GEO 384F: Finite Element Methods for Geophysics
Assignment 1—Fall 2016
Due Sept. 16, 2016

Consider a membrane occupying a square domain Ω of length l. The membrane is under constant tension t, rests on an elastic foundation of constant stiffness k, is acted upon by pressure $f(x, y)$, and is fixed on all four sides, i.e., the transverse displacement $u(x, y)$ is zero on the boundary Γ. We wish to determine $u(x, y)$ under the assumption of small displacements. The total potential energy of the membrane is given by

$$\Pi(u) := \frac{1}{2}\int_\Omega t\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\,dy + \frac{1}{2}\int_\Omega ku^2\,dx\,dy - \int_\Omega fu\,dxdy.$$

The first term of the potential energy represents the "strain" energy of the membrane, the second the strain energy of the foundation, and the third the loss of potential of applied forces.

1. Using variational calculus, derive the boundary value problem governing this problem. The following integration-by-parts formula will be useful: given two scalar functions f and g,

$$\int_\Omega f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right) dx\,dy = -\int_\Omega \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y}\right) dx\,dy + \int_\Gamma f\frac{\partial g}{\partial n} ds,$$

   where n is the outward unit normal to the boundary Γ, and s is the arc length.

2. Application of the Ritz method to this problem, using the approximate solution

$$u_N(x, y) = \sum_{i=1}^{N} \alpha_i \phi_i(x, y),$$

   produces a linear algebraic system of the form $\mathbf{K}\boldsymbol{\alpha} = \mathbf{F}$. Give expressions for typical elements K_{ij} and F_i of this system. Prove that **K** is symmetric positive semidefinite.

3. Prove that the Ritz method minimizes the error in the energy norm for this problem. Note that the energy norm is the square root of the strain energy, so if $e(x, y)$ is the error, then

$$\| e(x, y) \|_E^2 = \frac{1}{2}\int_\Omega \left\{ t\left[\left(\frac{\partial e}{\partial x}\right)^2 + \left(\frac{\partial e}{\partial y}\right)^2\right] + ke^2 \right\} dx\,dy.$$

4. Send an email to me at omar@ices.utexas.edu from your preferred address.