Adaptive Finite Element Simulations of Thermosolutal Convection in Porous Media

J. W. Peterson,[a]
B. T. Murray,[b] D. J. Knezevic,[c] G. F. Carey[d]

[a] Dept. of Aerospace Engineering & Eng. Mech., UT-Austin
[b] Dept. of Mech. Eng., SUNY Binghamton
[c] Numerical Analysis Group, Oxford University
[d] Inst. for Computational Engineering & Sciences (ICES), UT-Austin

July 26, 2005

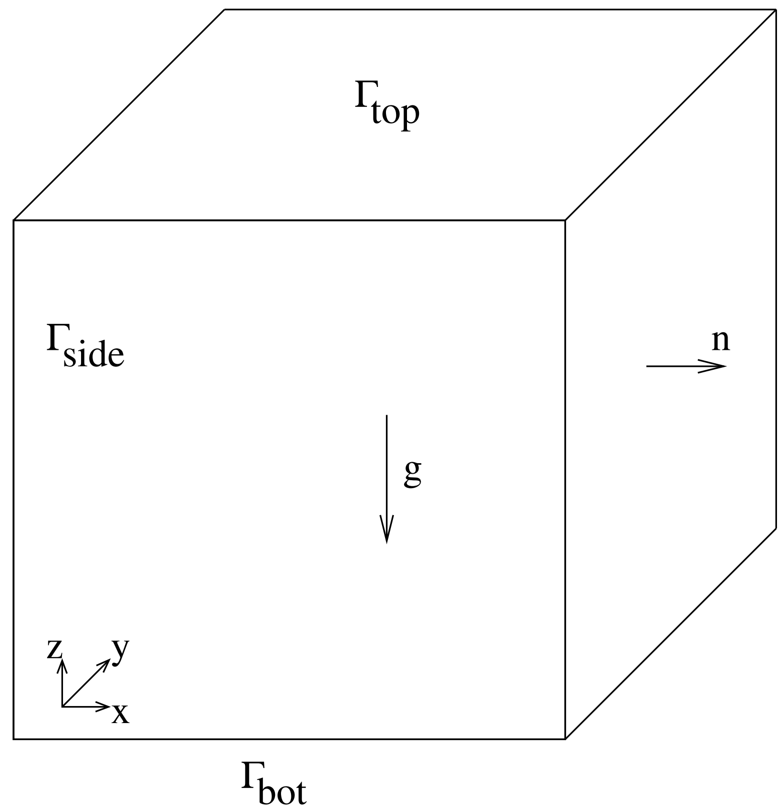
Introduction

- "Double-diffusive" effects occur whenever there are opposing gradients of two diffusing components, each of which affects the local density of a fluid.

- In thermosolutal convection, the competing components are solute concentration and heat (e.g. a differentially heated layer of sand saturated with brine)

- In this talk, we consider the stability of an extremely simple double-diffusive system, and develop an adaptive finite element method for computing accurate solutions under different parametric regimes.

Problem Description



- $\boldsymbol{u} \cdot \hat{n} = 0 \in \Gamma = \Gamma_{\text{top}} \cup \Gamma_{\text{side}} \cup \Gamma_{\text{bot}}$
- Solute and temperature values fixed on Γ_{top} and Γ_{bot}
- No solute or temperature flux from Γ_{side}

Governing Equations (1)

- The equations model slow flow through saturated porous media with buoyancy effects (see e.g. Nield & Bejan, 1992)

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1}$$

$$\frac{\rho_0}{\epsilon}\frac{\partial \boldsymbol{u}}{\partial t} = -\nabla p + \rho \boldsymbol{g} - \frac{\mu}{K}\boldsymbol{u} \tag{2}$$

$$\sigma\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \kappa_T \Delta T \tag{3}$$

$$\epsilon\frac{\partial S}{\partial t} + \boldsymbol{u} \cdot \nabla S = \kappa_S \Delta S \tag{4}$$

$$\rho = \rho_0[1 - \alpha(T - T_0) + \beta(S - S_0)] \tag{5}$$

Governing Equations (2)

- The equations are non-dimensionalized with length scale d (vertical depth), time scale $d^2\sigma/\kappa_T$, velocity scale κ_T/d, temperature scale δT, and solutal scale δS.
- Under this non-dimensionalization we obtain the thermal and solutal Rayleigh numbers

$$R_T = \frac{g\alpha K \delta T d}{\nu \kappa_T} \qquad R_S = \frac{g\beta K \delta S d}{\nu \kappa_S} \tag{6}$$

- A Poisson equation for the pressure is obtained by taking the divergence of Eqn. (2) and enforcing the $\nabla \cdot \boldsymbol{u} = 0$ condition.
- For simplicity, we also define the inverse Lewis number $\tau = 1/Le := \kappa_S/\kappa_T$, and the buoyant force vector $\boldsymbol{b} := (\tau R_S S - R_T T)\hat{e}_g$.

Governing Equations (3)

- The non-dimensional governing equations are

$$\nabla \cdot \boldsymbol{b} - \Delta p = 0 \tag{7}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{b} - \nabla p) \cdot \nabla T - \Delta T = 0 \tag{8}$$

$$\frac{\epsilon}{\sigma} \frac{\partial S}{\partial t} + (\boldsymbol{b} - \nabla p) \cdot \nabla S - \tau \Delta S = 0 \tag{9}$$

- Generally, $\tau < 1$ since solute diffuses in the medium more slowly than heat.

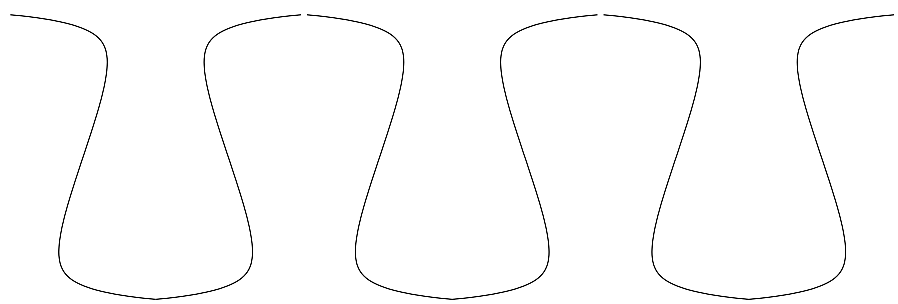
- $\epsilon / \sigma = \mathcal{O}(1)$, so the equations are not very "stiff"

Remarks (1)

- A steady, quiescent solution with $\hat{e}_g = -\hat{k}$, $0 \le z \le 1$ is given by

$$\begin{aligned}
p_0 &= \quad R_T \left[T_{\text{bot}} z + \frac{z^2}{2}(T_{\text{top}} - T_{\text{bot}}) \right] - \\
&\quad \tau R_S \left[S_{\text{bot}} z + \frac{z^2}{2}(S_{\text{top}} - S_{\text{bot}}) \right] \\
T_0 &= T_{\text{bot}} + z(T_{\text{top}} - T_{\text{bot}}) \\
S_0 &= S_{\text{bot}} + z(S_{\text{top}} - S_{\text{bot}})
\end{aligned}$$

- Depending on the parameters, this base solution may be stable or unstable to finite size perturbations.



Remarks (2)

- The "natural" boundary condition for Eqn. (7)

$$(\boldsymbol{b} - \nabla p) \cdot \hat{n} = 0 \in \Gamma \tag{10}$$

is equivalent to the no-penetration ($\boldsymbol{u} \cdot \hat{n} = 0$) condition imposed on the domain.

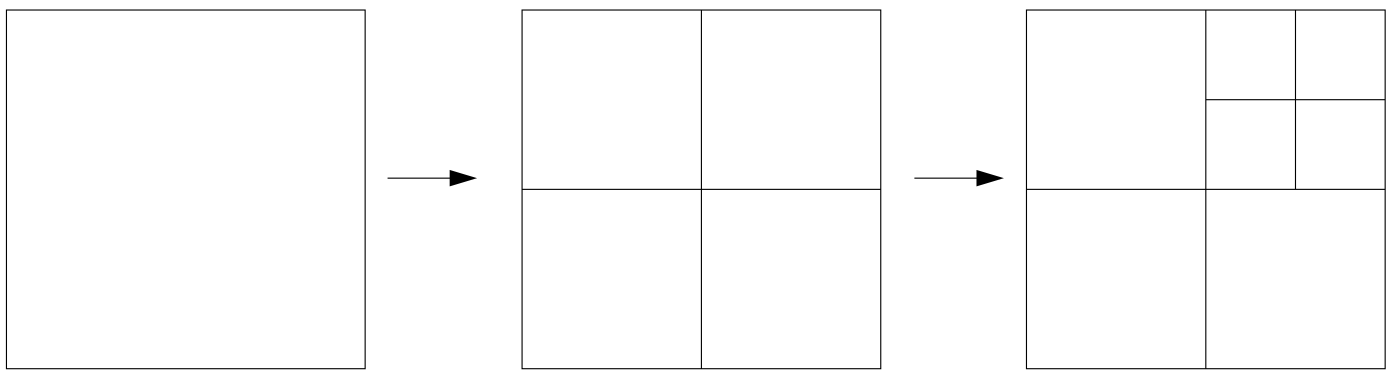
- An important quantity of interest will be the solutal Nusselt number

$$N_S := - \int_{\Gamma_D} (\nabla S \cdot \hat{n}) \, dx \tag{11}$$

This value will be used to compare solutions on different grids.

Numerical Method (1)

- Time discretization is via θ-method, (θ usually 1/2)
- Standard bilinear Lagrange elements are used to solve for the primary variables p, T, and S
- Hierarchical h-refinement with constrained hanging nodes, level-1 rule, and maximum refinement level of 2 with respect to the coarse mesh

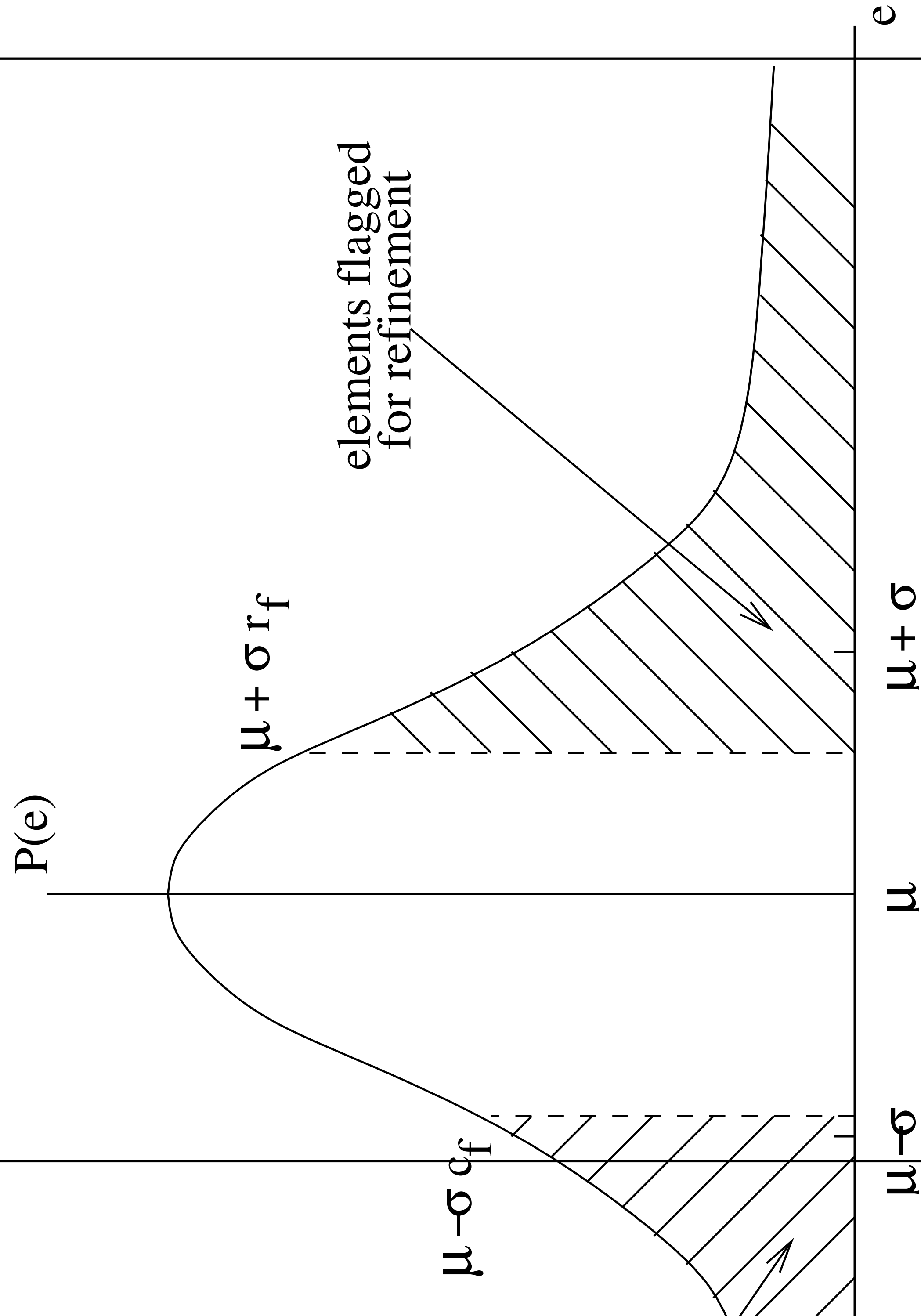


- One or two refinement steps per timestep

Numerical Method (2)

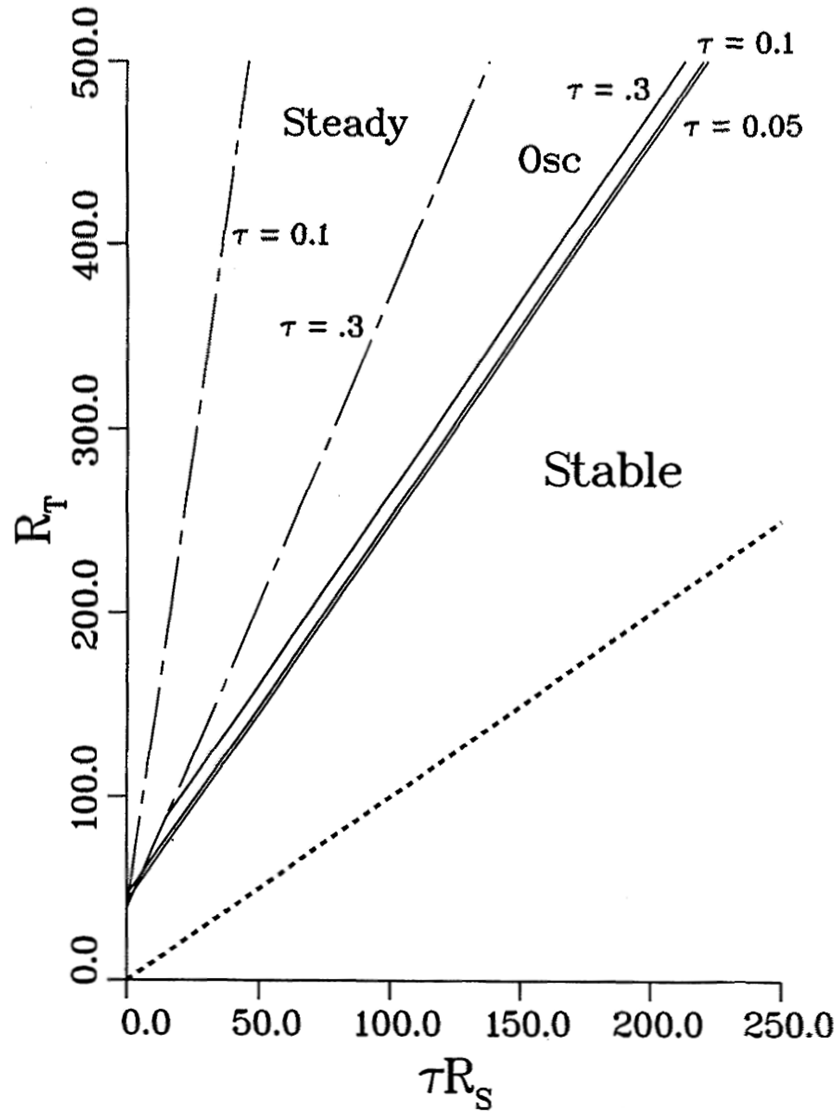
- Error indicator is the generic Kelly et. al indicator, jumps in the inter-element solutal flux drive the refinement patterns

- Refinement flagging scheme is based on assumed (log) normal distribution of the element errors


e
elements flagged
for refinement
μ + σ r f
P(e)
μ
μ + σ
μ−σ
μ−σ c f

Linear Stability Theory

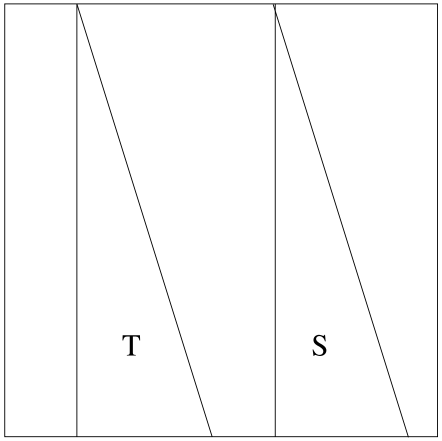
- Linear stability diagram for the destabilizing thermal / stabilizing solute configuration
- The diffusivity ratio affects the mechanism for onset of convection
- We conduct a 2D numerical experiment to determine the effect that varying τ has on the Nusselt number computations in the steady onset regime



2D Numerical Experiment (1)

- Parameter values in the steady onset regime were chosen

$$R_T = 200$$
$$R_S = 160$$
$$\epsilon/\sigma = 1/3$$

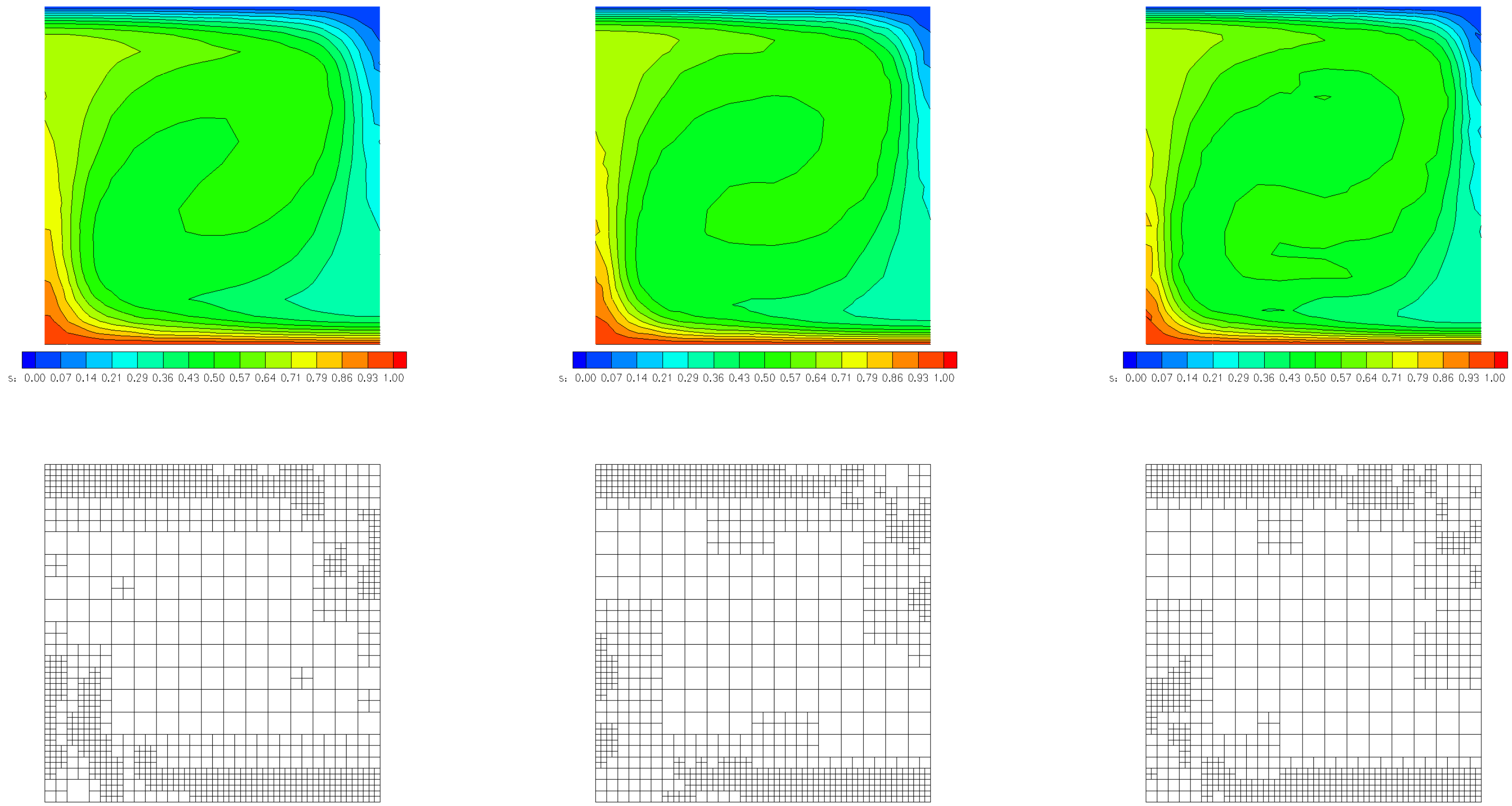


- The equations were solved for a range of different diffusivity ratios

$$0.01 \leq \tau \leq 0.25$$

- Maximum and steady state Nusselt numbers were recorded.

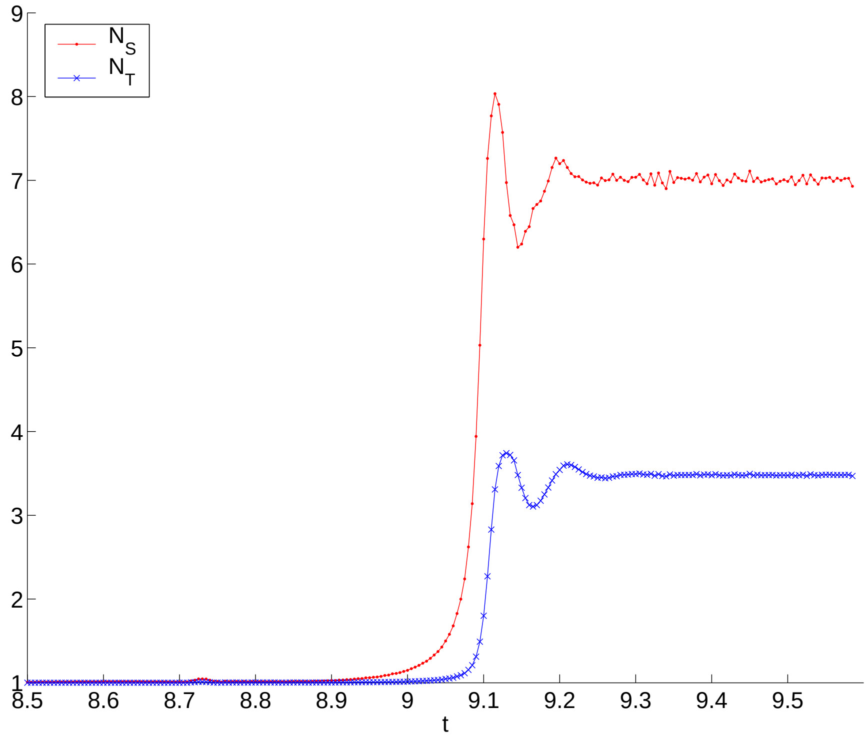
2D Numerical Experiment (2)



$\tau = 0.2$ $\tau = 0.15$ $\tau = 0.125$

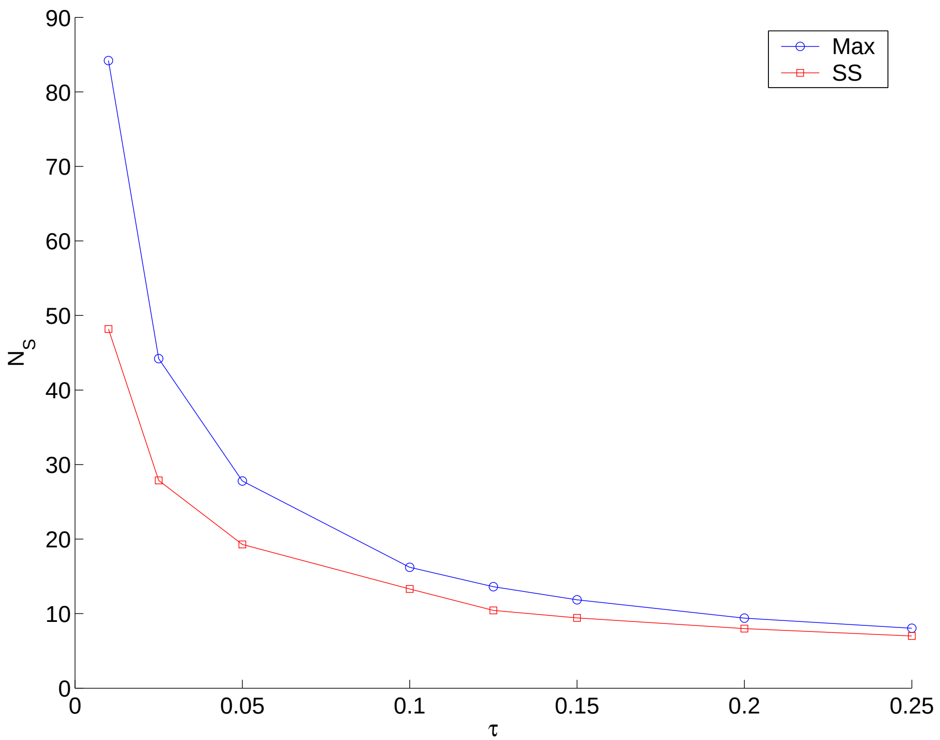
2D Numerical Experiment (3)

- Time history of solute and temperature flux, $\tau = 0.25$



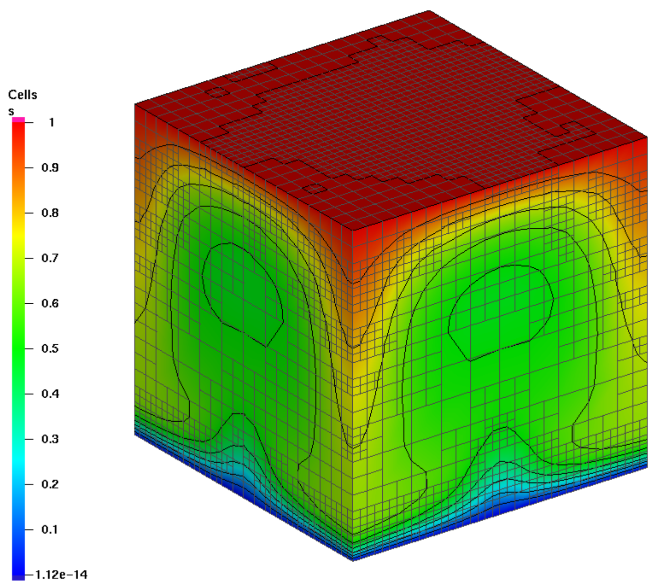
2D Numerical Experiment (4)

- Maximum and steady Nusselt numbers measured for different τ's

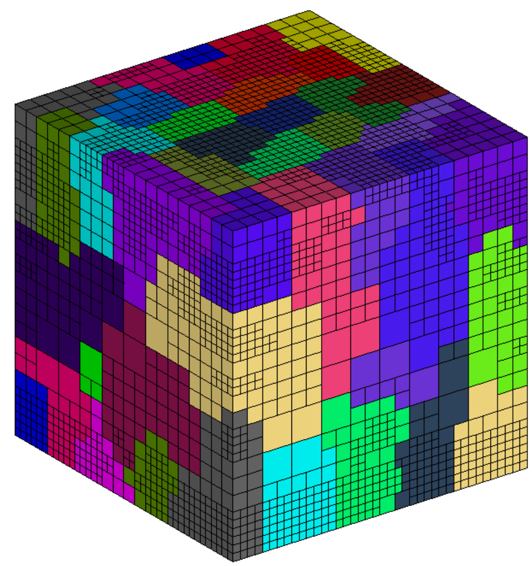


3D Adaptive Results

- Results in 3D for doubly-unstable layers
- $R_T = 50, R_S = 100$, $\tau = 0.05$
- $\max_t N_S \approx 7.33$



Solute Contours



Partitioning on 64 CPUs

Conclusions and Future Work

- An adaptive finite element code has been developed which solves the porous media equations in a relatively challenging parameter regime.
- The code works in two and three dimensions and is a promising tool for additional investigations of the parametric space. (Values of $\tau \approx 10^{-3}$ are physically realistic and computationally challenging.)
- Adaptivity is an attractive technique for handling layers in small τ (high Le) cases, since the locally varying mesh scales help prevent cell Peclet violations.
- Unfortunately, adaptivity is not ideal in small τ cases when it is required only to prevent oscillations in the initial *linear* profiles. A combined adaptive, stabilized (SUPG) method is currently under development to handle these cases.