Spring 2014:
Computational and Variational Methods for Inverse Problems
CSE 397/GEO 391/ME 397/ORI 397
Assignment 3 (due 24 March 2014)

The problems below require a mix of paper-and-pencil work and COMSOL Multiphysics implementation. Example implementations in COMSOL with MATLAB (as used for Problems 2 and 3 below) can be found on the class website, http://users.ices.utexas.edu/~omar/inverse_probs/index.html. Please hand in printouts of your COMSOL implementations together with the results.

1. The problem of removing noise from an image without blurring sharp edges can be formulated as an infinite-dimensional minimization problem. Given a possibly noisy image $u_0(x, y)$ defined within a square domain Ω, we would like to find the image $u(x, y)$ that is closest in the L_2 sense, i.e. we want to minimize

$$\mathcal{F}_{LS} := \frac{1}{2} \int_\Omega (u - u_0)^2 \, d\boldsymbol{x},$$

while also removing noise, which is assumed to comprise very "rough" components of the image. This latter goal can be incorporated as an additional term in the objective, in the form of a penalty, i.e.,

$$\mathcal{R}_{TN} := \frac{1}{2} \int_\Omega k(\boldsymbol{x}) \nabla u \cdot \nabla u \, d\boldsymbol{x},$$

where $k(\boldsymbol{x})$ acts as a "diffusion" coefficient that controls how strongly we impose the penalty, i.e. how much smoothing occurs. Unfortunately, if there are sharp edges in the image, this so-called *Tikhonov (TN) regularization* will blur them. Instead, in these cases we prefer the so-called *total variation (TV) regularization*,

$$\mathcal{R}_{TV} := \int_\Omega k(\boldsymbol{x}) (\nabla u \cdot \nabla u)^{\frac{1}{2}} \, d\boldsymbol{x}$$

where (we will see that) taking the square root is the key to preserving edges. Since $\mathcal{R}_{TV}$ is not differentiable when $\nabla u = \mathbf{0}$, it is usually modified to include a positive parameter ε as follows:

$$\mathcal{R}_{TV}^{\varepsilon} := \int_\Omega k(\boldsymbol{x}) (\nabla u \cdot \nabla u + \varepsilon)^{\frac{1}{2}} \, d\boldsymbol{x}.$$

We wish to study the performance of the two denoising functionals $\mathcal{F}_{TN}$ and $\mathcal{F}_{TV}^{\varepsilon}$, where

$$\mathcal{F}_{TN} := \mathcal{F}_{LS} + \mathcal{R}_{TN}$$

and

$$\mathcal{F}_{TV}^{\varepsilon} := \mathcal{F}_{LS} + \mathcal{R}_{TV}^{\varepsilon}.$$

We will prescribe the homogeneous Neumann condition $\nabla u \cdot \boldsymbol{n} = 0$ on the four sides of the square, which amounts to assuming that the image intensity does not change normal to the boundary of the image.

(a) For both $\mathcal{F}_{TN}$ and $\mathcal{F}_{TV}^{\varepsilon}$, derive the first-order necessary condition for optimality using calculus of variations, in both weak form and strong form. Use $\hat{u}$ to represent the variation of u.

(b) Show that when $\boldsymbol{\nabla} u$ is zero, $\mathcal{R}_{TV}$ is not differentiable, but $\mathcal{R}_{TV}^{\varepsilon}$ is.

(c) For both $\mathcal{F}_{TN}$ and $\mathcal{F}_{TV}^{\varepsilon}$, derive the infinite-dimensional Newton step, in both weak and strong form. For consistency of notation, please use $\tilde{u}$ as the differential of u (i.e. the Newton step). The strong form of the second variation of $\mathcal{F}_{TV}^{\varepsilon}$ will give an anisotropic diffusion operator of the form $-\text{div}(\boldsymbol{A}(u)\nabla\tilde{u})$, where $\boldsymbol{A}(u)$ is an anisotropic tensor that plays the role of the diffusivity coefficient[1]. (In contrast, you can think of the second variation of $\mathcal{F}_{TN}$ giving an *isotropic* diffusion operator, i.e. with $\boldsymbol{A} = \alpha\boldsymbol{I}$ for some α.)

(d) Derive expressions for the two eigenvalues and corresponding eigenvectors of $\boldsymbol{A}$. Based on these expressions, give an explanation of why $\mathcal{F}_{TV}^{\varepsilon}$ is effective at preserving sharp edges in the image, while $\mathcal{F}_{TN}$ is not. Consider a single Newton step for this argument.

(e) Show that for large enough ε, $\mathcal{R}_{TV}^{\varepsilon}$ behaves like $\mathcal{R}_{TN}$, and for $\varepsilon = 0$, the Hessian of $\mathcal{R}_{TV}^{\varepsilon}$ is singular. This suggests that ε should be chosen small enough that edge preservation is not lost, but not too small that ill-conditioning occurs.

2. An anisotropic Poisson problem in a two-dimensional domain Ω is given by the strong form

$$-\nabla \cdot (\boldsymbol{A}\nabla u) = f \qquad \text{in } \Omega, \tag{1a}$$

$$u = u_0 \qquad \text{on } \partial\Omega, \tag{1b}$$

where the conductivity tensor $\boldsymbol{A}(\boldsymbol{x}) \in \mathbb{R}^{2\times 2}$ is assumed to be symmetric and positive definite for all $\boldsymbol{x}$, $f(\boldsymbol{x})$ is a given distributed source, and $u_0(\boldsymbol{x})$ is the boundary source.[2]

(a) Derive the variational/weak form corresponding to the above problem, and give the energy functional that is minimized by the solution u of (1).

(b) Solve problem (1) in COMSOL using quadratic finite elements. Choose Ω to be a disc with radius 1 around the origin and take the source terms to be

$$f = \exp(-100(x^2 + y^2)) \quad \text{and} \quad u_0 = 0.$$

Use conductivity tensors $A(x)$ given by

$$A_1 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 1 & -5 \\ -5 & 100 \end{pmatrix}$$

and compare the results obtained using A_1 and A_2 in (1).

[1] Hint: For vectors $a, b, c \in \mathbb{R}^n$, note the identity $(a \cdot b)c = (ca^T)b$, where $a \cdot b \in \mathbb{R}$ is the inner product and $ca^T \in \mathbb{R}^{n\times n}$ is a matrix of rank one.

[2] One interpretation of Eqn. (1) is that it describes the steady state conduction of heat in a solid body. In this case, u is the temperature, $\boldsymbol{A}$ is the thermal conductivity, f is the distributed heat source, and the temperature at the boundary $\partial\Omega$ is maintained at u_0.

3. Implement the image denoising method from Problem 1 above using Tikhonov (TN) and total variation (TV) regularizations. To this end, set $k(\boldsymbol{x}) = \alpha$ with small $\alpha > 0$ in $\mathcal{R}_{TV}$ and $\mathcal{R}_{TN}$, choose small $\varepsilon > 0$, and take a homogeneous Neumann boundary condition for u (i.e., $\nabla u \cdot \boldsymbol{n} = 0$). A function that evaluates the noisy image $u_0(\boldsymbol{x})$ is provided in the file `ustar_fun.m`, while the file `TNTV.m` contains some lines to start up the implementation.

   (a) Solve the denoising inverse problem using TN regularization. Since for TN regularization, the gradient is linear in u, you can use COMSOL's linear solver `femlin`. Choose an $\alpha > 0$ such that you obtain a reasonable reconstruction[3], i.e., a reconstruction that removes noise from the image but does not overly smooth the image.

   (b) Solve the denoising inverse problem using TV regularization. Since the gradient is nonlinear in u, use the nonlinear solver `femnlin`[4]. Find an appropriate value for α[5]. You will have to increase the default number of nonlinear iterations in `femnlin`[6]. How does the number of nonlinear iterations behave for decreasing ε (e.g., between 10 and 10^{-4})? Try to explain this behavior[7].

   (c) Compare the denoised images obtained with TN and TV regularizations, using the insight derived from your answers to Problem 1.

[3] Either experiment manually with a few values for α or use the L-curve criterion. To evaluate terms in the cost functional, use the function `postint`.

[4] This function implements a Newton method with damping to solve the weak form of a nonlinear equation, using symbolic variations of the residual to obtain the Jacobian for Newton's method. While COMSOL's nonlinear solver will work for this problem (and you should use it), it does not work for every nonlinear system of PDEs, which is why it is often necessary to derive and implement Newton's method manually (via explicit solution of a sequence of linear problems). One situation where explicit implementation of Newton's method is useful is when the nonlinear equation corresponds to the vanishing of the gradient of an optimization problem (as it is in the denoising problem). In this case, COMSOL does not know the underlying optimization cost functional, so manual implementation allows one to globalize Newton's method via a line search based on a knowledge of the cost functional.

[5] Try out a few different values for α. Using the L-curve or Morozov's criterion in not justified here, since compared to TN regularization, TV regularization is not quadratic, which makes the automatic choice of α harder.

[6] Typing "`help femnlin`" will show you solver options. You should increase the value of `Maxiter`, which defaults to 25.

[7] There are more efficient so-called primal-dual Newton algorithms for TV-regularized problems (see, for instance, *T.F. Chan, G.H. Golub, and P. Mulet, A nonlinear primal-dual method for total variation-based image restoration, SIAM Journal on Scientific Computing, 20(6):1964–1977, 1999*). The efficient solution of TV-regularized problems is still an active field of research.