Bandgap phenomena in materials with periodic skeletal micro-structures

P.G. Martinsson

Yale University

Joint with with A. Movchan.

Contributors: I. Babuška, G. Rodin.

Goal: To find methods for designing structures that support no propagating waves in certain pre-scribed frequency bands.

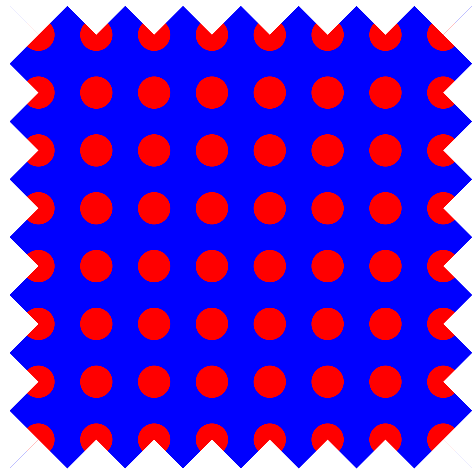
Applications: Most notably in optics: design of wave-guides, etc.

A representative problem:

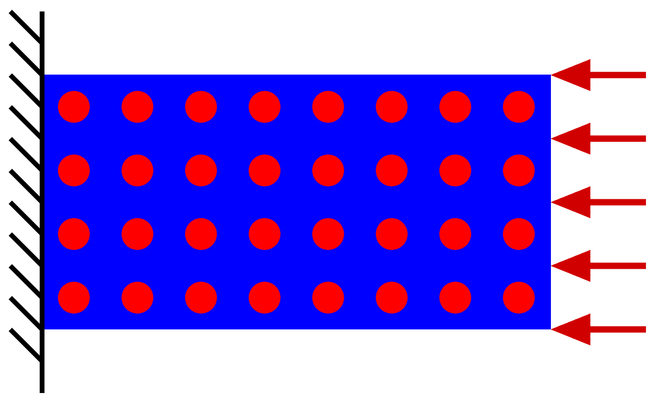
Design the blue box so that the amplitude of the vibration at the attachment point is minimized.

A simple solution strategy:

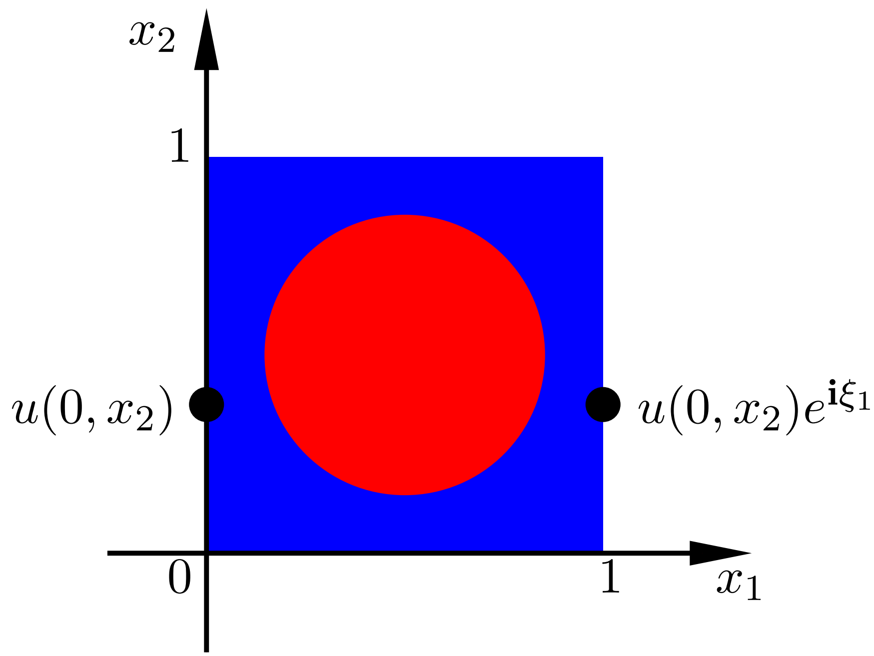
Step 1:
Solve a problem with a repeating unit cell in an infinite medium.



Step 2:
Construct the finite box using suitably many units of the cell constructed in Step 1.



This strategy is sub-optimal but leads to structures that are simple both to design and to construct.



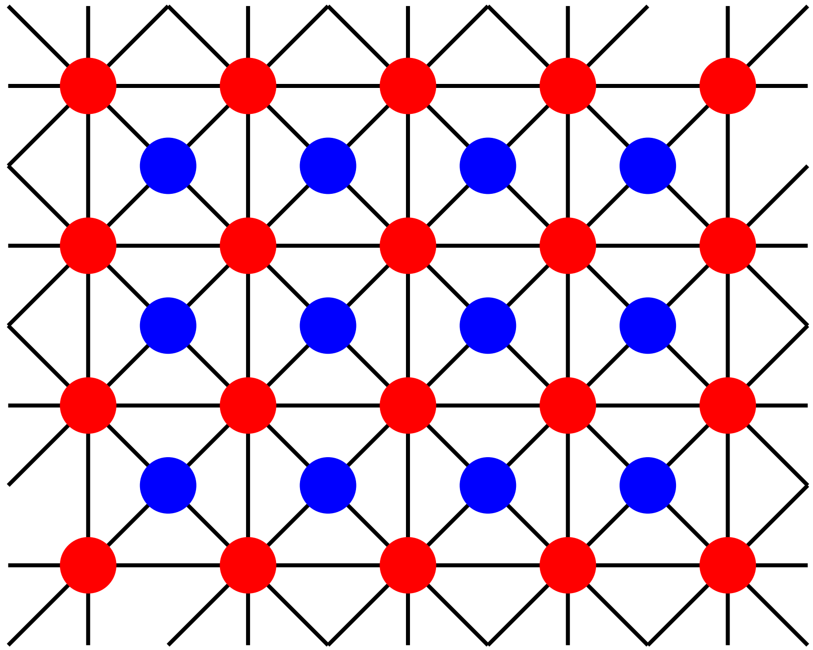
The wave propagation problem in the infinitely repeating unit cell is formulated as a Helmholtz problem with periodic boundary conditions:

$$
\begin{aligned}
-\nabla \cdot (A(x)\nabla u(x)) &= m\omega^2 u(x), && x \in (0,\, 1)^2 = \Omega,\\
u(x + e_1) &= u(x)e^{\mathbf{i}\xi_1}, && x \in \partial\Omega,\\
u(x + e_2) &= u(x)e^{\mathbf{i}\xi_2}, && x \in \partial\Omega.
\end{aligned}
$$

where $A(x)$ is the local stiffness of the material and

ω	is the frequency of the oscillation,
$\xi = (\xi_1,\, \xi_2)$	is the wave number ("Bloch vector")

Now consider the analogous problem on a mass-spring system.



For such structures:

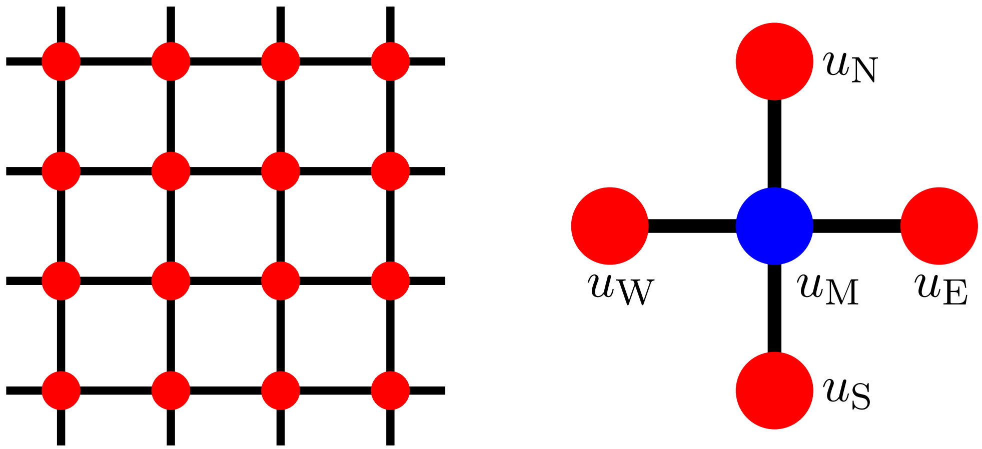
(a) the equation on the unit cell reduces to a matrix equation, and,

(b) such structures capture key phenomena of continuum problems.

Lattice materials form a good sand box for developing understanding.

We will demonstrate a simple heuristic method for placing bandgaps.

Let us determine the wave propagation modes for anti-plane waves of frequency ω with wave vector $\xi \in (-\pi, \pi)^2$ in a very simple 2D lattice.



$$
\begin{aligned}
u_{\mathrm{E}} &= e^{\mathbf{i}\xi_2} u_{\mathrm{M}}, \\
u_{\mathrm{W}} &= e^{-\mathbf{i}\xi_2} u_{\mathrm{M}}, \\
u_{\mathrm{N}} &= e^{\mathbf{i}\xi_1} u_{\mathrm{M}}, \\
u_{\mathrm{S}} &= e^{-\mathbf{i}\xi_1} u_{\mathrm{M}}.
\end{aligned}
$$

The equilibrium equation for the blue node reads

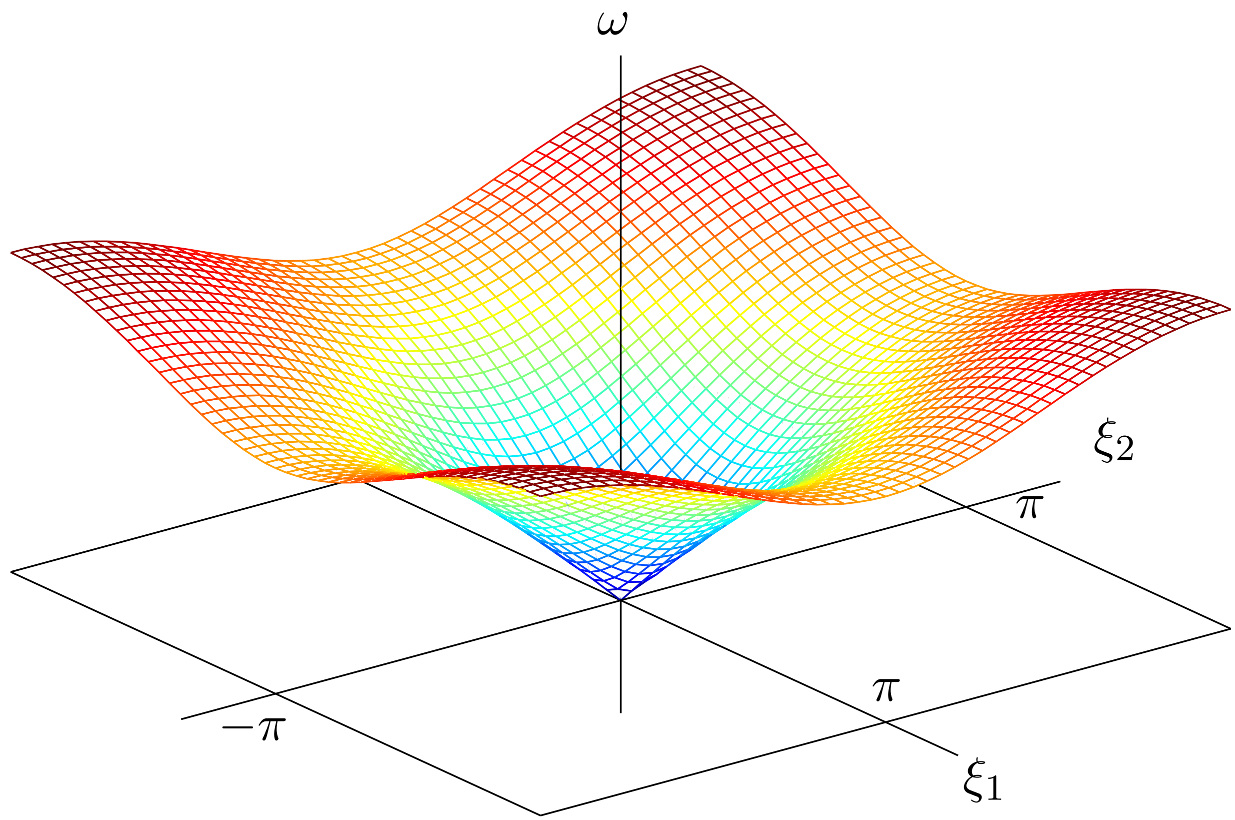
$$m\omega^2 u_{\mathrm{M}} = k\left(4u_{\mathrm{M}} - u_{\mathrm{S}} - u_{\mathrm{N}} - u_{\mathrm{W}} - u_{\mathrm{E}}\right),$$

where m is the mass of a node, and k is the spring constant of the bars. Inserting the periodicity assumption, we obtain the equation

$$m\omega^2 u_{\mathrm{M}} = k\sigma(\xi) u_{\mathrm{M}},$$

where

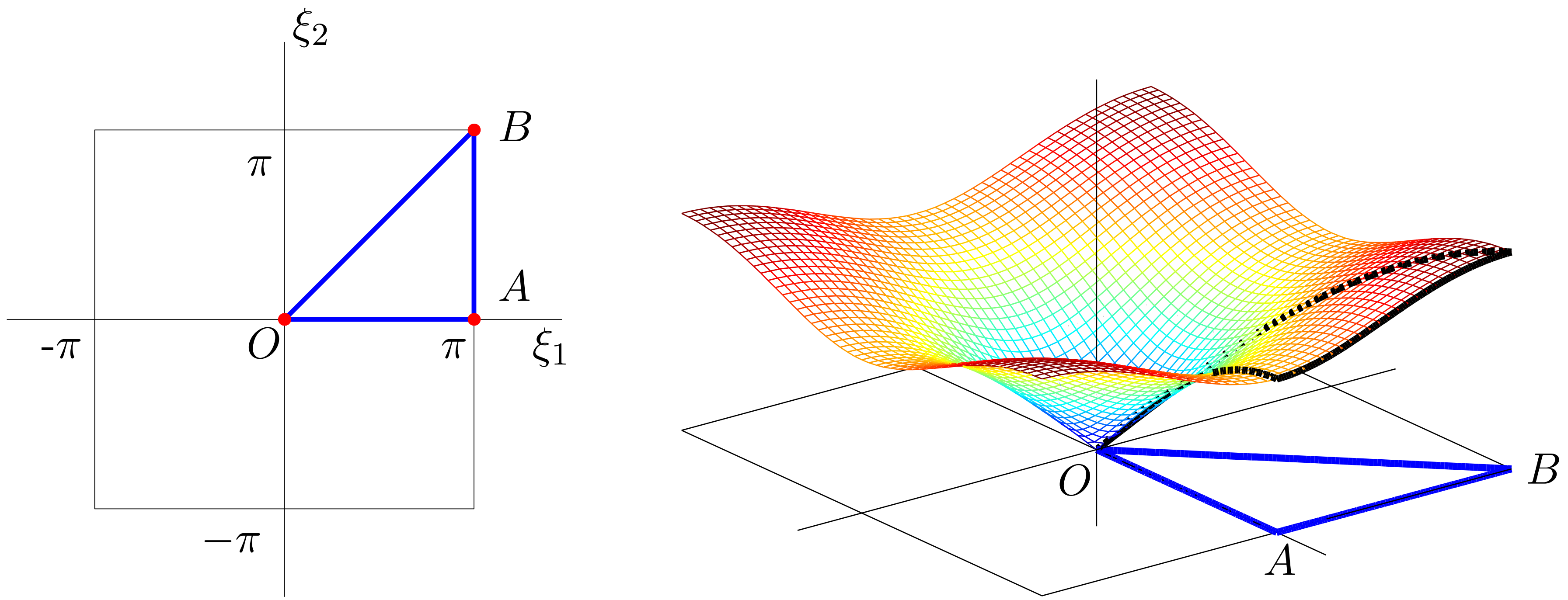
$$\sigma(\xi) = 4 - e^{\mathbf{i}\xi_1} - e^{-\mathbf{i}\xi_1} - e^{\mathbf{i}\xi_2} - e^{-\mathbf{i}\xi_2} = 4\sin^2\frac{\xi_1}{2} + 4\sin^2\frac{\xi_2}{2}.$$



The conclusion from the calculation is that for a given wave-vector $\xi = (\xi_1, \xi_2) \in (-\pi, \pi)^2$, only waves of frequency

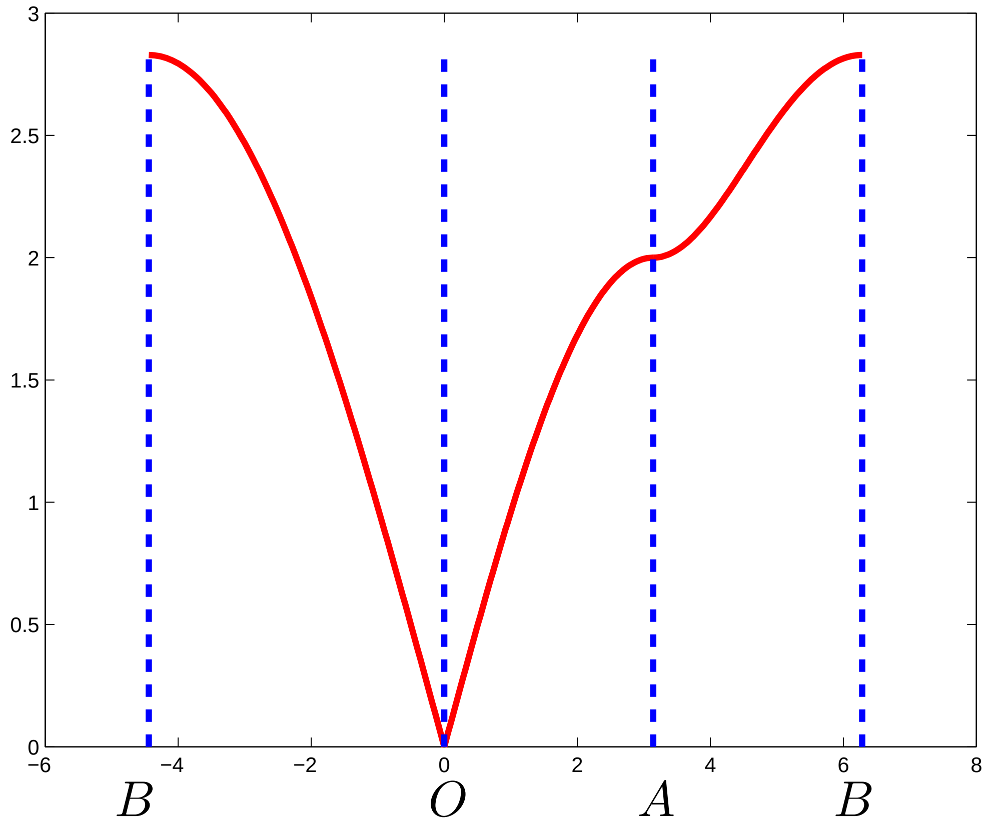
$$\omega = \sqrt{\frac{k}{m}\left(4\sin^2\frac{\xi_1}{2} + 4\sin^2\frac{\xi_2}{2}\right)}$$

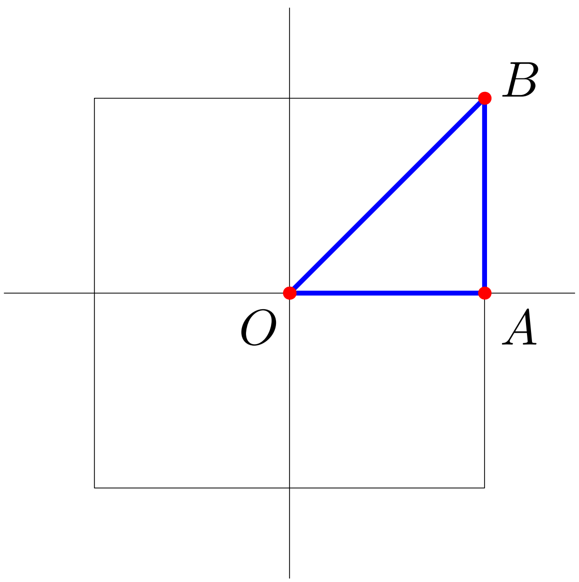
can propagate through the lattice.

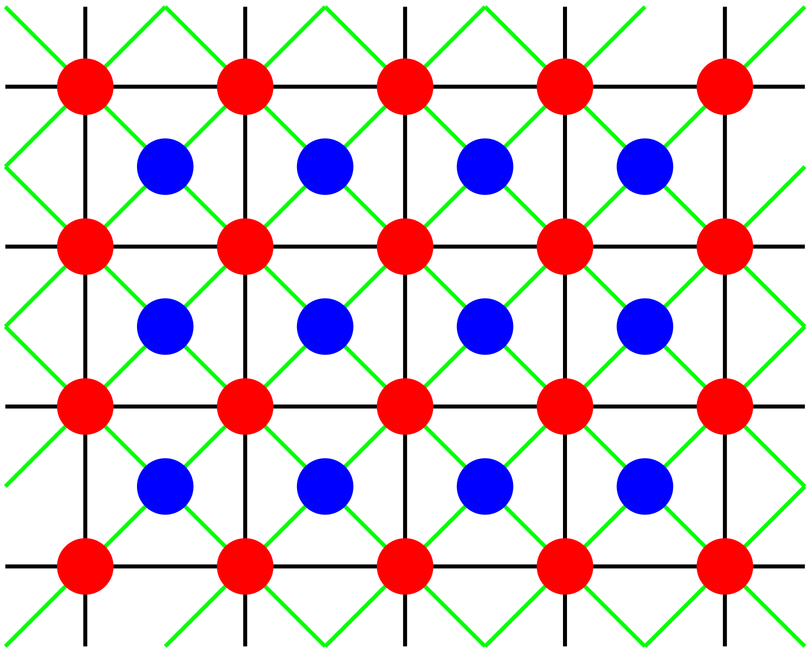


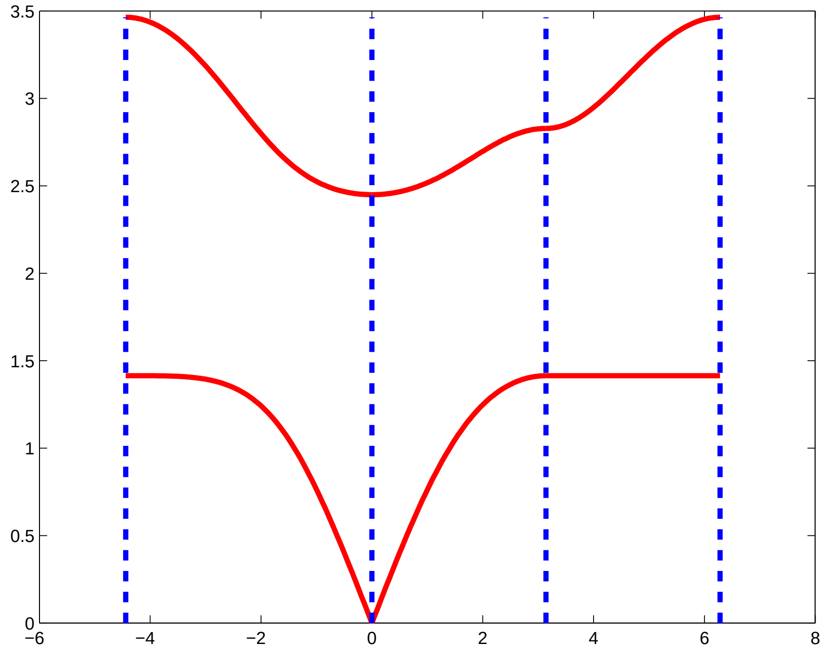
The symbol conveniently encodes all the relevant information about the lattice. Some of this information can be visualized by plotting the eigenvalues of $\sigma(\xi)$ as ξ travels along the path $B \to 0 \to A \to B$.

Dispersion diagram for the square lattice:





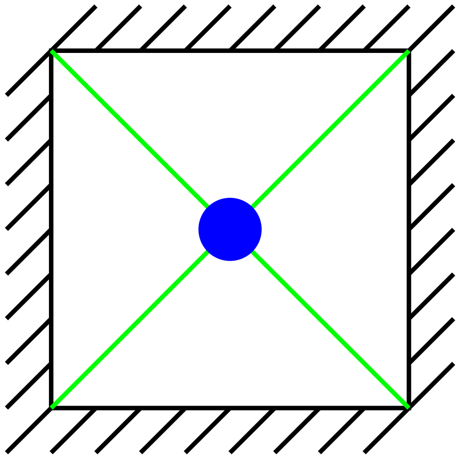


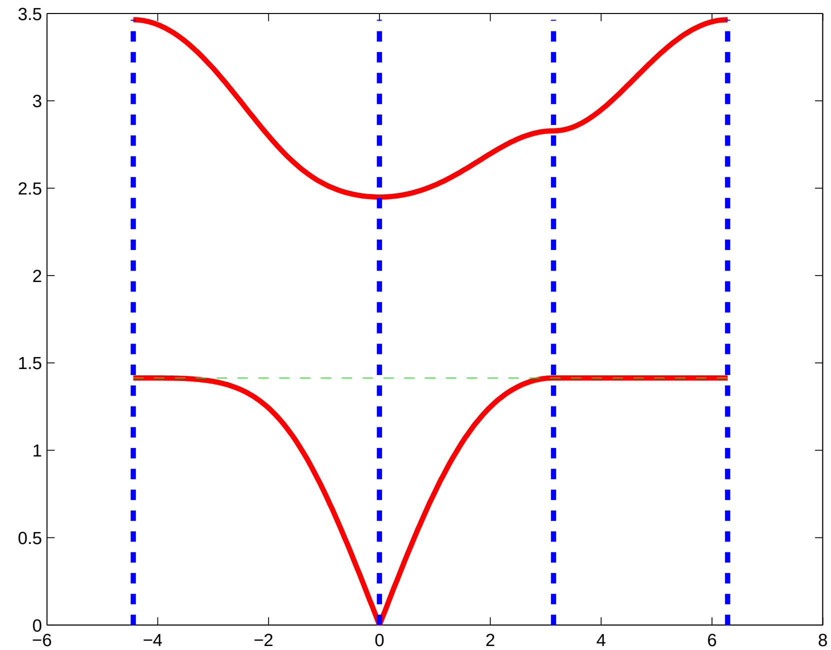


To the original lattice (black and red), we add the blue nodes and the green springs.

The corresponding dispersion diagram.
Note the band gap!

What determines the location of the bandgap?





Look at an isolated oscillator. Its equation is

$$m_{\text{osc}}\omega_{\text{osc}}^2 u = 4k_{\text{osc}}u,$$

so

$$\omega_{\text{osc}} = 2\sqrt{\frac{k_{\text{osc}}}{m_{\text{osc}}}}.$$

For the case plotted here,

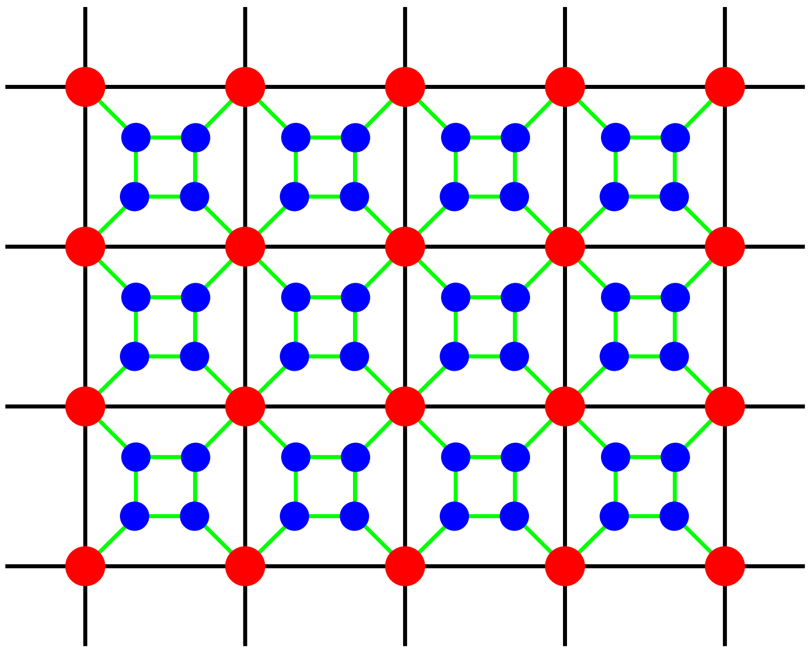
$$m_{\text{osc}} = 2,$$

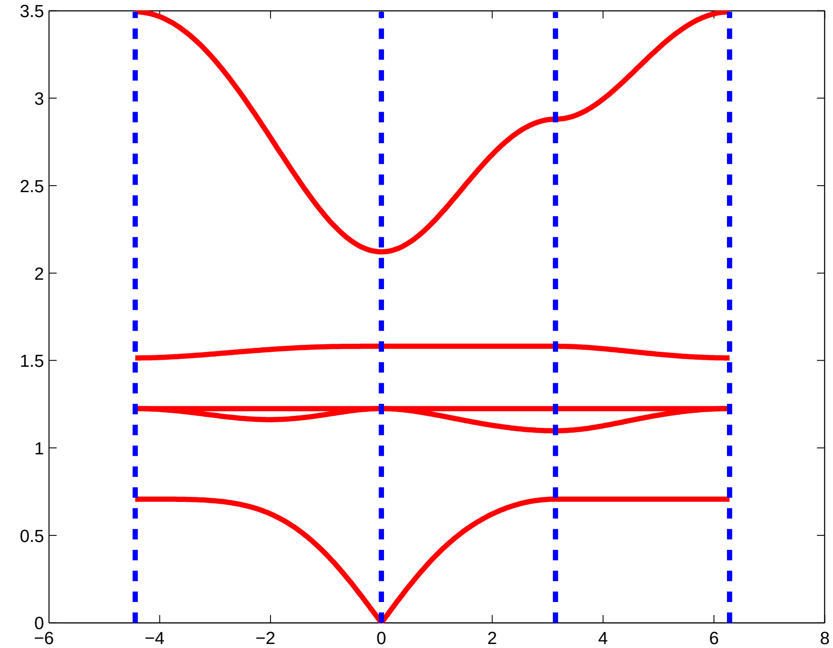
and

$$k_{\text{osc}} = 1,$$

so

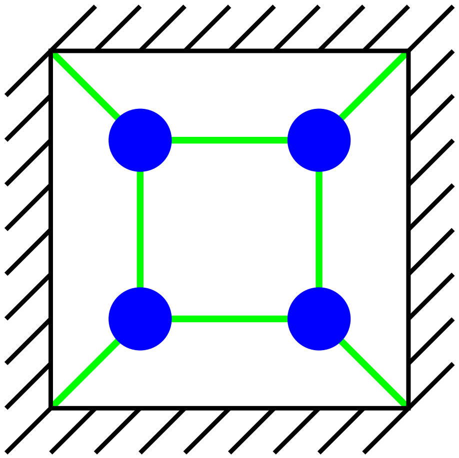
$$\omega_{\text{osc}} = \sqrt{2}.$$

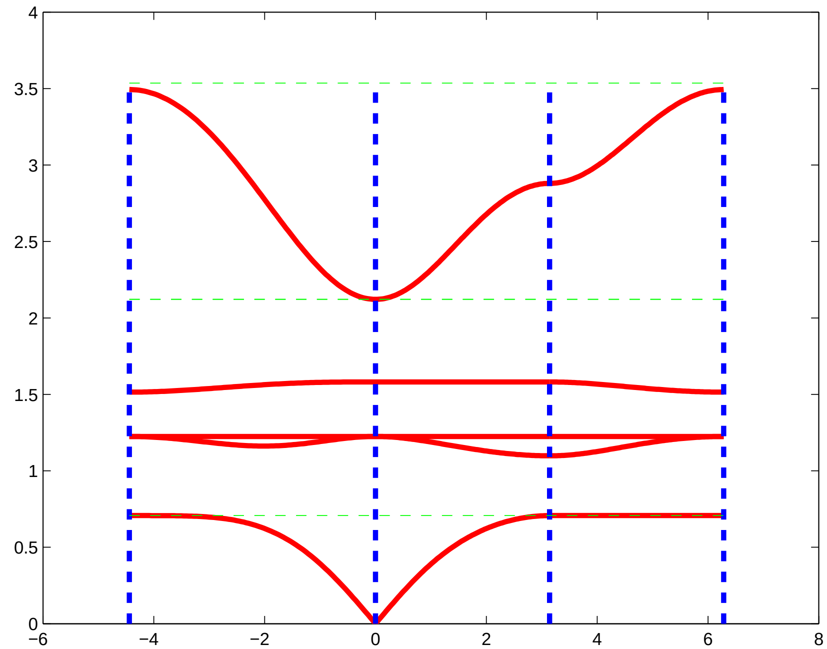




Now we add a more complicated oscillator.

The corresponding dispersion diagram.





$$m_{\text{osc}}\omega_{\text{osc}}^2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = k_{\text{osc}} \begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}.$$

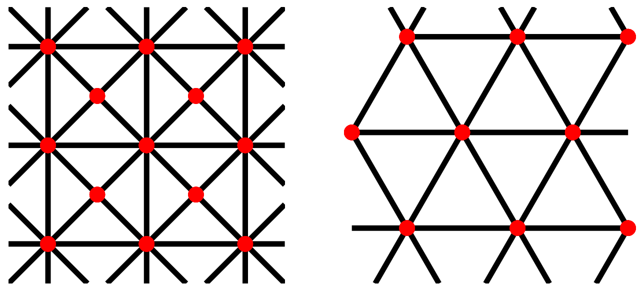
The eigenvalues of the matrix are $\{1, 3, 3, 5\}$, so

$$\omega_{\text{osc}} = \sqrt{\frac{k_{\text{osc}}}{m_{\text{osc}}}}, \quad 3\sqrt{\frac{k_{\text{osc}}}{m_{\text{osc}}}}, \quad 5\sqrt{\frac{k_{\text{osc}}}{m_{\text{osc}}}}.$$

In-plane vibrations — two distinct types of lattices

Truss Lattices

Strength from **axial** stiffness.



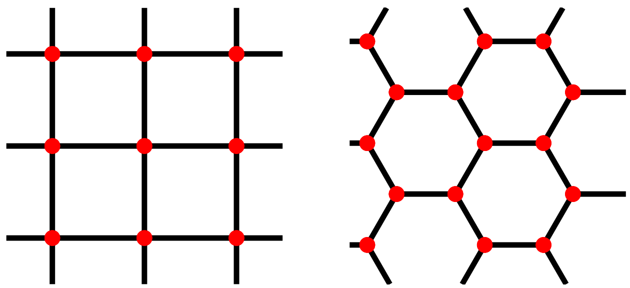
Symbol is always a matrix:
2 degrees of freedom per node.

Typically quite stiff.

Asymptotics: Classical elasticity.

Frame Lattices

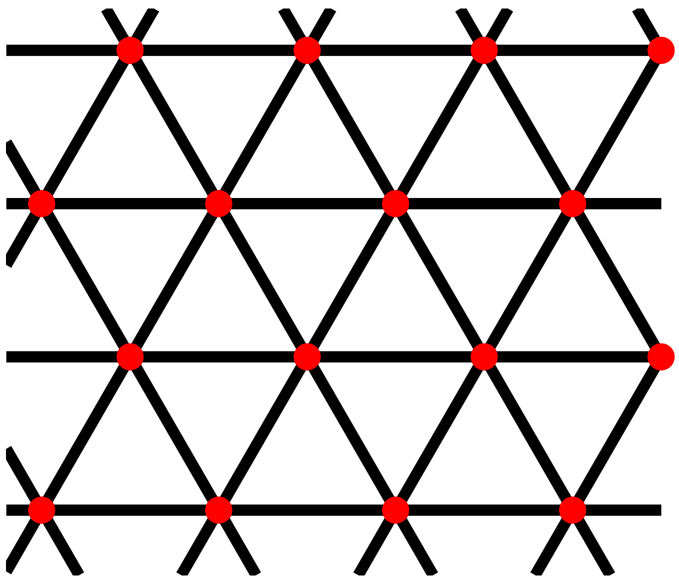
Strength from **bending** stiffness.



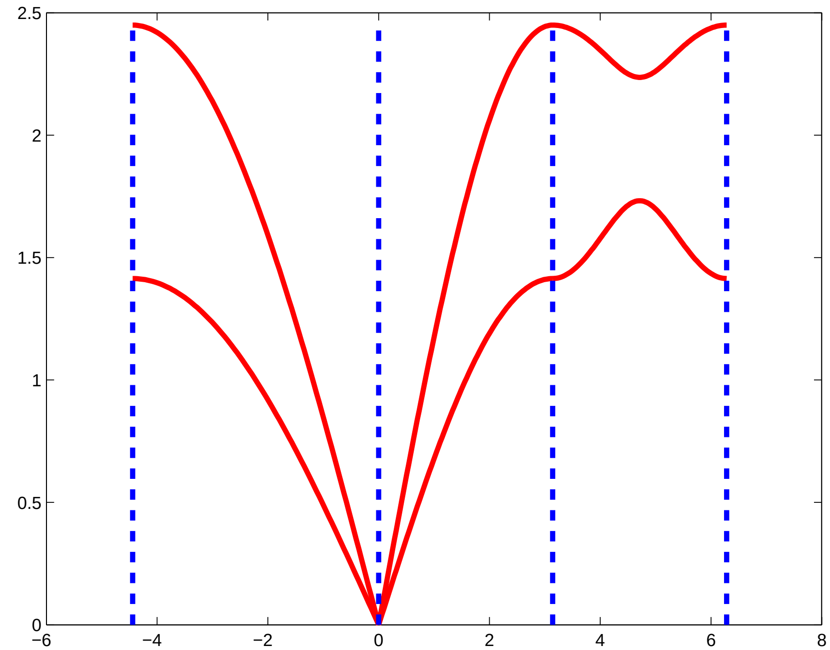
Symbol is always a matrix:
3 degrees of freedom per node.

Typically quite soft
(but anisotropic).

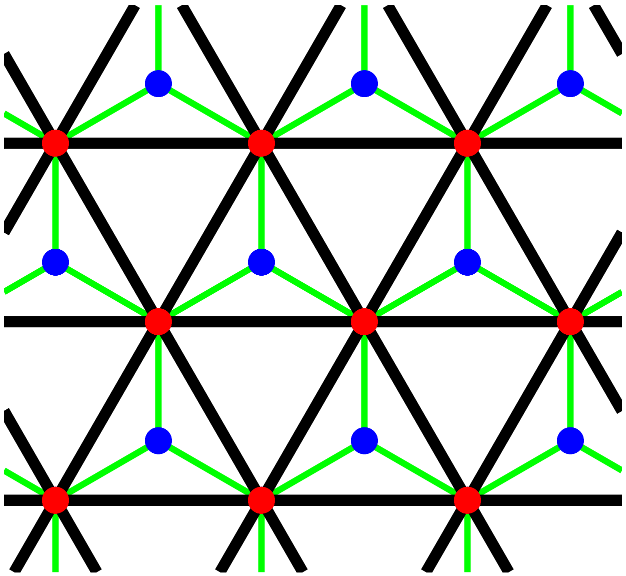
Asymptotics: Cosserat elasticity.



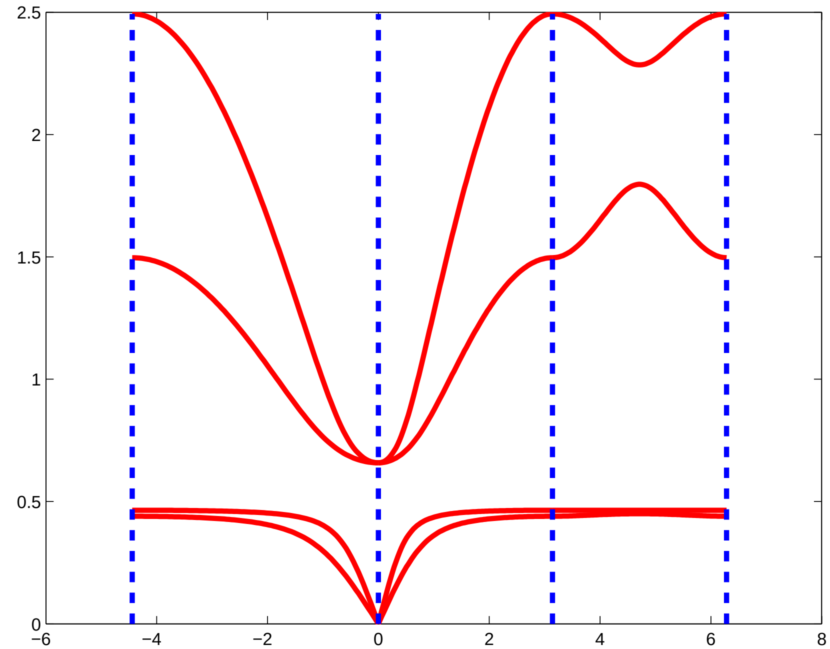
A triangular lattice.
Modelled as a **truss**.



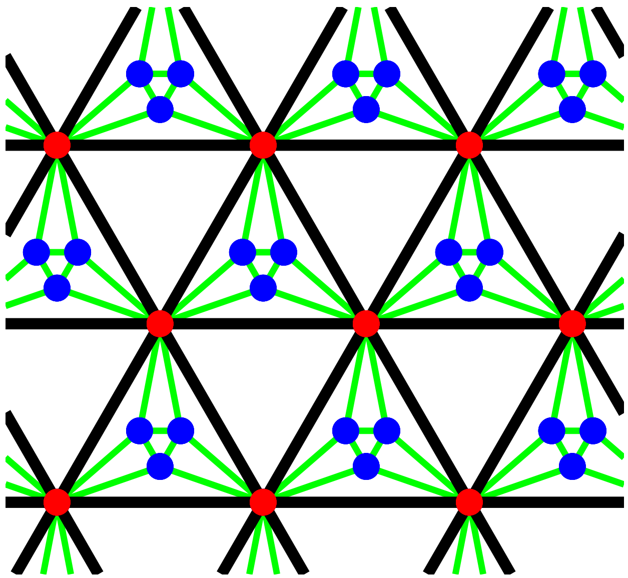
The corresponding dispersion diagram.



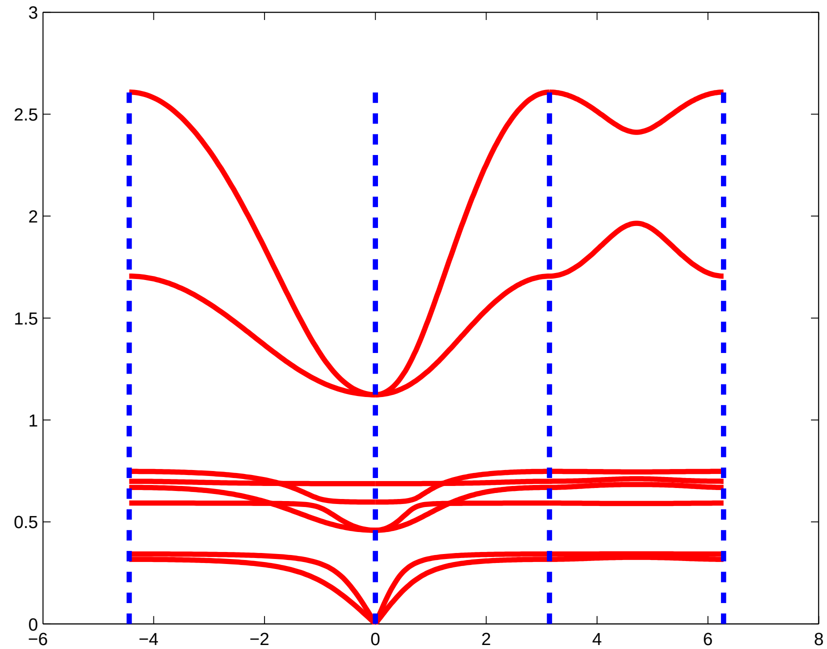
A triangular lattice with a simple oscillator.
Modelled as a **truss**.



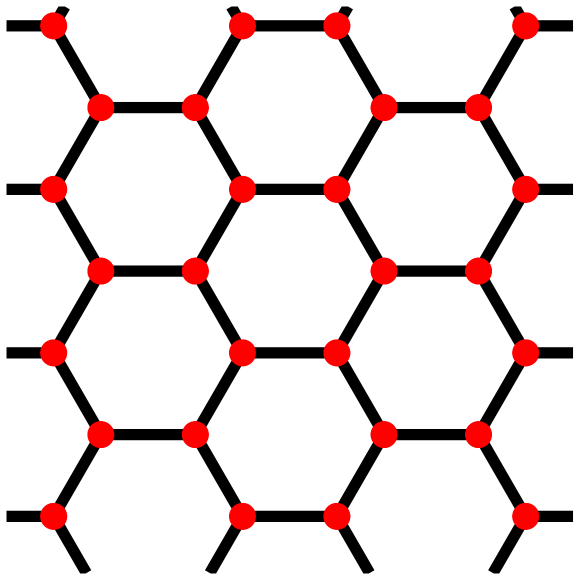
The corresponding dispersion diagram.

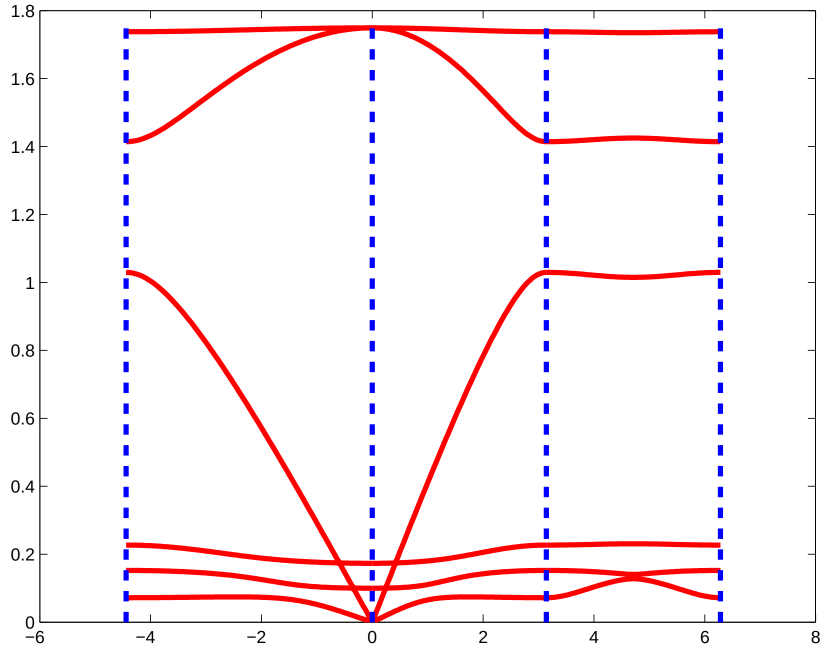


A triangular lattice with a complicated oscillator.
Modelled as a **truss**.



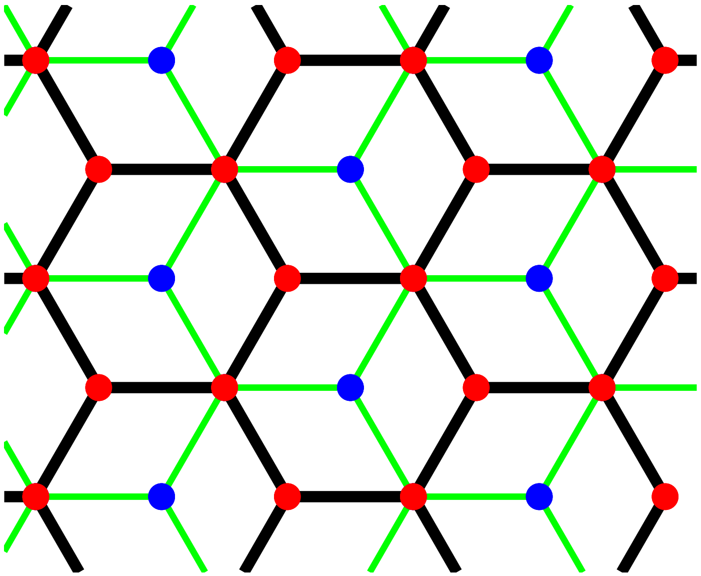
The corresponding dispersion diagram.





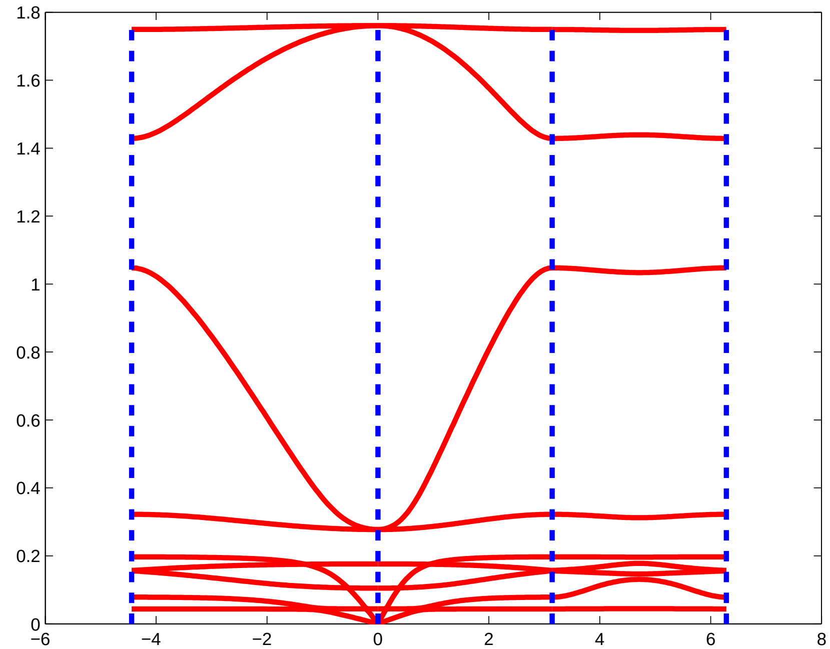
A honeycomb lattice.
Modelled as a **frame**.

The corresponding dispersion diagram.
Notice the “soft” modes.



A honeycomb lattice with an oscillator.

Modelled as a **frame**.



The corresponding dispersion diagram.

Summary.

- For infinite lattice structures, the problem on the unit cell simplifies to an eigenvalue problem for a smallish matrix.

- Due to the simplicity of the model, it is easy to “place” bandgaps. Heuristic understanding of the problem.

- It is possible to take advantage of the anisotropy of frame lattices to place bandgaps in low frequency bands.