Randomized algorithms for linear algebraic computations

Per-Gunnar Martinsson
Dept. of Mathematics & Oden Institute for Computational Sciences and Engineering
University of Texas at Austin

Collaborators: Robert van de Geijn, Francisco Igual, Yuji Nakatsukasa, Gregorio Quintana-Ortí, Vladimir Rokhlin, Joel Tropp, Mark Tygert.

Former and current students and postdocs: Ke Chen, Yijun Dong, Abinand Gopal, Nathan Halko, Nathan Heavner, James Levitt, Sergey Voronin, Heather Wilber, Bowei Wu, Anna Yesypenko.

Slides: `http://users.oden.utexas.edu/~pgm/main_talks.html`

Research support by:

Outline of talk

1. **Randomized low rank approximation**

   “Randomized singular value decomposition” or “RSVD”.

   Techniques based on randomized embeddings.

   Relatively well established material within numerical linear algebra.

2. **Variations of randomized algorithms for low rank approximation**

   Single pass and streaming algorithms.

   Structured random embeddings.

   Matrix approximation via sampling.

3. **Samples of current research directions**

   Least squares problems and linear system solvers.

   Sketching for data compression.

Low rank approximation — problem formulation:

Let **A** be a given $m \times n$ matrix, and let k be an integer such that $1 \leq k \ll n \leq m$.

We seek to compute approximate factors **E** and **F** such that

$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{E} & \mathbf{F}^*. \\ m \times n & & m \times k & k \times n \end{array}$$

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Why?

- Fitting a hyperplane to a given set of points. Or fitting a multivariate normal distribution to measurements ("principal component analysis").
- Model reduction in scientific computing.
- Spectral algorithms in data analysis.
- "Fast" algorithms of various types: Fast Multipole Methods, generalizations of the Fast Fourier Transform, fast direct solvers, etc.
- Many, many, many more.

Observe that from **E** and **F** you can compute approximate singular vectors, find dominant eigenvectors (when $\mathbf{A} = \mathbf{A}^*$), find spanning rows/columns, etc.

We seek only to control the residual error $\|\mathbf{A} - \mathbf{E}\mathbf{F}^*\|$.

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Existing methods for this task are well established. Textbook methods include:

1. Compute the full singular value decomposition of **A**, and then truncate:
   — Resulting approximation is in many regards ”optimal” — best possible fit.
   — Expensive! Cost is $O(mn^2)$. Good for small n, or “expensive” data.
2. Krylov methods:
   — Standard technique for large sparse matrices.
   — Interacts with **A** only through its action on vectors.
   — Theoretically optimal in important regards.
3. Execute Gram-Schmidt on the columns of **A** (“column pivoted QR”):
   — Simple and practical for medium size dense matrices.
   — Not entirely optimal, but often good enough.
   — Cost is $O(mnk)$ since you can stop after k steps.

These methods work great! But room for improvement in important environments.

Randomized SVD:

Objective: Given an $m \times n$ matrix $\mathbf{A}$, find an approximate rank-k partial SVD:

$$\begin{array}{ccccc} \mathbf{A} & \approx & \mathbf{U} & \mathbf{D} & \mathbf{V}^* \\ m \times n & & m \times k & k \times k & k \times n \end{array}$$

where $\mathbf{U}$ and $\mathbf{V}$ are orthonormal, and $\mathbf{D}$ is diagonal. (We assume $k \ll \min(m, n)$.)

(A) *Randomized sketching:*

A.1 Draw an $n \times k$ Gaussian random matrix $\mathbf{\Omega}$. `Omega = randn(n,k)`

A.2 Form the $m \times k$ sample matrix $\mathbf{Y} = \mathbf{A\Omega}$. `Y = A * Omega`

A.3 Form an $m \times k$ orthonormal matrix $\mathbf{Q}$ such that $\text{ran}(\mathbf{Q}) = \text{ran}(\mathbf{Y})$. `[Q, ~] = qr(Y)`

(B) *Deterministic post-processing:*

B.1 Form the $k \times n$ matrix $\mathbf{B} = \mathbf{Q}^*\mathbf{A}$. `B = Q' * A`

B.2 Form the full SVD of the small matrix $\mathbf{B}$: $\mathbf{B} = \hat{\mathbf{U}}\mathbf{D}\mathbf{V}^*$. `[Uhat, Sigma, V] = svd(B,'econ')`

B.3 Form the matrix $\mathbf{U} = \mathbf{Q}\hat{\mathbf{U}}$. `U = Q * Uhat`

The objective of Stage A is to compute an ON-basis that approximately spans the column space of $\mathbf{A}$. The matrix $\mathbf{Q}$ holds these basis vectors and $\mathbf{A} \approx \mathbf{QQ}^*\mathbf{A}$.

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Stage B is exact: $\|\mathbf{A} - \mathbf{Q}\underbrace{\mathbf{Q}^*\mathbf{A}}_{=\mathbf{B}}\| = \|\mathbf{A} - \mathbf{Q}\underbrace{\mathbf{B}}_{=\hat{\mathbf{U}}\mathbf{D}\mathbf{V}^*}\| = \|\mathbf{A} - \underbrace{\mathbf{Q}\hat{\mathbf{U}}}_{=\mathbf{U}}\mathbf{D}\mathbf{V}^*\| = \|\mathbf{A} - \mathbf{UDV}^*\|$.

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We claim that the columns of $\mathbf{Y}$ form a good approximate basis for $\text{ran}(\mathbf{A})$.

Observe that $\text{ran}(\mathbf{Y}) \subseteq \text{ran}(\mathbf{A})$ automatically.

Loss of accuracy can happen if $\text{ran}(\mathbf{Y})$ does not capture important directions.

To avoid this, we draw p extra samples, for, say, $p = 5$ or $p = 10$.

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Important: You only need to ensure that you do not undersample.

Over-sampling is unproblematic, since excess data gets "filtered out" in Stage B.

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Input: An $m \times n$ matrix $\mathbf{A}$, a target rank k, and an over-sampling parameter p (say $p = 5$).	
Output: Rank-$(k+p)$ factors $\mathbf{U}$, $\mathbf{D}$, and $\mathbf{V}$ in an approximate SVD $\mathbf{A} \approx \mathbf{UDV}^*$.	
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- It is simple to adapt the scheme to the situation where the *tolerance is given,* and the rank has to be determined adaptively.

- Observe how simple the interaction with $\mathbf{A}$ is.

  Two matrix-matrix products only.

  This often leads to very high practical execution speed on modern hardware.

- Accuracy of the basic scheme is good when the singular values decay reasonably fast. When they do not, the scheme can be combined with Krylov-type ideas:

  Taking one or two steps of subspace iteration vastly improves the accuracy.

  For instance, use the sampling matrix $\mathbf{Y} = \mathbf{AA}^*\mathbf{A\Omega}$ instead of $\mathbf{Y} = \mathbf{A\Omega}$.

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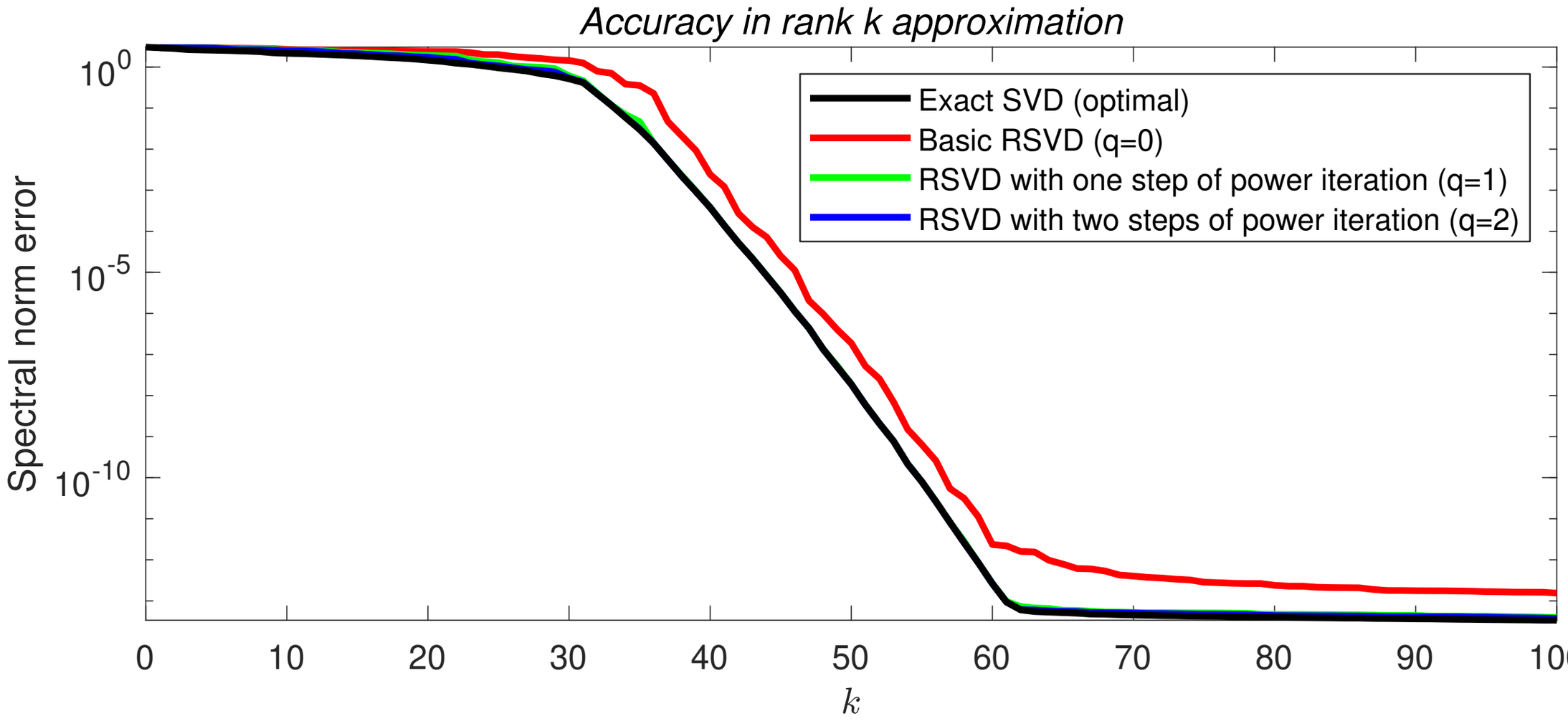
Let us next investigate the accuracy of the method.

To illustrate the errors, we set $p = 0$ (no over-sampling), and then define

$$e_k = \|\mathbf{A} - \mathbf{UDV}^*\| = \|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|.$$

Eckart-Young theorem: $e_k \geq \sigma_{k+1}$, where σ_j is the j'th singular value of $\mathbf{A}$.

Randomized SVD:



The plot shows the errors from the randomized SVD. To be precise, we plot

$$e_k = \|\mathbf{A} - \mathbf{P}_k\mathbf{A}\|,$$

where $\mathbf{P}_k$ is the orthogonal projection onto the first k columns of

$$\mathbf{Y} = \left(\mathbf{A}\mathbf{A}^*\right)^q \mathbf{A}\mathbf{\Omega},$$

and where $\mathbf{\Omega}$ is a Gaussian random matrix. (For clarity, no oversampling is done.) The matrix $\mathbf{A}$ is an approximation to a scattering operator for a Helmholtz problem.

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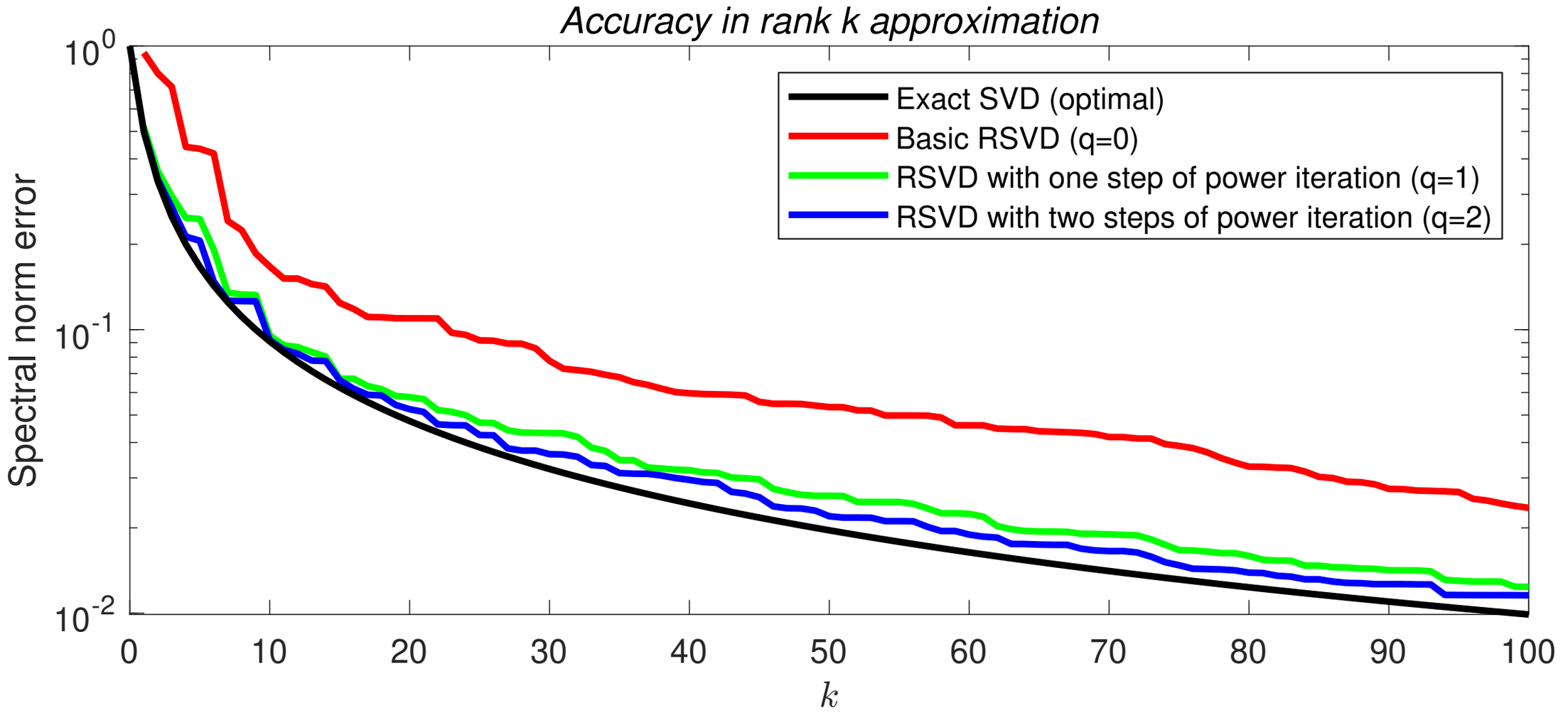

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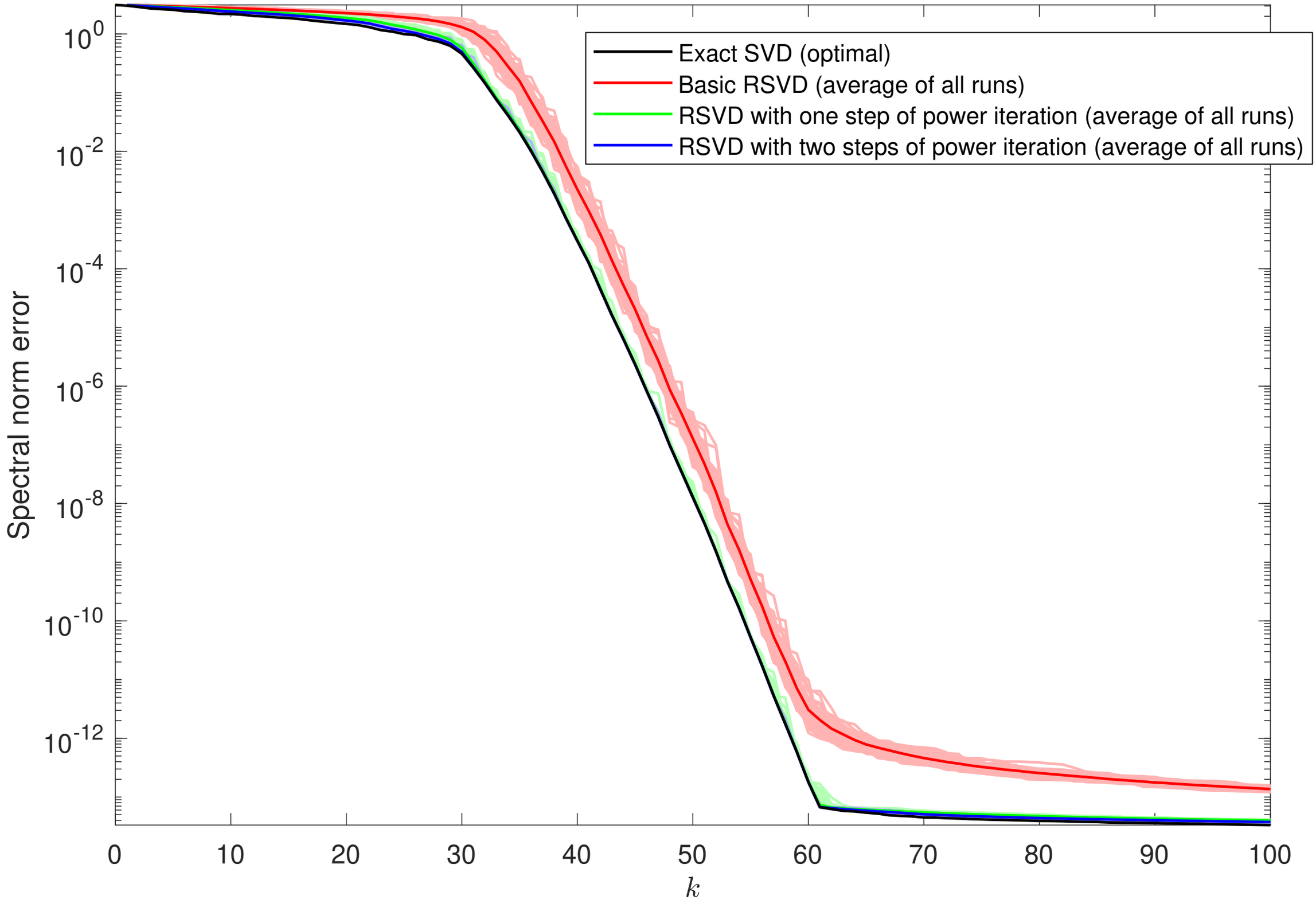
and where $\mathbf{\Omega}$ is a Gaussian random matrix. (For clarity, no oversampling is done.) The matrix $\mathbf{A}$ now has singular values that decay slowly.

Randomized SVD: The same plot as before, but now showing 100 instantiations.



The darker lines show the mean errors across the 100 experiments.

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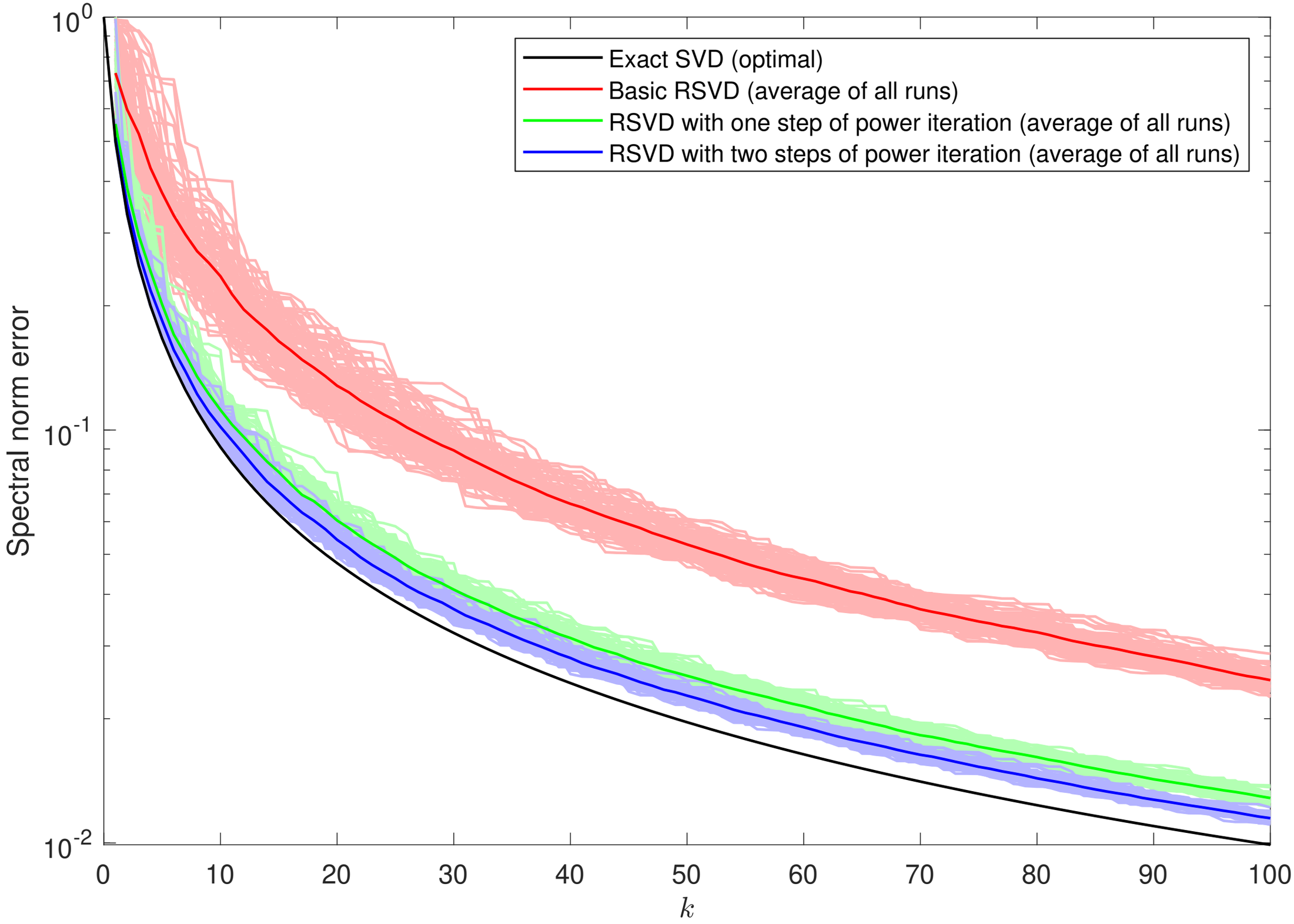

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Since the error in the RSVD is a random variable (it depends on the draw of $\mathbf{\Omega}$), any theoretical analysis needs to describe the *probability distribution* of the error.

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For instance, we can bound the expectation of the error:

Theorem: Let $\mathbf{A}$ be an $m \times n$ matrix with singular values $\{\sigma_j\}_{j=1}^{\min(m,n)}$. Let k be a target rank, and let p be an over-sampling parameter such that $p \geq 2$ and $k + p \leq \min(m, n)$. Let $\boldsymbol{\Omega}$ be a Gaussian random matrix of size $n \times (k+p)$ and set $\mathbf{Q} = \texttt{orth}(\mathbf{A}\boldsymbol{\Omega})$. Then the average error satisfies

$$\mathbb{E}\big[\|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|_{\mathrm{Fro}}\big] \leq \left(1 + \frac{k}{p-1}\right)^{1/2} \left(\sum_{j=k+1}^{\min(m,n)} \sigma_j^2\right)^{1/2},$$

$$\mathbb{E}\big[\|\mathbf{A} - \mathbf{QQ}^*\mathbf{A}\|\big] \leq \left(1 + \sqrt{\frac{k}{p-1}}\right) \sigma_{k+1} + \frac{e\sqrt{k+p}}{p} \left(\sum_{j=k+1}^{\min(m,n)} \sigma_j^2\right)^{1/2}.$$

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Since the error in the RSVD is a random variable (it depends on the draw of $\boldsymbol{\Omega}$), any theoretical analysis needs to describe the *probability distribution* of the error.

There are also bounds on the likelihood of a large deviation from the expectation.
(It turns out to decay super-exponentially fast as p increases!)

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Algorithm: (Merely a restatement of what has already been described.)

Fix an over-sampling parameter p, say $p = 5$.

Draw a Gaussian random matrix $\mathbf{\Omega} \in \mathbb{R}^{m \times (k+p)}$.

Form the sample matrix $\mathbf{Y} = \mathbf{A\Omega}$.

Orthonormalize the columns of $\mathbf{Y}$ to form an ON matrix $\mathbf{Q}$.

Then $\mathbf{A} \approx \mathbf{QQ}^*\mathbf{A}$.

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We seek to eliminate the intermediate matrices $\mathbf{Y}$ and $\mathbf{Q}$ from the description.

We will use the notion of a *Moore-Penrose pseudoinverse*. Brief recap:

Let $\mathbf{X}$ be an $m \times n$ matrix of rank k, with singular value decomposition

$$\begin{array}{ccccc} \mathbf{X} & = & \mathbf{U} & \mathbf{D} & \mathbf{V}^*. \\ m \times n & & m \times k & k \times k & k \times n \end{array}$$

Then the *Moore-Penrose pseudoinverse* is

$$\begin{array}{ccccc} \mathbf{X}^\dagger & = & \mathbf{V} & \mathbf{D}^{-1} & \mathbf{U}^*. \\ n \times m & & n \times k & k \times k & k \times m \end{array}$$

The matrix $\mathbf{X}^\dagger$ is a generalization of an inverse for an invertible matrix.

The property we need is: $\mathbf{XX}^\dagger$ is the orthogonal projection onto col($\mathbf{X}$).

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Using the notion of a *Moore-Penrose pseudoinverse*, we can eliminate the intermediate matrices $\mathbf{Y}$ and $\mathbf{Q}$ from the description:

$$\mathbf{A} \approx \mathbf{Q}\mathbf{Q}^*\mathbf{A} = \mathbf{Y}\mathbf{Y}^\dagger\mathbf{A} = (\mathbf{A\Omega})\,(\mathbf{A\Omega})^\dagger\mathbf{A}.$$

Key claim: The columns of $\mathbf{A\Omega}$ approximately span the column space of $\mathbf{A}$.

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Next, let us also approximate the row space: Draw a $(k+p) \times m$ Gaussian matrix $\mathbf{\Psi}$, then the rows of $\mathbf{\Psi A}$ approximately span the row space of $\mathbf{A}$. Then:

$$\mathbf{A} \approx (\mathbf{A\Omega})\,(\mathbf{A\Omega})^\dagger\mathbf{A}(\mathbf{\Psi A})^\dagger\,(\mathbf{\Psi A}).$$

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Next, let us also approximate the row space: Draw a $(k+p) \times m$ Gaussian matrix $\mathbf{\Psi}$, then the rows of $\mathbf{\Psi A}$ approximately span the row space of $\mathbf{A}$. Then:

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Simplify: $\mathbf{A} \approx (\mathbf{A\Omega})\,(\mathbf{A\Omega})^\dagger\mathbf{A}(\mathbf{\Psi A})^\dagger\,(\mathbf{\Psi A}) = \cdots = (\mathbf{A\Omega})\,\big(\mathbf{\Psi A\Omega}\big)^\dagger\,(\mathbf{\Psi A}).$

Randomized SVD: Double-sided approximation ("generalized Nyström")

Task: Find a rank-k approximation to a given $m \times n$ matrix $\mathbf{A}$.

Algorithm: Draw Gaussian random matrices $\mathbf{\Omega} \in \mathbb{R}^{m\times(k+p)}$ and $\mathbf{\Psi} \in \mathbb{R}^{(k+p)\times n}$.

Form approximation $\mathbf{A} \approx (\mathbf{A\Omega})\left(\mathbf{\Psi A\Omega}\right)^{\dagger}(\mathbf{\Psi A}) =: \mathbf{A}_{\text{approx}}$.

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Observation 1:

The matrix $\mathbf{A}_{\text{approx}}$ can be built *in a single pass over* $\mathbf{A}$.

In other words, we need to view each matrix entry of $\mathbf{A}$ only once.

This cannot be done using deterministic methods (as far as I know).

"Streaming" or "single-view" algorithm.

Note: *Using different over-sampling parameters for the row and column spaces is often better.*

References: Alon, Gibbons, Matias and Szegedy (2002); Woolfe, Liberty, Rokhlin, and Tygert (2008); Clarkson and Woodruff (2009); Li, Nguyen and Woodruff (2014); Boutsidi, Woodruff and Zhong (2016), Tropp, Yurtsever, Udell and Cevher (2017); Pourkamali-Anaraki and Becker (2019); Wang, Gittens and Mahoney (2019); Nakatsukasa, *arXiv:2009.11392*, 2020; Dong & Martinsson, *arXiv:2104.05877*, 2021; . . . many more . . .

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Observation 2:

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$O(mnk)$ matches the flop count of Gram-Schmidt, or a Krylov method applied to a dense matrix.

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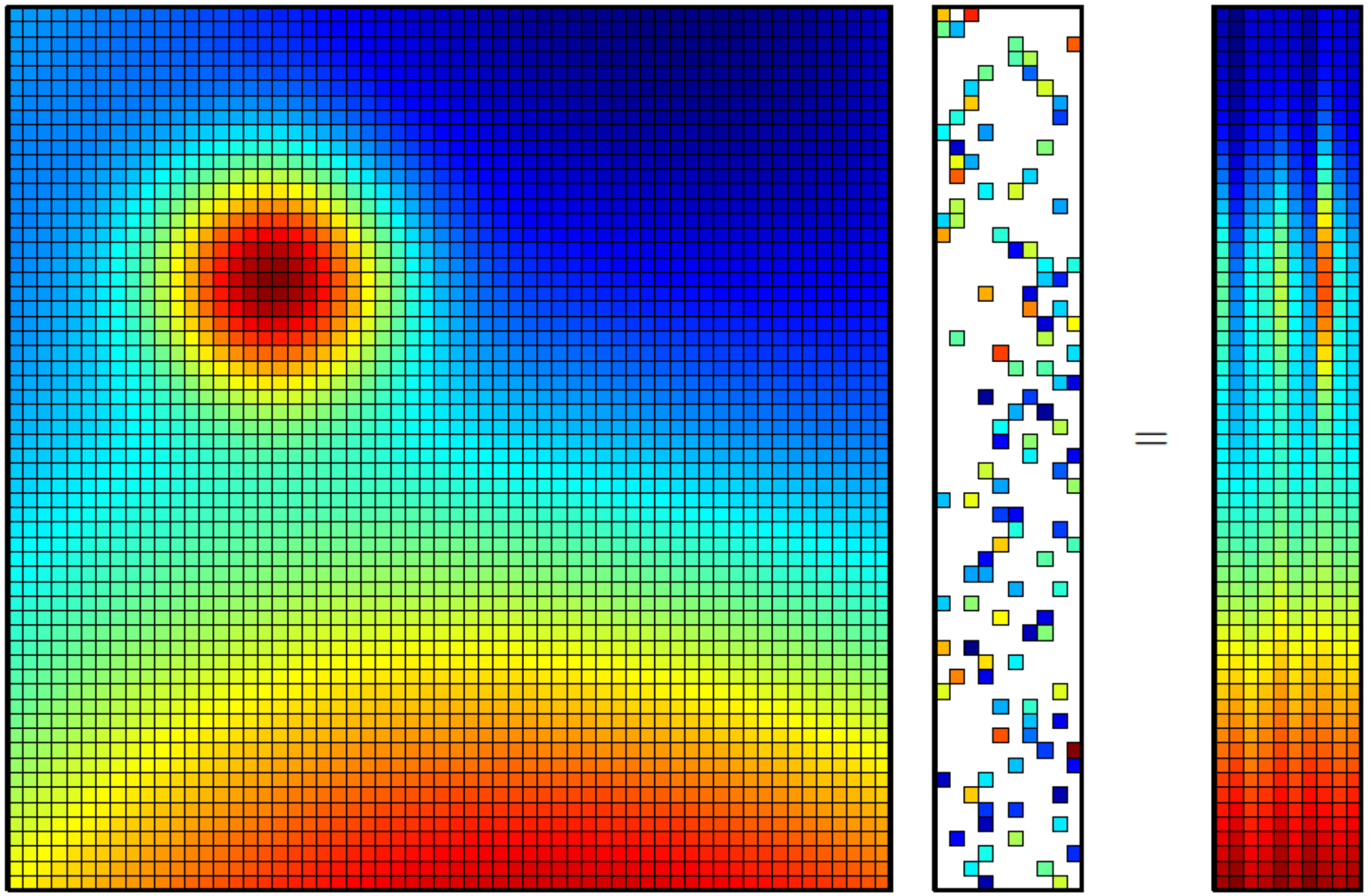

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flops!

$n \log(k)$).

or $r = 4$.)

The matrix **Ω** *is a sparse random matrix. Two nonzero entries are placed randomly in each row. In consequence, each column of* **A** *contributes to precisely two columns of the sample matrix* $\mathbf{Y} = \mathbf{A\Omega}$*. This structured random map has* $O(mn)$ *complexity, is easy to work with practically, and often provides good accuracy.*

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When a structured random matrix is used, overall cost can be reduced to $O(mn + k^3)$.

Despite the pseudo-inverse, this can be done in a numerically stable way.

References: Ailon & Chazelle (2006); Liberty, Rokhlin, Tygert, and Woolfe (2006); Halko, Martinsson, Tropp (2011); Clarkson & Woodruff (2013); Meng & Mahoney (2013); Nelson & Nguyen (2013); Urano (2013); Nakatsukasa, *arXiv:2009.11392*, 2020; Dong & Martinsson, *arXiv:2104.05877*, 2021. Much subsequent work — "Fast Johnson-Lindenstrauss transform."

Low rank approximation via sampling:

Task: Find a rank-k approximation to a given $m \times n$ matrix $\mathbf{A}$.

Question: Can we be even *more* aggressive? Complexity *less* than $O(mn)$?

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Sampling approach:

1. Draw vectors J and I holding k samples from the column and row indices, resp.
2. Form matrices $\mathbf{C}$ and $\mathbf{R}$ consisting of the corresponding columns and rows

$$\mathbf{C} = \mathbf{A}(:, J), \qquad \text{and} \qquad \mathbf{R} = \mathbf{A}(I, :).$$

3. Use as your approximation $\mathbf{A}_{\text{approx}} = \mathbf{CUR}$ where $\mathbf{U}$ is computed from information in $\mathbf{A}(I, J)$. (It should be an approximation to the optimal choice $\mathbf{U} = \mathbf{C}^{\dagger}\mathbf{A}\mathbf{R}^{\dagger}$.)

The computational profile depends crucially on the probability distribution that is used.

Uniform probabilities: Can be *very* cheap. But in general not reliable.

Probabilities from "leverage scores": Optimal distributions can be computed using the information in the top left and right singular vectors of $\mathbf{A}$. Then quite strong theorems can be proven on the quality of the approximation. Problem: Computing the probability distribution is as expensive as computing a partial SVD.

References: Drineas, Kannan & Mahoney (2006); Drineas, Mahoney & Muthukrishnan (2008); Kannan & Vempala (2017); Drineas & Mahoney (2018); Martinsson & Tropp (2020); ...

Low rank approximation via sampling: Connection to structured embeddings

Task: Find a rank k approximation to a given $m \times n$ matrix $\mathbf{A}$.

Sampling approach (now single sided again): Draw a subset of k columns $\mathbf{C} = \mathbf{A}(:, J)$ where J is drawn at random. Consider the approximant

$$\mathbf{A}_{\text{approx}} = \mathbf{C}\mathbf{C}^{\dagger}\mathbf{A}.$$

As we have seen, this in general does not work very well. But it does work well for the class of matrices for which uniform sampling is optimal.

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As we have seen, this in general does not work very well. But it does work well for the class of matrices for which uniform sampling is optimal. *We can turn* $\mathbf{A}$ *into such a matrix!* Let $\mathbf{U}$ be a matrix drawn from a uniform distribution on the set of $n \times n$ unitary matrices (the "Haar distribution"). Then form

$$\tilde{\mathbf{A}} = \mathbf{A}\mathbf{U}.$$

Now each column of $\tilde{\mathbf{A}}$ has exactly the same distribution! We may as well pick $J = 1 : k$, and can then pick a well behaved sample through

$$\mathbf{C} = \tilde{\mathbf{A}}(:, J) = \mathbf{A}\mathbf{U}(:, J).$$

The $n \times k$ "slice" $\mathbf{U}(:, J)$ is in a sense an optimal random embedding.

Fact: Using a Gaussian matrix is mathematically equivalent to using $\mathbf{U}(:, J)$.

Observation: A structured random embedding seeks to mimic the action of $\mathbf{U}(:, J)$.

Finding spanning rows and columns — CUR and interpolatory decompositions

In some applications, finding sets of columns/rows that span the column/row space is the primary task. To illustrate, suppose that we are given an $m \times n$ matrix $\mathbf{A}$, a rank $k < \min(m, n)$, and seek to compute a factorization

$$\underset{m \times n}{\mathbf{A}} \quad \approx \quad \underset{m \times k}{\mathbf{C}} \quad \underset{k \times n}{\mathbf{Z}},$$

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This problem can be solved via index sampling using leverage scores, as discussed.

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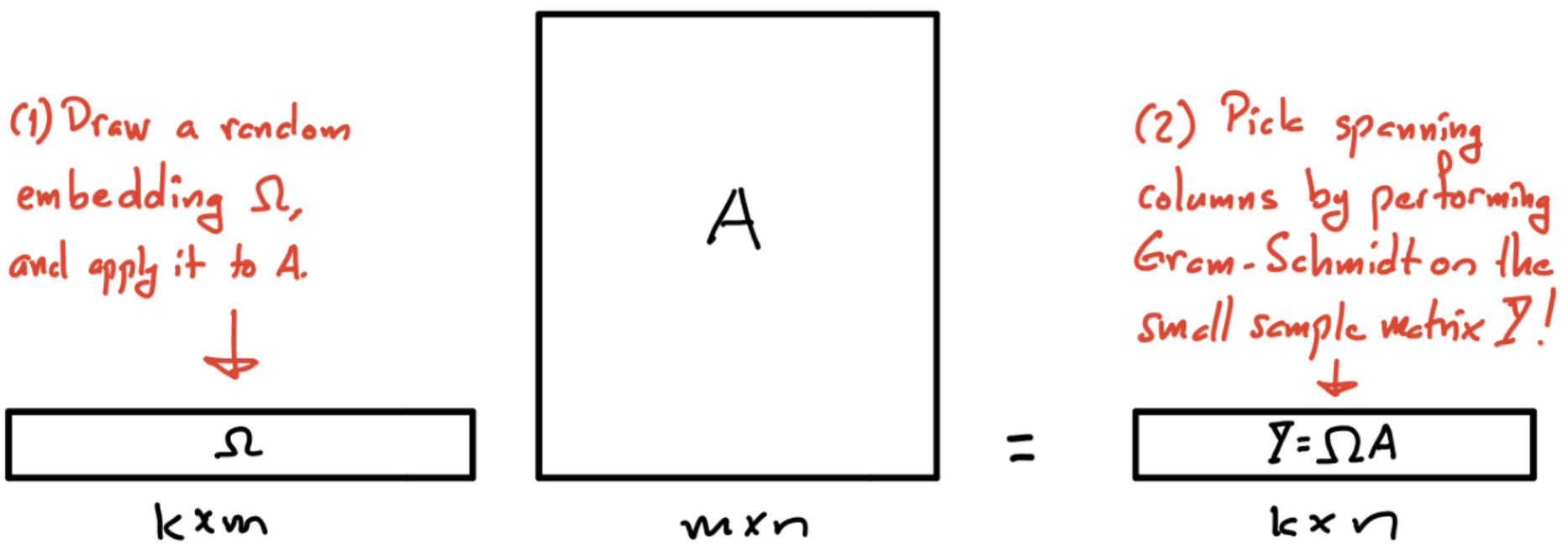
Alternatively, the following two stage approach works very well:

(1) Apply a random embedding $\mathbf{\Omega}$ to the columns of $\mathbf{A}$ (sparse random is perfect).

(2) Execute a classical pivoting method on $\mathbf{\Omega A}$ (*partially* pivoted LU is ideal).



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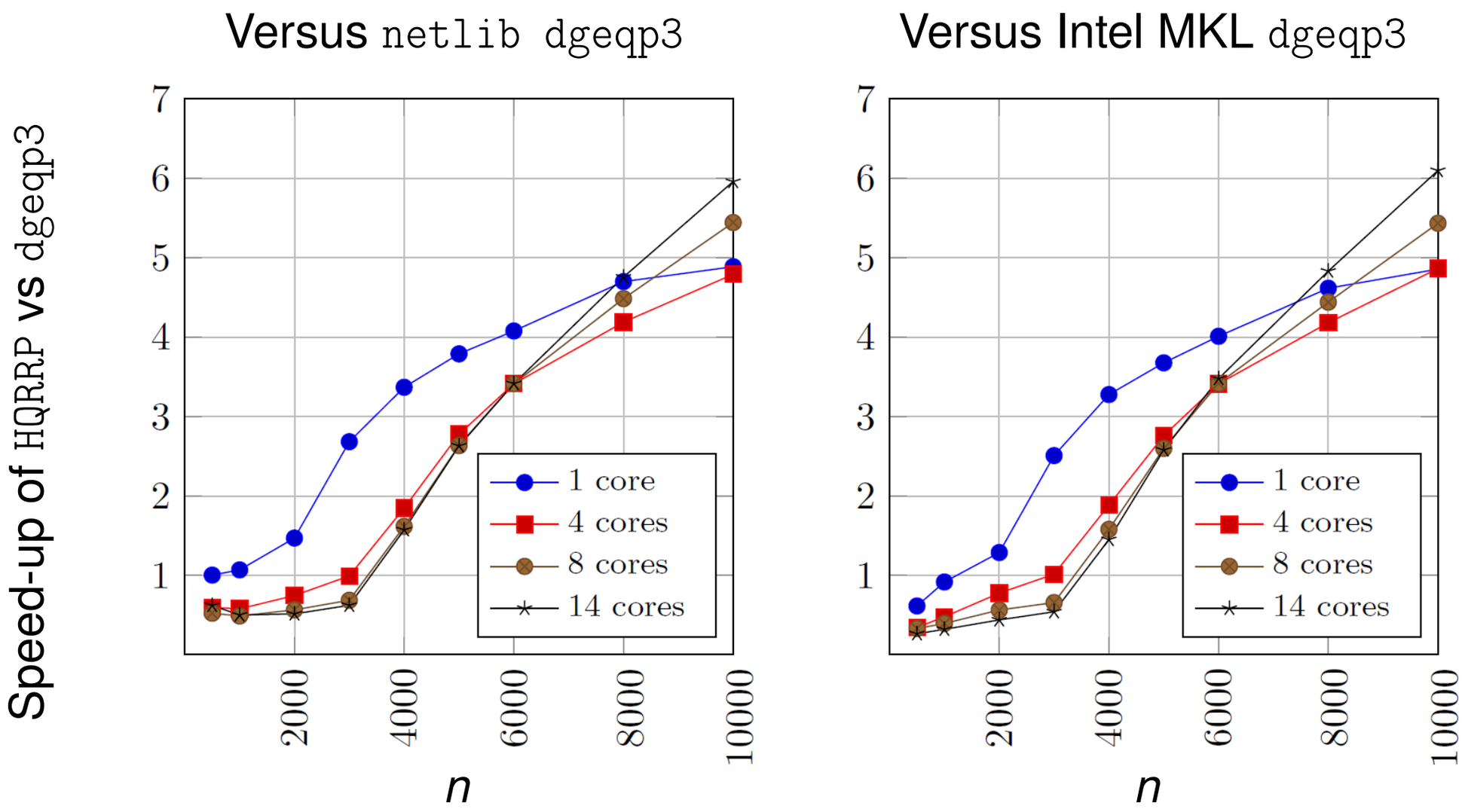
Computing full factorizations — accelerated column pivoted QR: The column selection strategy on the previous slide has been applied to resolve a classical problem in numerical linear algebra: How do you pick *groups* of pivots when executing column pivoted QR? The purpose is to move flops from BLAS2 to BLAS3 operations.



Speedup attained by a randomized algorithm for computing a full column pivoted QR factorization of an $n \times n$ matrix. The speed-up is measured versus LAPACK's faster routine `dgeqp3` as implemented in Netlib (left) and Intel's MKL (right). Our implementation was done in C, and was executed on an Intel Xeon E5-2695. Joint work with G. Quintana-Ortí, N. Heavner, and R. van de Geijn (SISC 2017). Closely related work by Duersch and Gu, SISC 2017 / SIREV 2020.

Outline of talk

1. **Randomized low rank approximation** *[Done!]*

   "Randomized singular value decomposition" or "RSVD".

   Techniques based on randomized embeddings.

   Relatively well established material within numerical linear algebra.

2. **Variations of randomized algorithms for low rank approximation** *[Done!]*

   Single pass and streaming algorithms.

   Structured random embeddings.

   Matrix approximation via sampling.

3. **Samples of current research directions**

   Least squares problems and linear system solvers.

   Sketching for data compression.

Solving least squares problems, linear systems, . . .

- **Overdetermined least squares problems.**

  Suppose $\mathbf{A} \in \mathbb{R}^{m \times n}$ for $m \gg n$, and that you seek to solve $\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$.

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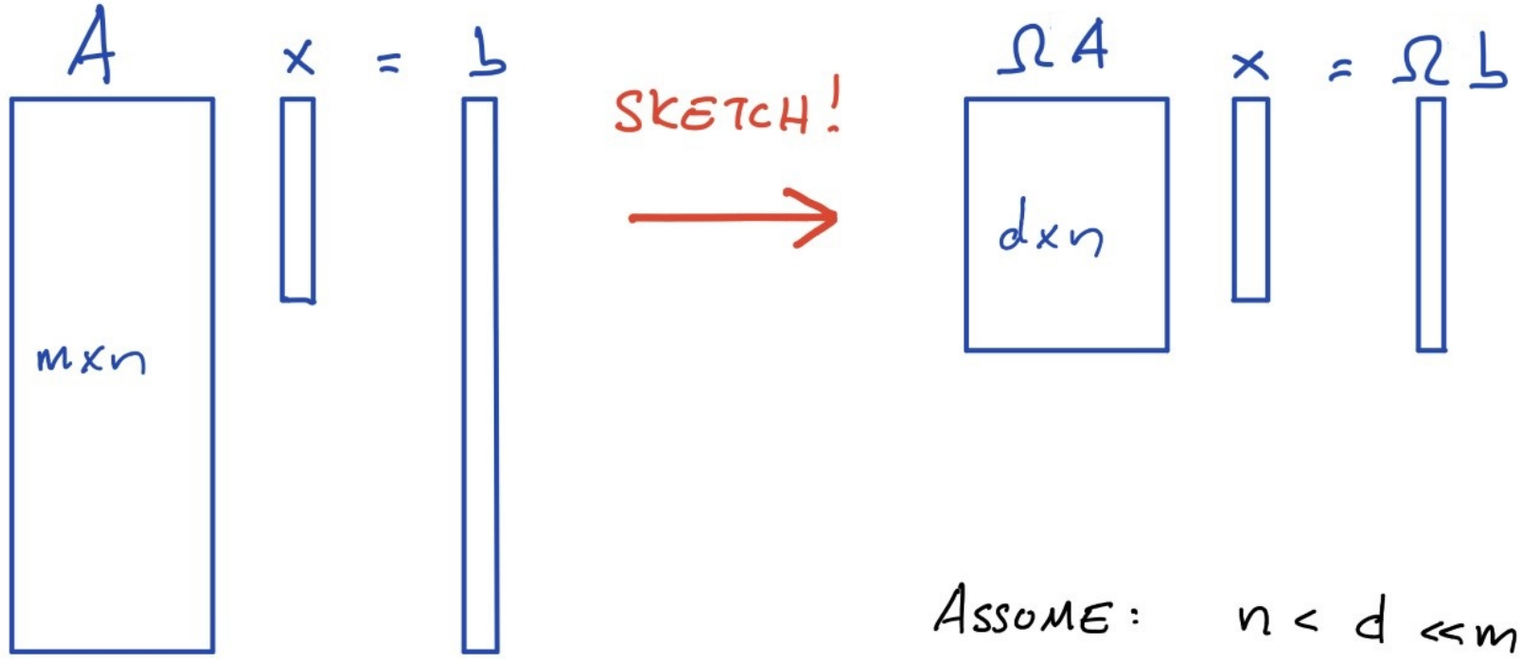
  Draw a random embedding $\mathbf{\Omega} \in \mathbb{R}^{d \times m}$ and construct a smaller *sketched* system.



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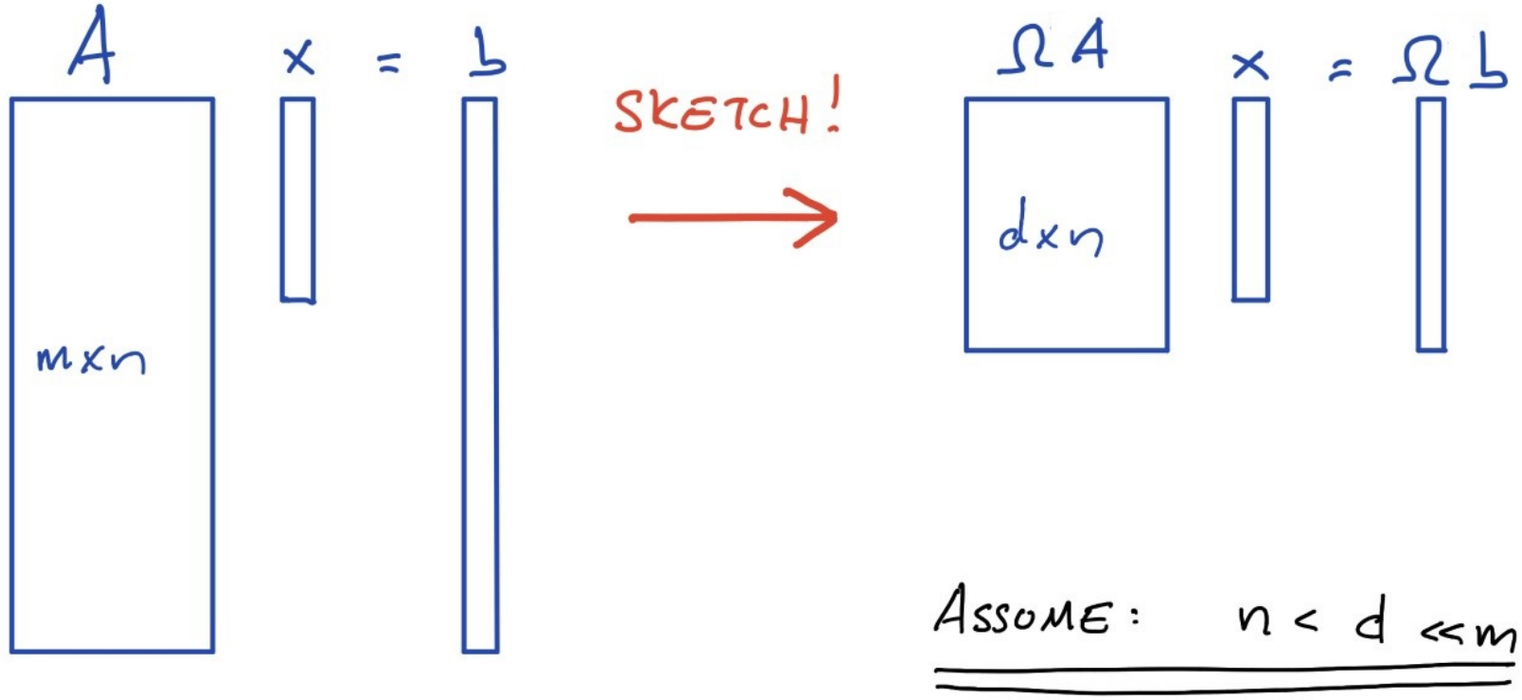

A bold approach — "sketch-to-solve":

Find the vector $\mathbf{x}$ that solves the sketched system.

A safe approach — "sketch-to-precondition":

Build a *preconditioner* $\mathbf{M} \in \mathbb{R}^{n \times n}$ by factorizing $\mathbf{\Omega A}$ so that $\mathbf{\Omega A} = \mathbf{QM}$.

Iterate on the preconditioned linear system $(\mathbf{AM}^{-1})(\mathbf{Mx}) = \mathbf{b}$.

Rokhlin/Tygert (2008), Avron/Maymounkov/Toledo (2010), many more

Solving least squares problems, linear systems, . . .

- **Overdetermined least squares problems.** *Sketch-to-precondition paradigm.*
- **Randomized Kaczmarz:** Randomization in a classical algorithm. Particularly effective when randomized embeddings are incorporated. *(Strohmer/Vershynnin 2009, Needell/Ward/Srebro 2014, Gower/Richtarik 2015, Liu/Wright 2016, . . .)*
- **Eliminating pivoting in Gaussian elimination:** D. Stott Parker showed in 1995 that you can eschew pivoting in Gaussian elimination if you first "scramble" the coefficient matrix through "pre-conditioning" via random unitary maps. Early example of fast J-L transform! Related work by Demmel/Dumitriu/Holtz (2007).
- **Graph Laplacians:** Linear systems whose coefficient matrix is a graph Laplacian can be solved using randomized methods in close to linear (in the number of edges) complexity. The idea is to compute an approximate Cholesky factorization $\mathbf{A} \approx \mathbf{C}\mathbf{C}^*$, and then use $\mathbf{C}$ as a preconditioner in conjugate gradients. Can be hybridized with ideas from nested dissection to achieve high practical speed for problems in scientific computing. *(Spielman/Teng 2004, Kyng/Sachdeva 2016, Koutis/Miller/Tolliver 2011, Livne/Brandt 2012, Spielman 2020, Chen/Liang/Biros 2020, . . .).*

Solution of huge linear systems via randomized sampling

Randomized sampling has enabled the solution of some systems $\mathbf{A}\mathbf{x} = \mathbf{b}$ that would otherwise be intractable. Think of systems so large that even forming $\mathbf{A}$ is impossible.

In such situations, we must have some à priori information about structure or regularity in $\mathbf{A}$ that can be exploited. A common situation is that $\mathbf{A}(i,j) = k(\mathbf{x}_i, \mathbf{x}_j)$, where k is a given *kernel function*, and $\{\mathbf{x}_i\}_{i=1}^N$ is a given set of points in $\mathbb{R}^d$.

- **Kernel ridge regression:** Need to solve a linear system $\big(\mathbf{A} + \mu\mathbf{I}\big)\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is a kernel matrix. Techniques based on Nyström approximation combined with randomized sampling have proven highly effective. *(Alaoui/Mahoney 2015, Avron/Clarkson/Woodruff 2017, Musco/Musco 2017, Rudi/Calandriello/Carratino/Rosasco 2018, . . .)*

- **Rank-structured representations of kernel matrices:** Multifrontal sparse direct solvers; FMMs; boundary integral equation methods; covariance matrices in Gaussian processes; log-determinants; etc. *(Martinsson 2011, March/Xiao/Biros 2015, Ambikasaran/Foreman-Mackey/Greengard/Hogg/O'Neil 2015, Ghysels/Li/Gorman/Rouet 2016, Yu/Levitt/Reiz/Biros 2017, Rebrova/Chávez/Liu/Ghysels/Li 2018, Geoga/Anitescu/Stein 2019, . . .)*

- **Electronic structure calculations:** Impossible even to store $\mathbf{x}$. *(Lim/Weare 2017)*

Randomized embeddings: "Sketching" for data compression

Storage of scientific data is often a core challenge:

- Massive sensor arrays.
- High resolution and multiband imaging.
- Time stepping in scientific simulations — fluid dynamics, molecular dynamics, etc.

In some environments, well-established techniques based on harmonic analysis (Fourier bases, wavelets, etc) are helpful.

When less is known about the data à priori, computing *randomized sketches* has proven to be valuable. Many different regimes have been pursued:

- Low rank matrices.
- Various tensor decompositions.
- Sparse Fourier transforms.

References: *Woolfe/Liberty/Rokhlin/Tygert (2008), Clarkson/Woodruff (2009), Halko/Martinsson/Tropp (2011), Li/Nguyen/Woodruff (2014), Ghashami/Liberty/Phillips/Woodruff (2016), Kressner/Periša (2017), Tropp/Yurtsever/Udell/Cevher (2017–), . . .*

Key points on randomized singular value decomposition (RSVD):

- High practical speed — interacts with **A** only through matrix-matrix multiplication.
- Highly *communication efficient.*

  Acceleration of classical algorithms such as column pivoted QR.

  Particularly efficient for GPUs, out-of-core computing, distributed memory, etc.
- Reduction in complexity from $O(mnk)$ to $O(mn\log k)$ or even less via *structured random embeddings.*
- Single pass algorithms have been developed for *streaming environments.*

  Not possible with deterministic methods!

Current research directions:

- High performance implementations.
- Faster algorithms for computing *full* matrix factorizations.
- Solvers for linear systems and for least squares problems.
- Applications of randomized embeddings outside of linear algebra: Faster nearest neighbor search, faster clustering algorithms, data compression on the fly, etc.

Surveys:

- P.G. Martinsson and J. Tropp, "Randomized Numerical Linear Algebra: Foundations & Algorithms". *Acta Numerica*, 2020. (Arxiv report 2002.01387)

  Long survey summarizing major findings in the field in the past decade.

- P.G. Martinsson, "Randomized methods for matrix computations." The Mathematics of Data, IAS/Park City Mathematics Series, 25(4), pp. 187 - 231, 2018.

  Book chapter that is written to be accessible to a broad audience. Focused on practical aspects.

- N. Halko, P.G. Martinsson, J. Tropp, "Finding structure with randomness: Probabilistic algorithms for constructing approximate matrix decompositions." *SIAM Review*, 53(2), 2011, pp. 217-288.

  Survey that describes the randomized SVD and its variations.

Tutorials, summer schools, etc:

- 2020: 3 lecture mini course on randomized linear algebra, KTH, Stockholm. Videos available.
- 2016: Park City Math Institute (IAS): *The Mathematics of Data.*
- 2014: CBMS summer school at Dartmouth College. 10 lectures on YouTube.
- 2009: NIPS tutorial lecture, Vancouver, 2009. Online video available.

Software:

- ID: `http://tygert.com/software.html` (ID, SRFT, CPQR, etc)
- RSVDPACK: `https://github.com/sergeyvoronin` (RSVD, randomized ID and CUR)
- HQRRP: `https://github.com/flame/hqrrp/` (LAPACK compatible randomized CPQR)
- Randomized UTV: `https://github.com/flame/randutv`

DOE report on randomized algorithms: `https://arxiv.org/abs/2104.11079` (2021)