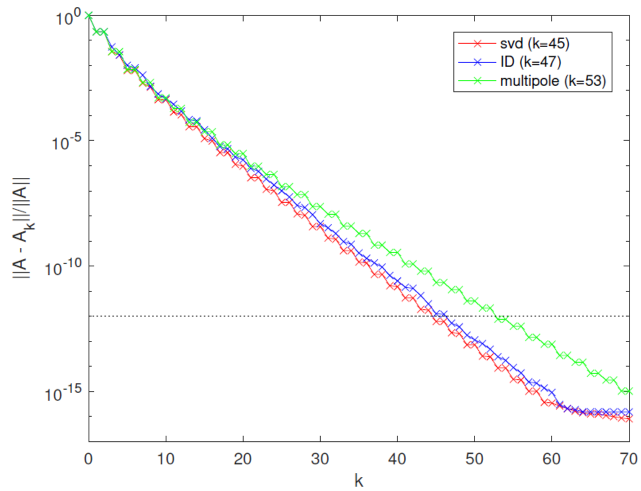
(a)



(b)



(c)



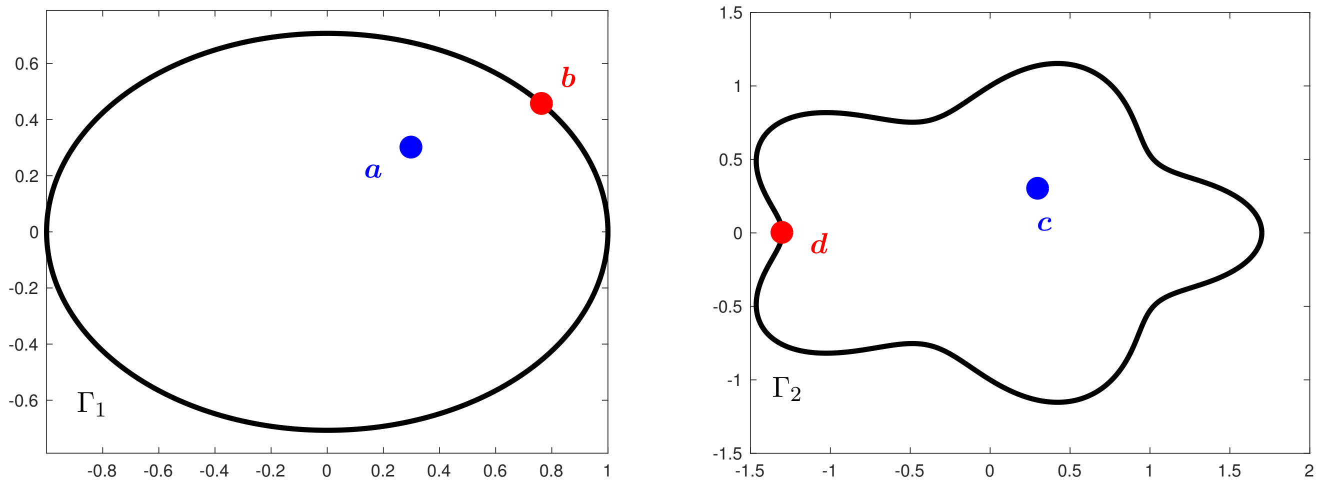
(d)

Using the techniques outlined in Chapter 4 of the course notes, compute the first few singular vectors associated with the integral operator

$$A: \ L^2(\Omega_\sigma) \to L^2(\Omega_\tau): \ q \mapsto f(\boldsymbol{x}) = \int_{\Omega_\sigma} \log|\boldsymbol{x} - \boldsymbol{y}| \, q(\boldsymbol{y}) \, d\boldsymbol{y}, \tag{1}$$

where Ω_σ and Ω_τ are as shown in the Figure 4.3(a) and 4.3(c). Hand in plots of the first 6 singular vectors.

Optional: Replace $\log|\boldsymbol{x} - \boldsymbol{y}|$ by $H_0^{(1)}(\kappa|\boldsymbol{x} - \boldsymbol{y}|)$. Plot the singular values for a few different values of κ. What differences to you observe?



Geometry of Problem 4.2 *Geometry of Problem 4.3*

Problem 4.2: Define a contour Γ_1 via

$$\Gamma_1 = \{x = (x_1,\, x_2) \in \mathbb{R}^2 :\; x_1^2 + 2x_2^2 = 1\}.$$

Let Ω_1 denote the domain interior to Γ_1. Define points $\boldsymbol{a} \in \Omega_1$ and $\boldsymbol{b} \in \Gamma_1$ via

$$\boldsymbol{a} = (0.3,\, 0.3), \qquad \boldsymbol{b} = (\cos(0.7),\, (1/\sqrt{2})\,\sin(0.7)).$$

Let u be the unique solution to

$$\begin{cases} -\Delta u(x) = 0, & x \in \Omega_1, \\ \quad u(x) = f(x), & x \in \Gamma_1, \end{cases} \tag{2}$$

where

$$f(x_1,\, x_2) = x_1^2\, e^{\sin(10\, x_2)}.$$

Let u have the representation

$$u(\boldsymbol{x}) = [S\sigma](\boldsymbol{x}) = \int_{\Gamma_1} -\frac{1}{2\pi}\,\log\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,\sigma(\boldsymbol{y})\,ds(\boldsymbol{y}).$$

Your task is to form an equation for σ, discretize this equation, solve the equation, and then to evaluate the function u. Use just a plain Trapezoidal rule.

Let N denote the number of degrees of freedom in your approximation, let σ_N denote the corresponding solution, and include in your solution the following table (with values filled in where the question marks are):

N	$u_N(\boldsymbol{a})$	$\sigma_N(\boldsymbol{b})$	
100	?	?	
200	?	?	
400	?	?	
800	?	?	
⋮	⋮	⋮	⋮

Include as large N as your computer can handle in a reasonable amount of time, and estimate the convergence rate for each column.

Estimate the rates of convergence.

Problem 4.3: Repeat Problem 4.2, but now set

$$G_1(t) = 1.5\,\cos(t) + 0.1\,\cos(6\,t) + 0.1\,\cos(4\,t),$$
$$G_2(t) = \sin(t) + 0.1\,\sin(6\,t) - 0.1\,\sin(4\,t),$$

and define

$$\Gamma_2 = \{x = (G_1(t),\, G_2(t))\,:\, t \in [0,\, 2\pi)\}.$$

The Dirichlet data f is the same. Report $\sigma(\boldsymbol{c})$ and $u(\boldsymbol{d})$ for the points

$$\boldsymbol{c} = (0.3,\, 0.3), \qquad \boldsymbol{d} = (-1.3,\, 0).$$

Problem 4.4: Repeat Problem 4.2 (with the contour Γ_1) but now use the double layer potential

$$u(\boldsymbol{x}) = [D\sigma](\boldsymbol{x}) = \int_{\Gamma_1} \frac{\boldsymbol{n}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{2\pi|\boldsymbol{x}-\boldsymbol{y}|^2}\,\sigma(\boldsymbol{y})\,ds(\boldsymbol{y}),$$

where $\boldsymbol{n}(\boldsymbol{y})$ is the outwards pointing unit normal at $\boldsymbol{y}$.

Hint: Recall from class that the double layer kernel is smooth. Some work is required in determining its value on the diagonal, though!

Problem 4.5: Repeat Problem 4.3 (with the contour Γ_2) but now use the double layer potential

$$u(\boldsymbol{x}) = [D\sigma](\boldsymbol{x}) = \int_{\Gamma_1} \frac{\boldsymbol{n}(\boldsymbol{y})\cdot(\boldsymbol{x}-\boldsymbol{y})}{2\pi|\boldsymbol{x}-\boldsymbol{y}|^2}\,\sigma(\boldsymbol{y})\,ds(\boldsymbol{y}),$$

where $\boldsymbol{n}(\boldsymbol{y})$ is the outwards pointing unit normal at $\boldsymbol{y}$.

Problem 4.6: Repeat problems 4.3 and 4.5, but now use the dirichlet data

$$f(\boldsymbol{x}) = \log\bigl((x_1 - 1.5)^2 + (x_2 - 0.5)^2\bigr), \qquad \boldsymbol{x} \in \Gamma.$$

In this case, you of course know that the exact analytic solution is simply

$$u(\boldsymbol{x}) = \log\bigl((x_1 - 1.5)^2 + (x_2 - 0.5)^2\bigr), \qquad \boldsymbol{x} \in \Omega.$$

Estimate the rate of convergence of your computed solution at the point $\boldsymbol{c}$ when you use the single and the double layer formulations, respectively.