Numerical stability and error analysis for the incompressible Navier–Stokes equations

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SUMMARY

This paper describes a strategy to control errors in finite element approximations of the time-dependent incompressible Navier–Stokes equations. The approach involves estimating the errors due to the discretization in space, using information from the residuals in the momentum and continuity equations. Following a numerical stability analysis of channel flows past a cylinder, it is concluded that the errors due to the residual in the continuity equation should be carefully controlled since it appears to be the source of unphysical perturbations artificially created by the spatial discretization. The performance of the adaptive strategy is then tested for lid-driven oblique cavity flows. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: incompressible Navier–Stokes equations; numerical stability; mesh adaptivity; residuals; reliability; a posteriori error estimation

1. INTRODUCTION

Error control and adaptation of numerical processes has been and will be a major concern in computational science and engineering [1], not only to limit computational costs, but mainly to assess the reliability of predictions. This is particularly true for time-dependent flow simulations. One objective here is then to devise an algorithm which allows for preserving the dynamical mechanisms of the differential equation when performing numerical simulations. In other words, the solution process should be able to reproduce the main features during the time evolution of the solutions within reasonable accuracy. One concern for example is to make sure that the discrete solutions of the Navier–Stokes equations evolve to the same attractor as the ‘exact’ solutions. If numerical errors are viewed as perturbations artificially created in numerical flows, one then agrees that error control becomes fundamental to help reproduce...
stability/instability mechanisms in such flow evolutions. Literature on a posteriori error estimation and adaptive schemes for the Navier–Stokes equations is understandably sparse. We cite nevertheless the contribution by Zienkiewicz et al. [2], which proposed an error estimate based on the recovery method [3,4], and the paper by Verf"{u}rth [5], which is concerned by the theoretical derivation of error estimates. We propose here a residual method, based on our previous work [6], in which we relate the residual to the errors in some appropriate norms, and then efficiently compute the norm of the residual to be used as the error estimators and refinement indicators. The adaptive strategy adopted here is derived based on the conclusions drawn from numerical experiments on flow stability.

The presentation is laid out as follows. In Section 2, we introduce the functional setting for the Navier–Stokes equations. In Section 3, we analyse the stability of finite element incompressible laws, and, in particular, we investigate whether the stability properties are modified due to the discretization in space. The investigation essentially relies on numerical simulations of channel flows past a cylinder. We briefly outline in Section 4 the error estimates which serve as a basis for mesh adaptation and present a numerical example. This is followed by concluding remarks.

2. PRELIMINARIES

The flow of a viscous fluid in an open bounded domain $\Omega$ in $\mathbb{R}^d$, $d = 2$, is modelled by the Navier–Stokes equations

$$\begin{align*}
\partial_t u + u \cdot \nabla u - Re^{-1} \Delta u + \nabla p &= f \\
\nabla \cdot u &= 0 \\
&\text{in } \Omega \times (0, T)
\end{align*}$$

(1)

where $u = u(x, t)$ and $p = p(x, t)$, respectively, denote the velocity vector and the pressure at $x \in \Omega$ and $t \in (0, T)$. $Re$ is the Reynolds number, and $f = f(x, t)$ is a prescribed body force. The velocity is assumed, for the sake of simplicity here, to satisfy $u = 0$ on the boundary $\partial \Omega$, and to be identically equal to $u_0$ at $t = 0$.

Let $V = [H^1_0(\Omega)]^2$ and $Q = \{q \in L^2(\Omega): \int_{\Omega} q \, dx = 0\}$ be the spaces of trial velocities and pressures, respectively, and let $\|v\|_0$ and $|v|$ denote the usual $[L^2(\Omega)]^d$-norm and $[H^1_0(\Omega)]^d$-norm in $V$, and $\|q\|_0$ the $L^2(\Omega)$-norm in $Q$. A weak formulation of the Navier–Stokes problem consists, for $f$ and $u_0$ given, in finding $u \in L^2(0, T; V)$ and $p \in L^2(0, T; Q)$, such that $u = u_0$ at $t = 0$, and, for almost every time $t \in (0, T):

$$(\partial_t u, v) + c(u,u,v) + Re^{-1}a(u,v) + b(v,p) = F(v), \quad \forall v \in V$$

$$b(u,q) = 0, \quad \forall q \in Q$$

(2)

where the forms $a$, $b$, $c$ and $F$ are defined, for any $u,v,w \in V$ and any $q \in Q$, as

$$a(u,v) = \int_{\Omega} \nabla u : \nabla v \, dx, \quad c(u,v,w) = \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx$$

$$b(v,q) = -\int_{\Omega} q \nabla \cdot v \, dx, \quad F(v) = \int_{\Omega} f \cdot v \, dx$$
We assume here that there exists a unique solution \((u, p)\) of the Navier-Stokes problem. This has been actually proven for \(d = 2\) under mild conditions on \(f\) and \(u_0\), but has not yet been proved or disproved for \(d = 3\) (see Reference [7]).

The above equations are discretized in space using \(h-p\) finite element spaces \(V^h \subset V\) and \(Q^h \subset Q\) (see Reference [8]). It is well-known [9] that \(V^h\) and \(Q^h\) must satisfy the discrete version of the inf-sup condition. In other words, the discrete formulation is stable if there exists a constant \(\beta_h > 0\) such that

\[
\sup_{v \in V^h, q \in Q^h} \frac{|b(v, q)|}{|v|} \geq \beta_h \|q\|_0, \quad \forall q \in Q^h
\]

(3)

In the case of hierarchical shape functions [10], we use the following rule inspired by the work of Suri et al. [11, 12] on locking-free \(h-p\) elements: given an element \(K\) in the mesh and a spectral order \(p_K\) for the velocity variable, the pressure is chosen of degree at most \(p_K - 1\) for the edge functions and at most \(p_K - 2\) for the interior bubble functions. Note that the Taylor–Hood element [9] (biquadratic in \(v\), bilinear in \(p\)) is consistent with this rule.

We select here the Adams–Bashforth Crank–Nicolson time discretization scheme (ABCN) to derive a fully discrete problem of the Navier–Stokes equations. Let \(\Delta t\) denote the time step. Then, given some initial conditions \(u_0^h\) and \(u_1^h\), the discrete solution \((u_n^h, p_n^h) \in V^h \times Q^h\) at \(t^n = n\Delta t, n = 2, 3, \ldots\) is advanced in time by solving

\[
\frac{1}{\Delta t}(u_n^h, v) + \frac{1}{2} Re^{-1}a(u_n^h, v) + b(v, p_n^h) = \mathcal{F}_h(v), \quad \forall v \in V^h
\]

\[
b(u_n^h, q) = 0, \quad \forall q \in Q^h
\]

(4)

where \(\mathcal{F}_h\) is a functional which depends on the solutions at the previous discrete times, \(i.e.\)

\[
\mathcal{F}_h(v) = \frac{1}{2} F^n(v) + \frac{1}{2} F^{n-1}(v) + \frac{1}{\Delta t}(u_n^{n-1}, v) - \frac{1}{2} Re^{-1}a(u_n^{n-1}, v)
\]

\[
- \frac{3}{2} c(u_n^{n-1}, u_n^{n-1}, v) + \frac{1}{2} c(u_n^{n-2}, u_n^{n-2}, v)
\]

(5)

The ABCN scheme is implicit with respect to the linear terms and explicit for the non-linear convective terms. Therefore, it is conditionally stable and results in a system of linear equations to be solved. The resulting equations are moreover linear. Note that \(u_0^h\) is taken equal to a projection of \(u_0\) on the finite element space \(V^h\) and that \(u_1^h\) is obtained using a first-order scheme.

3. NUMERICAL STABILITY

The Navier–Stokes equations (1) or (2) define a dissipative autonomous dynamical system with control parameter \(Re\). Assuming that the flow evolves to an attractor at any Reynolds number (see \(i.e.\) References [13, 14]), the type of attractor is expected to change as \(Re\) increased. However, the fully discrete system of equations (4) defines a new dynamical system with the mesh size \(h\), the polynomial degree \(p\), and the time step \(\Delta t\) as additional control parameters. The fundamental issue which we want to address here is whether the attractors of the Navier–Stokes equations and of the discrete system are the same at a given \(Re\). In
other words, can we expect the bifurcations to occur at the same critical Reynolds numbers? In [15-17], numerical simulations of the Navier-Stokes equations were performed with the objective of showing the existence of strange attractors and of unveiling a possible route to chaos. Our motivation here is rather to show how and why attractors are sensitive to the mesh discretization in order to design a strategy for the control of the error and stability. We emphasize that our primary goal in the following numerical experiments is not to obtain accurate solutions, but rather to analyse their stability in critical situations.

The numerical study is performed on channel flows past a cylinder since these undergo the first bifurcation from steady-state to periodic vortex shedding flows at low Reynolds numbers. The Reynolds number is defined here as $Re = U_c d / v$, where $d$ is the diameter of the cylinder and $U_c$ the inflow velocity at the centreline of the channel. Boundary conditions are prescribed as shown in Figure 1, and $u_0$ is set to zero everywhere in $\Omega$. Four meshes using the Taylor-Hood element are constructed. In view of approaching the attractors, i.e. the long-time behaviour of flows, simulations are run during the time interval $[0, 500]$ and the time step is chosen small ($\Delta t = 0.02$). We note however that the present numerical investigation focuses on the effect of space discretization accuracy with respect to the dynamical system given by the ABCN time discretization. In order to get a complete picture, we would need to investigate other time integration schemes, including implicit schemes which preserves stability properties of the Navier-Stokes equations. We expect the choice of an integration scheme to affect the numerical results and this should be studied in the future.

In order to characterize the type of attractors, we extract from the output data time series signals based on the kinetic energy; in particular, at each time $t^n$, we compute the kinetic energy $K_e(t^n) = \frac{1}{2} \int_{\Omega} (u^n \cdot u^n)^2 \, dx$ in the ‘triangular’ subregion $\Omega_s$ of $\Omega$ represented by the shaded area in Figure 1. The time series signals are then post-processed into time delay reconstruction diagrams in order to be able to classify the attractors. Time delay reconstruction diagrams are obtained in our case by plotting $K_e(t^n + \tau)$ versus $K_e(t^n)$, where $\tau$ is a user-defined constant. Note that the time delay diagram of the signal defined by $\sin(t)$ is represented by a perfect circle if $\tau$ is chosen as the fourth of the period of $\sin(t)$. In the following experiments, the value of $\tau$ will vary depending on the time series signal $K_e(t^n)$ which is analysed. We summarize our results in Table I and observe that the attractor depends on the spatial discretization as expected (at fixed $\Delta t$ and $Re$). For instance, for $Re = 75$, the flow
Table 1. Types of attractor versus the Reynolds number and the number of mesh elements: (-) fixed point, (O) periodic orbit with on fundamental, (X) other, (-) no attractor.

<table>
<thead>
<tr>
<th>No. of Elements</th>
<th>Reynolds number</th>
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<tbody>
<tr>
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<tr>
<td>112</td>
<td></td>
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<td>160</td>
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</tr>
<tr>
<td>192</td>
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<td>262</td>
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evolves to a steady state on the coarsest mesh, and to a periodic state on the other meshes. On the other hand, for Re = 200, the flow is still periodic on the two finer meshes while it has changed to a more complex state on the other two. Finally, the numerical flow on the coarsest mesh does not reach any attractors at Re = 250 and 300 as the solution 'blows up' after a finite time.

In the next set of experiments, we study the influence of the choice $V^h$ and $Q^h$ on the flow stability. We take advantage of the $h$–$p$ data structure of our code to independently increase the spectral orders of the velocity and pressure (while preserving the inf–sup condition (3)), and run the experiments on the coarsest of the meshes used previously. Let $q = (q_v, q_p)$ denote the pair of polynomial degrees $q_v$ and $q_p$ for the velocity and the pressure, respectively (we use here the notation $q$ for the polynomial degree to avoid any confusion with the pressure denoted by $p$). We then consider $q = (3, 1)$ and $q = (3, 2)$.

We display in Figure 2 the time delay reconstruction diagrams determined from the signals based on twice the kinetic energy computed in $\Omega$. At Re = 75, in the case $q = (2, 1)$, the solution evolves towards a steady state. When the dimension of $V^h$ is increased, taking $q = (3, 1)$, we observe a periodic state, but with frequency $f = 0.095$. When enriching $Q^h$ using $q = (3, 2)$, the new periodic signal has frequency $f = 0.145$, which is a value that compares better with the one obtained on the finest mesh. At Re = 200, the signal looks chaotic for $q = (2, 1)$. When we employ the discretization $q = (3, 1)$, we obtain a periodic signal with $f = 0.084$. On the other hand, the periodic signal obtained with $q = (3, 2)$ has frequency $f = 0.173$. Finally, the solution at Re = 250 blows up at even earlier time when using $q = (3, 1)$ instead of $q = (2, 1)$. A stable solution is finally obtained when the dimension of $Q^h$ is eventually increased to $q = (3, 2)$ and is periodic with $f = 0.171$. We also observe that the amplitude of the signals varies from one discretization to the other.

In conclusion, these experiments reveal that the solutions have better stability properties when the dimension of $Q^h$ is increased. Indeed, the larger dim $Q^h$, the better the incompressibility constraint is enforced. In other words, the divergence of $u_h$, which is not necessarily zero in classical finite element approximations, can be viewed as a source of unphysical perturbations artificially generated by the spatial discretization. It suggests that this quantity should be carefully controlled in order to control the numerical stability and hence the overall accuracy of the solutions. The control of the error essentially relies on a posteriori error estimation and mesh adaptation during the flow evolution. We shall see in the next section that the quantity $\nabla \cdot u_h$ naturally appears in our error estimates.
4. ERROR ESTIMATION AND ADAPTIVITY

The numerical errors at time $t^n$, due to the spatial discretization, are defined as the pair $(\varepsilon^n, \varepsilon^n)$ in $V \times Q$, $\varepsilon^n = u^n - u_h^n$ and $\varepsilon^n = p^n - p_h^n$, and are governed, for all $n = 2, 3, \ldots, N_{\Delta t}$, by

$$\frac{1}{\Delta t}(\varepsilon^n, v) + \frac{1}{2} Re^{-1} a(\varepsilon^n, v) + b(v, \varepsilon^n) = \mathcal{R}_h^n(v) + \mathcal{R}_h^n(q), \quad \forall v \in V$$

where $\mathcal{R}_h^n$ and $\mathcal{R}_h^n$ are the residuals in the momentum and continuity equations

$$\mathcal{R}_h^n(v) \equiv \frac{1}{2} F^n(v) + \frac{1}{2} F^{n-1}(v) - \frac{1}{\Delta t}(u_h^n - u_h^{n-1}, v) - \frac{1}{2} Re^{-1} a(u_h^n + u_h^{n-1}, v)$$

$$- b(v, p_h^n) - \frac{3}{2} c(u_h^{n-1}, u_h^{n-1}, v) + \frac{1}{2} c(u_h^{n-2}, u_h^{n-2}, v)$$

The residual $\mathcal{R}_h^n$ combines all the terms involving $\varepsilon^{n-1}$ and $\varepsilon^{n-2}$, and thus, governs the accumulation of errors after long periods of time. It is conjectured that the accumulated errors do not reflect the mesh inadequacy since errors created at a given time are later convected away from their source. Consequently, for the purpose of mesh adaptation, we only focus on the errors which are local in time by setting $\mathcal{R}_h^n = 0$. 

Figure 2. Time delay reconstruction diagrams: cases where (top) $Re = 75$ and $p = (2, 1)$, (3,1), or (3,2); (bottom) $Re = 200$ and $p = (2, 1)$, (3,1), or (3,2). These diagrams are obtained by plotting $2K_e(t^n + \tau)$ versus $2K_e(t^n)$ (symbolically written $2K_e(t^n + T)$ and $2K_e(t^n)$ on the plots). Note that $\tau$ is a user-defined constant which may vary from one diagram to the other.
The key idea in our approach is to characterize the respective influence of $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$ on the errors $(e^m, e^c)$. Let $(e_m, e_m)$ and $(e_c, e_c)$ denote the errors governed by $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$, respectively. Thanks to the linearity of the ABCN scheme, these pairs of errors can be treated independently. By the superposition principle, the governing equations for the error component $(e_m, e_m) \in \mathbf{V} \times \mathbf{Q}$ are deduced from (6) as

$$\frac{1}{\Delta t}(e_m, v) + \frac{1}{2} \mathcal{R}_h^m \mathbf{a}(v, e_m) + b(v, e_m) = \mathcal{R}_h^m(v), \quad \forall v \in \mathbf{V}$$

$$b(e_m, q) = 0, \quad \forall q \in \mathbf{Q}$$

(7)

Defining the norms of the error and residual as

$$\|v\|_{\Delta t, \mathcal{R}_h} = \sqrt{\frac{1}{\Delta t} \mathbf{a}(v, v) + \frac{1}{2} \mathcal{R}_h^m \mathbf{a}(v, v), \quad \|\mathcal{R}_h^m\|_* = \sup_{v \in \mathbf{V} \setminus \{0\}} \frac{|\mathcal{R}_h^m(v)|}{\|v\|_{\Delta t, \mathcal{R}_h}}$$

it is shown in References [6, 18] that

$$\|e_m\|_{\Delta t, \mathcal{R}_h} \lesssim \|\mathcal{R}_h^m\|_*.$$  

(8)

In a similar manner, the errors $(e_c, e_c) \in \mathbf{V} \times \mathbf{Q}$ are governed by

$$\frac{1}{\Delta t}(e_c, v) + \frac{1}{2} \mathcal{R}_h^c \mathbf{a}(v, e_c) + b(v, e_c) = 0, \quad \forall v \in \mathbf{V}$$

$$b(e_c, q) = \mathcal{R}_h^c(q), \quad \forall q \in \mathbf{Q}$$

(9)

This time however, we cannot directly relate $e_c$ to $\mathcal{R}_h^c$. On the other hand, decomposing the vector $e_c$ into the divergence-free component $e_d$ and its orthogonal vector $e_p$ (w.r.t. the inner product $\mathbf{a}(\cdot, \cdot)$), we showed in References [18, 19] that

$$\beta |e_p| \leq \|\mathcal{R}_h^c\|_* \leq |e_p|$$

(10)

where $\beta$ is the constant from the inf-sup condition (3) with respect to $\mathbf{V}$ and $\mathbf{Q}$. The norm of $\mathcal{R}_h^c$ is simply defined and computed as

$$\|\mathcal{R}_h^c\|_* = \sup_{q \in \mathbf{Q} \setminus \{0\}} \frac{|\mathcal{R}_h^c(q)|}{\|q\|_0} = \|\nabla \cdot u_\mathcal{R}^c\|_0$$

(11)

The residuals were given as the source terms in the equations governing the errors. Here, we have studied the effect of $\mathcal{R}_h^m$ and $\mathcal{R}_h^c$ on the error over one time step. The connection between residuals and long-time errors are of course more complicated as it also involved the residual $\mathcal{R}_h^c$ which controls the error accumulation during long-time ranges. In other words, the connection is intimately related to the stability of the solutions. Nevertheless, as shown above, the residuals provide easily computable measures of the discretization error and we use in our adaptation strategy the information provided by the norms of the residuals in order to control the errors. The global quantities $\eta_m$ and $\eta_c$, where $\eta_m$ is an inexpensive approximation of $\|\mathcal{R}_h^m\|_*$ (we refer to Reference [18] for more details) and $\eta_c$ is simply given by $\|\nabla \cdot u_\mathcal{R}^c\|_0$, are decomposed into elementwise quantities $\eta_{m,K}$ and $\eta_{c,K}$ on each element $K$. Whenever the relative errors, associated with $|e_p|$ and $\|e_m\|_{\Delta t, \mathcal{R}_h}$ and estimated by $\eta_m$ and $\eta_c$, become too large with respect to preset tolerances $C^\text{tol}_{m,K}$ and $C^\text{tol}_{c,K}$, the elements for which the contributions...
Figure 3. Shown clockwise: initial mesh, velocities \( u_1 \) and \( u_2 \), vorticity, pressure, adapted mesh.

\( \eta_{m,K} \) or \( \eta_{c,K} \) are the largest are refined. In this procedure, we always check \( \eta_c \) first, since failing to enforce the divergence-free constraint is supposedly held responsible for stronger instabilities in numerical flows, as seen in the previous section.

The adaptation strategy is now applied to the simulation of a flow in an oblique cavity (obtained from the square cavity by rotating the vertical walls by an angle of 20°, while keeping the area of the cavity to one; see Figure 3). The velocities are zero everywhere on \( \partial \Omega \) except on the top part where \( u = (1, 0) \) and the fluid is initially at rest. This test case has actually been studied in Reference [20] for various Reynolds numbers and the authors have identified three possible steady-state solutions in the range \( 1449.7 \leq Re \leq 2002.8 \). The Reynolds number is set here to \( Re = 2000 \) to see whether the adaptive strategy allows the computed flow field to approach the most stable state. We select the time step \( \Delta t = 0.002 \) and time range \([0, 40]\). The tolerances are chosen as \( C_{ml} = 0.016 \) and \( C_{col} = 0.015 \). The contour plots of the velocities, pressure and vorticity are shown in Figure 3 along with the initial and adapted meshes. We also measure the square-root of the kinetic energy \( K_e \) and find \( K_e = 0.219 \), which is close to the value of the most stable of the three possible steady states (see Figure 5.2(a) in Reference [20]).

5. CONCLUDING REMARKS

In this paper, we have proposed a new adaptive strategy to automatically control the numerical error of finite element approximations of the Navier–Stokes equations, and \textit{a fortiori}, to control the flow stability with respect to the mesh discretization. In this adaptive scheme, \textit{a posteriori} error estimates are used to evaluate the accuracy of the solutions and to indicate where to refine the mesh whenever the relative errors exceed preset tolerances. These estimates are based on the residuals \( \mathcal{R}_h^m \) and \( \mathcal{R}_h^c \), and we suggest to control the former before the latter.
The advantages of such a strategy are twofold: the norm of the residual $R$, which is simply the $L^2$-norm of the divergence of the discrete velocity, is inexpensive to compute. Moreover, this also helps to better control the stability of the numerical flows as this residual creates unphysical perturbations. The validity of the adaptive scheme is successfully demonstrated for a flow in a lid-driven oblique cavity.

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