A posteriori error estimation and error control for finite element approximations of the time-dependent Navier–Stokes equations

S. Prudhomme, J.T. Oden*

Texas Institute of Computational and Applied Mathematics, The University of Texas at Austin, SHC 304, 105 W. 26th St., Austin TX 78712, USA

Abstract

We present an approach to estimate numerical errors in finite element approximations of the time-dependent Navier–Stokes equations along with a strategy to control these errors. The error estimators and the error control procedure are based on the residuals of the Navier–Stokes equations, which are shown to be comparable to error components in the velocity variable. The present methodology applies to the estimation of numerical errors due to the spatial discretization only. Its performance is demonstrated for two-dimensional channel flows past a cylinder in the periodic regime. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: A posteriori estimation; Navier–Stokes equations; Mesh adaptation

1. Introduction

Progress in computer performances have justified the use of finite element to investigate more and more complex flow problems (e.g. see [1,2]). However, time-dependent simulations produce large amounts of output information, and one faces the major issue of assessing the accuracy, and a fortiori the validity, of such numerical approximations. A posteriori error estimation proves to be an unavoidable adjunct to the computation of numerical solutions in order to design reliable adaptive methods. Two major approaches for error estimation have emerged over the past twenty years, namely the Error Residual Methods and the Recovery Methods. Error residual methods were
originally developed for linear elliptic problems [3], and were later extended for flow problems to the Stokes [4–7], and to the steady-state Navier–Stokes equations [8–11]. Recovery methods, or commonly called ZZ-error estimators, were proposed by Zienkiewicz and Zhu [12–14] and are based on the extraction of more accurate derivatives than the ones directly available from the finite element approximations. Error estimators of this type were derived in [15] in the case of the time-dependent Navier–Stokes equations.

The error estimation technique we propose here belongs to the family of error residual methods. In the present approach, two residuals, one deriving from the momentum equation and the other from the continuity equation are introduced. These residuals constitute the sources of errors due to the finite element discretization, and, as such, are shown to provide reliable estimates of the actual errors in some appropriate norms. We specify that the methodology is developed with respect to time-discretized Navier–Stokes equations and that we do not estimate the errors due to the time discretization. We suppose here that they are small compared to the ones produced by the finite element discretization. Moreover, we focus on the study of local errors in time, namely the errors generated after one timestep; in other words, we are not concerned by the errors convected throughout the computational domain.

The objective in error control is to contain the numerical errors within preset tolerances over the whole simulation. The errors are controlled by acting on the sources, that is, by reducing the intensity of the residuals. This is accomplished in finite element methods by locally refining the mesh where the residuals have large contributions. The advantage of the present approach lies in the possibility to independently control the respective effects of the two residuals by prescribing two different tolerances.

Following the introduction, we present in Section 2 the Navier–Stokes equations and set the preliminaries of the analysis for a posteriori error estimation. We describe in Section 3 the time discretization scheme used in this paper. Theoretical results and error estimators are presented in Section 4. Then, we propose in Section 5 the error control strategy. Finally, numerical results are described in Section 6, followed by a summary of our major conclusions.

2. Preliminaries

We consider a viscous incompressible Newtonian fluid moving in an open bounded domain \( \Omega \subset \mathbb{R}^d, d = 2 \) or 3, with boundary \( \partial \Omega \). The flow is characterized by the time-dependent Navier–Stokes equations, given here in the nondimensionalized form,

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u - \frac{1}{\text{Re}} \Delta u + \nabla p &= f \quad \text{in } \Omega \times (0, T), \\
\nabla \cdot u &= 0 \quad \text{in } \Omega \times (0, T),
\end{align*}
\]

and is subject to the following boundary and initial conditions:

\[
\begin{align*}
    u(x, t) &= 0 \quad \forall x \in \partial \Omega \quad \forall t \in (0, T), \\
    u(x, 0) &= u^0(x) \quad \forall x \in \Omega
\end{align*}
\]

Here \( u \) and \( p \) denote the velocity vector and the pressure, respectively, \( \text{Re} \) is the Reynolds number, and \( f = f(x, t) \) a prescribed body force. We intentionally restrict ourselves to the case of
homogeneous Dirichlet boundary condition on $\partial \Omega$ to simplify somewhat the theoretical analysis while retaining all the interesting features.

We introduce the function spaces $V$ and $Q$, related to the velocities and pressures, respectively, and the associated norms:

$$V = [H_0^1(\Omega)]^d,$$

$$\|v\|^2 = \int_{\Omega} v \cdot v \, dx, \quad |v|^2 = \int_{\Omega} \nabla v : \nabla v \, dx,$$

$$Q = \{ q \in L^2(\Omega); \int_{\Omega} q \, dx = 0 \},$$

$$\|q\|^2_0 = \int_{\Omega} q^2 \, dx.$$

In the following, we assume there exists a unique solution $(u, p) \in V \times Q$, for all $t \in [0, T)$ (which is a legitimate assumption under certain conditions, see Temam [16,17] or Heywood [18,19]), satisfying the weak form of the Navier-Stokes equations:

$$\begin{align*}
(\partial_t u, v) + c(u, u, v) + Re^{-1}a(u, v) + b(v, p) &= \langle f, v \rangle \quad \forall v \in V, \\
b(u, q) &= 0 \quad \forall q \in Q, \\
u &= u^0 \quad \text{at } t = 0,
\end{align*}$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ and the trilinear form $c(\cdot, \cdot, \cdot)$ are defined for $u, v, w \in V$ and for $q \in Q$ as

$$\begin{align*}
a(u, v) &= \int_{\Omega} \nabla u : \nabla v \, dx, \\
b(v, q) &= -\int_{\Omega} q \nabla \cdot v \, dx, \\
c(u, v, w) &= \int_{\Omega} (u \cdot \nabla)v \cdot w \, dx.
\end{align*}$$

We obtain a finite element approximation of the solution $(u, p)$ of problem (2) by applying the classical Galerkin approach. Let $V^h$ and $Q^h$ define $h$-$p$ finite element spaces (see [20]), $V^h \subset V$ and $Q^h \subset Q$, constructed in such a way that the bilinear form $b(\cdot, \cdot)$ satisfies the standard LBB or inf-sup condition with respect to $V^h$ and $Q^h$ (see [21,22]). Let $(u_h, p_h) \in V^h \times Q^h$ denote the resulting finite element approximation. The numerical errors in $(u_h, p_h)$ are then defined as the pair

$$(e, E) = (u - u_h, p - p_h) \in V \times Q,$$
for all $t \in [0, T)$. Replacing $u$ and $p$, respectively, by $(u_h + e)$ and $(p_h + E)$ in (2), they are shown to be governed by the following time evolution equation and constraints:

$$
(\partial_t e, v) + c(e, u_h, v) + c(e, e, v) + Re^{-1}a(e, v) + b(v, E) = R_h^m(v) \quad \forall v \in V,
$$

$$
b(e, q) = R_h^c(q) \quad \forall q \in Q,
$$

$$
e = u^0 - u_h^0 \quad \text{at } t = 0.
$$

The initial condition $u_h^0$ above is defined as the $L^2$ projection of $u^0$ on the finite element space $V^h$. The functionals $R_h^m$ and $R_h^c$ are called the residuals in the momentum and continuity equations, respectively,

$$
R_h^m(v) \equiv \langle f, v \rangle - (\partial_t u_h, v) - c(u_h, u_h, v) - Re^{-1}a(u_h, v) - b(v, p_h),
$$

$$
R_h^c(q) \equiv -b(u_h, q).
$$

The residuals indicate how the approximations $(u_h, p_h)$ fail to satisfy the momentum and continuity equations. In other words, assuming $e = 0$ at $t = 0$, the errors $e$ and $E$ are identically zero at all subsequent times $t \in (0, T)$ as long as the residuals remain zero. Obviously, the residuals represent the sources of error. It implies that the approximations should be controlled by reducing the effects of the residuals, which means that the refinement indicators should be derived with respect to $R_h^m$ and $R_h^c$. On the other hand, the decision about when refinement is necessary during the simulation should be based on the size of the numerical error, hence the need of error estimators. Because the global problem (7) is infinitely more expensive to solve than the problem for $(u_h, p_h)$, the objective in error residual method is to judiciously postprocess the residuals in order to derive meaningful estimates of the errors in an inexpensive manner. However, since the problem (2) is never approximated as is, we first repeat the analysis above with respect to the time-discretized Navier–Stokes equations.

### 3. Time-discretization scheme for the Navier–Stokes equations

The Navier–Stokes equations in their weak form (2) are usually discretized in time using a finite difference-type scheme. Here, we choose the Adams–Bashforth Cranck–Nicolson (ABCN) scheme, which is implicit for the viscous term and explicit for the convection term. Let $\Delta t$ denote the timestep and let $u^n$ denote the value of a function $u$ at time $t^n = n\Delta t$, that is $u^n = u(t^n) = u(n\Delta t)$. Then, the solution of the discretized Navier–Stokes equations is advanced in time to the state $(u^n, p^n) \in V \times Q$ from the previous states $(u^{n-1}, p^{n-1})$ and $(u^{n-2}, p^{n-2})$ by solving the following system of equations:

$$
\frac{1}{\Delta t} (u^n, v) + \frac{1}{2} Re^{-1}a(u^n, v) + b(v, p^n) = F^{n-1,n-2}(v) \quad \forall v \in V,
$$

$$
b(u^n, q) = 0 \quad \forall q \in Q,
$$

$$
(\partial_t u^n, v) + c(u^n, u^n, v) + c(e, e, v) + Re^{-1}a(e, v) + b(v, E) = R_h^m(v) \quad \forall v \in V,
$$

$$
b(e, q) = R_h^c(q) \quad \forall q \in Q,
$$

$$
e = u^0 - u_h^0 \quad \text{at } t = 0.
$$

The functionals $R_h^m$ and $R_h^c$ are called the residuals in the momentum and continuity equations, respectively,

$$
R_h^m(v) \equiv \langle f, v \rangle - (\partial_t u_h, v) - c(u_h, u_h, v) - Re^{-1}a(u_h, v) - b(v, p_h),
$$

$$
R_h^c(q) \equiv -b(u_h, q).
$$

The residuals indicate how the approximations $(u_h, p_h)$ fail to satisfy the momentum and continuity equations. In other words, assuming $e = 0$ at $t = 0$, the errors $e$ and $E$ are identically zero at all subsequent times $t \in (0, T)$ as long as the residuals remain zero. Obviously, the residuals represent the sources of error. It implies that the approximations should be controlled by reducing the effects of the residuals, which means that the refinement indicators should be derived with respect to $R_h^m$ and $R_h^c$. On the other hand, the decision about when refinement is necessary during the simulation should be based on the size of the numerical error, hence the need of error estimators. Because the global problem (7) is infinitely more expensive to solve than the problem for $(u_h, p_h)$, the objective in error residual method is to judiciously postprocess the residuals in order to derive meaningful estimates of the errors in an inexpensive manner. However, since the problem (2) is never approximated as is, we first repeat the analysis above with respect to the time-discretized Navier–Stokes equations.
where $F^{n-1,n-2}$ defines a linear functional from $V$ to $\mathbb{R}$ which depends on $u^{n-1}, u^{n-2}$ and $f^{n-1}$:

$$F^{n-1,n-2}(v) = \langle f^{n-1}, v \rangle + \frac{1}{\Delta t} (u^{n-1}, v) - \frac{1}{2} \text{Re}^{-1} a(u^{n-1}, v)$$

$$- \frac{3}{2} c(u^{n-1}, u^{n-1}, v) + \frac{1}{2} c(u^{n-2}, u^{n-2}, v).$$

(9)

We assume, in what follows, that the solution $(u^n, p^n)$ is a very accurate approximation of $(u, p)$ if the timestep $\Delta t$ is chosen sufficiently small, so that the numerical errors are essentially due to the spatial discretization. Once again, we apply the Galerkin method to obtain a finite element approximation $(u^n_h, p^n_h) \in V^h \times Q^h$ for all $n = 0, 1, \ldots, N_{\Delta t}$, where $N_{\Delta t} = t/\Delta t$ and we obtain $(u^n_h, p^n_h)$ at the discrete time $t^n$ by solving the finite system of equations:

$$\frac{1}{\Delta t} (u^n_h, v) + \frac{1}{2} \text{Re}^{-1} a(u^n_h, v) + b(v, p^n_h) = F^{n-1,n-2}_h(v) \quad \forall v \in V^h,$$

$$b(u^n_h, q) = 0 \quad \forall q \in Q^h,$$

(10)

where $F^{n-1,n-2}_h$ is now given by

$$F^{n-1,n-2}_h(v) = \langle f^{n-1}, v \rangle + \frac{1}{\Delta t} (u^n_h, v) - \frac{1}{2} \text{Re}^{-1} a(u^n_h, v)$$

$$- \frac{3}{2} c(u^n_h, u^n_h, v) + \frac{1}{2} c(u^{n-2}_h, u^{n-2}_h, v).$$

(11)

We also define the numerical errors due to the finite element space discretization as the pair:

$$(e^n, e^n) = (u^n - u^n_h, p^n - p^n_h) \in V \times Q.$$

(12)

for all $n = 1.2, \ldots, N_{\Delta t}$. Then the errors at time $t^n$ are shown to satisfy the following system of equations:

$$\frac{1}{\Delta t} (e^n, v) + \frac{1}{2} \text{Re}^{-1} a(e^n, v) + b(v, E^n) = \mathcal{R}^n_h(v) + \mathcal{R}^e_h(v) \quad \forall v \in V,$$

$$b(e^n, q) = \mathcal{R}^e_h(q) \quad \forall q \in Q,$$

(13)

where $\mathcal{R}^n_h, \mathcal{R}^e_h$ and $\mathcal{R}^r_h$ define the new residuals with respect to the discretized equations:

$$\mathcal{R}^n_h(v) \equiv \langle f^{n-1}, v \rangle - \frac{1}{\Delta t} (u^n_h - u^n_h, v) - \frac{1}{2} \text{Re}^{-1} a(u^n_h + u^{n-1}_h, v) - b(v, p^n_h)$$

$$- \frac{3}{2} c(u^n_h, u^n_h, v) + \frac{1}{2} c(u^{n-2}_h, u^{n-2}_h, v).$$

$$\mathcal{R}^e_h(q) \equiv - b(u^n_h, q),$$
We identify \( \mathcal{R}_h^n(v) \) with the residual in the momentum equation and \( \mathcal{R}_h^c \) with the residual in the continuity equation. Meanwhile, the new residual \( \mathcal{R}_h^c \) corresponds to the source of errors induced from the errors committed at the previous timesteps, as it combines all the terms containing \( e^{n-1} \) and \( e^{n-2} \). Thus, \( \mathcal{R}_h^c \) governs the errors accumulated after long periods of time. We stipulate here that these errors do not reflect the mesh inadequacy; indeed, errors created at time \( t < t^n \) are later convected away from their source. We then set \( \mathcal{R}_h^c = 0 \), i.e. \( e^{n-1} = e^{n-2} = 0 \), for the remainder of the analysis. In other words, we investigate here the short-term errors only.

### 4. A posteriori error estimation

In this section, we describe the approach to relate the residuals \( \mathcal{R}_h^n \) and \( \mathcal{R}_h^c \) to the errors. In particular, we derive relationships between specific norms of the residuals and norms of the error in the velocity variable. We start the analysis by distinguishing the respective influence of the residuals \( \mathcal{R}_h^n \) and \( \mathcal{R}_h^c \) onto the errors \( (e^n, E^n) \) obtained after one timestep assuming that the errors at the previous times \( t^{n-1} \) and \( t^{n-2} \) are zero. Let \( (e_m, E_m) \in V \times Q \) be the errors generated by \( \mathcal{R}_h^n \) and \( (e_c, E_c) \in V \times Q \) be those generated by \( \mathcal{R}_h^c \) (the superscript \( n \) is dropped for the sake of clarity). Thanks to the linearity of the time-discretization scheme, we have:

\[
(e^n, E^n) = (e_m + e_c, E_m + E_c). \tag{14}
\]

We proceed by analyzing the two error components separately.

#### 4.1. Numerical error generated by \( \mathcal{R}_h^n \)

From Eq. (13), we deduce that the errors \( (e_m, E_m) \in V \times Q \) are governed by the system of equations

\[
\frac{1}{\Delta t} (e_m, v) + \frac{1}{2} \Re^{-1} a(e_m, v) + b(v, E_m) = \mathcal{R}_h^n(v) \quad \forall v \in V,
\]

\[
b(e_m, q) = 0 \quad \forall q \in Q. \tag{15}
\]

We immediately observe that, since \( E_m \in Q \), the continuity equation implies

\[
b(e_m, E_m) = 0. \tag{16}
\]

Thus, substituting \( e_m \) for \( v \) in the momentum equation yields

\[
\frac{1}{\Delta t} (e_m, e_m) + \frac{1}{2} \Re^{-1} a(e_m, e_m) = \mathcal{R}_h^n(e_m), \tag{17}
\]
that is, by definition of the norms,
\[
\frac{1}{\Delta t} \|e_m\|^2 + \frac{1}{2} \text{Re}^{-1} \alpha |e_m|^2 = R_h^n(e_m).
\]  
(18)

Introducing the new norm \(\|\cdot\|\) for the error \(e_m\)
\[
\|e_m\| = \sqrt{\frac{1}{\Delta t} \|e_m\|^2 + \frac{1}{2} \text{Re}^{-1} |e_m|^2},
\]  
(19)

and the norm \(\|\cdot\|_*\) associated with the dual space \(V^*\) of \(V\)
\[
\|R_h^n\|_* = \sup_{v \in V \setminus \{0\}} \frac{|R_h^n(v)|}{\|v\|},
\]  
(20)

one readily obtains, by the Cauchy–Schwartz inequality, the following upper bound on the error \(e_m\):
\[
\|e_m\| \leq \|R_h^n\|_*.
\]  
(21)

Unfortunately, the norm of the residual \(\|R_h^n\|_*\) cannot be computed exactly. However, it is possible to calculate, in an inexpensive manner, functions \(\phi_h \in \bar{V}^h = V^h \oplus W^h \subset V\) and \(\psi_h \in W^h\), which deliver good approximations of \(\|R_h^n\|_*\). We refer the reader to [7] for a detailed analysis. Briefly, the finite element space \(W^h\), often referred to as the space of bubble functions, is defined as the space which consists of piecewise polynomials of degree between \(p + 1\) and \(p + q\), \(q \geq 1\), where \(p\) denotes the maximal degree used in \(V^h\). Then, we can show that the norms of \(\phi_h\) and \(\psi_h\) are equivalent to the norm of the residual in the sense that there exist positive constants \(0 \leq \sigma < 1\) and \(0 \leq \gamma < 1\) such that
\[
\sqrt{(1 - \sigma^2)} \|\phi_h\| \leq \|\psi_h\| \leq \|\phi_h\|,
\]  
(22)

and
\[
\sqrt{(1 - \sigma^2)(1 - \gamma^2)} \|\phi_h\| \leq \|\psi_h\| \leq \|\phi_h\|.
\]  
(23)

The constant \(\sigma\) depends on the “richness” of the space \(W^h\), that is, the larger the number \(q\), the smaller \(\sigma\) is. The constant \(\gamma\) represents a measure of the angle between the spaces \(V^h\) and \(W^h\) with respect to the inner product \((V^h, V^h)\).

The function \(\phi \in \bar{V}^h\) is obtained by solving the following system of equations:
\[
\frac{1}{\Delta t} (\phi_h, v) + \frac{1}{2} \text{Re}^{-1} \alpha(\phi_h, v) = R_h^n(v) \quad \forall v \in \bar{V}^h,
\]  
(24)

and \(\psi_h \in W^h\) is obtained by solving
\[
\frac{1}{\Delta t} (\psi_h, v) + \frac{1}{2} \text{Re}^{-1} \alpha(\psi_h, v) = R_h^n(v) \quad \forall v \in W^h.
\]  
(25)

These global problems are symmetric positive definite, so we can efficiently solve them by performing only a few iterations of a preconditioned Conjugate-Gradient method.

Consequently, we propose the error estimators \(\tilde{\eta}_m\) and \(\eta_m\) of the quantity \(\|e_m\|\), which are computed as
\[
\tilde{\eta}_m = \|\phi_h\| = \sqrt{\frac{1}{\Delta t} \|\phi_h\|^2 + \frac{1}{2} \text{Re}^{-1} |\phi|^2},
\]  
(26)
and
\[ \eta_m = \sqrt{\frac{1}{\Delta t} \|\psi_h\|^2 + \frac{1}{2} \text{Re}^{-1} |\psi_h|^2}. \] (27)

The estimate \( \eta_m \) is expected to produce a smaller value than \( \tilde{\eta}_m \), but is in turn cheaper to compute. Therefore, we would prefer to use \( \eta_m \) rather than \( \tilde{\eta}_m \); nevertheless, we will first study how they compare to \( \|e_m\| \) in the numerical experiments.

4.2. Numerical error generated by \( \mathcal{R}_h \)

In the same manner as above, the numerical errors \( (e_c, E_c) \in V \times Q \) satisfy
\[ \frac{1}{\Delta t} (e_c, v) + \frac{1}{2} \text{Re}^{-1} a(e_c, v) + b(v, E_c) = 0 \quad \forall v \in V, \] (28)
\[ b(e_c, q) = \mathcal{R}_h(q) \quad \forall q \in Q. \]

The key point in the present analysis consists in decomposing the vector \( e_c \) into two components \( e_d \in J \) and \( e_{\perp} \in J^\perp \), the decomposition being unique, such that
\[ e = e_d + e_{\perp}, \] (29)
where
\[ J = \{ v \in V; b(v, q) = 0, \quad \forall q \in Q \}, \]
\[ J^\perp = \{ v \in V; a(v, w) = 0, \quad \forall w \in J \}. \]

The space \( J \) is simply the subspace of \( V \) of divergence-free functions, while \( J^\perp \) is the orthogonal complement of \( J \) with respect to the inner product \( (V^*, V^*) \) (see [22]). It readily follows that:
\[ |e_c|^2 = |e_d|^2 + |e_{\perp}|^2. \] (30)

Introducing the following norm associated to the dual space \( Q' \) of \( Q \) for the residual \( \mathcal{R}_h \):
\[ \|\mathcal{R}_h\|_* = \sup_{q \in Q \setminus \{0\}} \frac{\|\mathcal{R}_h(q)\|}{\|q\|_0}, \] (31)
we are able to show the following result:

**Theorem 1.** Let \( e_{\perp} \in J^\perp \) be defined as above. Then,
\[ \beta |e_{\perp}|_1 \leq \|\mathcal{R}_h\|_* \leq \sqrt{d} |e_{\perp}|_1. \] (32)

where \( \beta \) denotes the LBB constant for the bilinear form \( b(\cdot, \cdot) \) and \( d \) is the geometrical dimension of \( \Omega \).

**Proof.** The proof follows the same steps as for the proof of Lemma 2 in [7]. Starting from the second equation in (28), we have for all \( q \in Q \):
\[ \mathcal{R}_h(q) = b(e_c, q) = b(e_d + e_{\perp}, q) = b(e_{\perp}, q) = b(e_{\perp}, q) \leq \sqrt{d} |e_{\perp}|_1 \|q\|_0. \]
The upper bound follows as
\[
\| \mathcal{R}_h \|_* = \sup_{q \in Q; 0} \frac{\| \mathcal{R}_h (q) \|}{\| q \|_0} \leq \sup_{q \in Q; 0} \frac{\sqrt{d} \| e_\perp \|_1}{\| q \|_0} \leq \sqrt{d} \| e_\perp \|_1.
\]

Then, observing that \( V \cdot e_\perp \in Q \), one gets
\[
\| V \cdot e_\perp \|_0^2 = \mathcal{R}_h (V \cdot e_\perp) \leq \| \mathcal{R}_h \|_* \| V \cdot e_\perp \|_0,
\]
which yields \( \| V \cdot e_\perp \|_0 \leq \| \mathcal{R}_h \|_* \). Moreover, from Girault and Raviart [22, pp. 24,81], we show that
\[
\beta | e_\perp |_0 \leq \| V \cdot e_\perp \|_0.
\]
The lower bound is proved. \( \Box \)

Theorem 1 indicates that the norm of the residual \( \mathcal{R}_h \) provides an equivalent measure to the error component \( e_\perp \) expressed in the norm \( | \cdot |_1 \). Next, we show that the computation of \( \| \mathcal{R}_h \|_* \) is exact and cheap. Indeed,

\section*{Theorem 2}

\textit{Let \( u_n^\alpha \) be the finite element solution at time \( t^n \). Then,}
\[
\| \mathcal{R}_h \| = \| V \cdot u_n^\alpha \|_0. \tag{33}
\]

\textbf{Proof.} From the definition of the residual \( \mathcal{R}_h \), we have
\[
\mathcal{R}_h (q) = - b(u_n^\alpha, q) = \int_\Omega q V \cdot u_n^\alpha \, dx.
\]
Therefore,
\[
\| \mathcal{R}_h \|_* = \sup_{q \in Q; 0} \frac{\mathcal{R}_h (q)}{\| q \|_0} \leq \sup_{q \in Q; 0} \frac{\| V \cdot u_n^\alpha \|_0}{\| q \|_0} \leq \| V \cdot u_n^\alpha \|_0.
\]
Moreover, since \( V \cdot u_n^\alpha \in Q \), we also have
\[
\| V \cdot u_n^\alpha \|_0^2 = \int_\Omega (V \cdot u_n^\alpha)^2 \, dx = \mathcal{R}_h (V \cdot u_n^\alpha) \leq \| \mathcal{R}_h \|_* \| V \cdot u_n^\alpha \|_0,
\]
so that
\[
\| V \cdot u_n^\alpha \|_0 \leq \| \mathcal{R}_h \|_e.
\]
Theorem 2 has just been proven. \( \Box \)

In view of the results above, we define the new error estimator \( \eta_\epsilon \) such as
\[
\eta_\epsilon = \| V \cdot u_n^\alpha \|_0. \tag{34}
\]
The numerical experiments will show that it is an accurate estimate of the error \( | e_\perp |_1 \).
5. Adaptive control of the numerical error

The objective in adaptive control is to contain numerical errors within some preset tolerances while minimizing the cost of the computations. In finite element methods, we know that accuracy is directly related to the size $h$ of the elements as predicted by the numerous a priori error estimates (we refer the reader, for instance, to [9, Theorem 2] in the case of the steady-state Navier–Stokes equations). The smaller $h$ is, the more accurate the finite element solutions are. However, efficiency can be improved by generating small elements only in the subregions of the domain $\Omega$ where these are necessary.

Here, we assert that the finite element mesh should be refined where the residuals take on large values. In other words, the quantities $\eta_c$ and $\eta_m$ (or $\tilde{\eta}_m$), which measure the respective magnitude of the residuals $R_h^c$ and $R_h^m$, should provide sufficient information to design the mesh refinement strategy. Indeed, $\eta_c$ and $\eta_m$ can be decomposed into elementwise contributions $\eta_{c,K}$ and $\eta_{m,K}$ associated with each element of the mesh such as

$$\eta_c^2 = \sum_{K=1}^{N_e} \eta_{c,K}^2 = \sum_{K=1}^{N_e} \| \nabla \cdot u_h^m \|_0^2,$$

and

$$\eta_m^2 = \sum_{K=1}^{N_e} \eta_{m,K}^2 = \sum_{K=1}^{N_e} \| \psi_h \|_K^2,$$

where $N_e$ denotes the number of elements and $\| \cdot \|_0$ and $\| \cdot \|_K$ the restrictions of the norms $\| \cdot \|_0$ and $\| \cdot \|_K$ on any element $\Omega_K$ of the mesh. The local quantities $\eta_{c,K}$ and $\eta_{m,K}$ are interpreted as local measures of the residuals $R_h^c$ and $R_h^m$. We therefore propose, as an adaptation strategy, to locally refine the mesh by subdividing the elements for which $\eta_{c,K}$ or $\eta_{m,K}$ contribute the most to the global residuals. The refinement criteria for each element then read

$$\frac{\eta_{c,K}}{\max_K(\eta_{c,K})} \geq C_{c,\text{adp}},$$

and

$$\frac{\eta_{m,K}}{\max_K(\eta_{m,K})} \geq C_{m,\text{adp}},$$

where $C_{c,\text{adp}}$ and $C_{m,\text{adp}}$ are user-prescribed constants ranging from 0 to 1.

The next issue is to determine a criterion to decide when mesh adaptation is required during the flow simulation. For instance, given the error $\varepsilon$, the exact solution $u$ and a norm $\| \cdot \|$ for an arbitrary computational process, the criterion for refinement, based on the relative error $\varepsilon$, would read

$$\varepsilon = \frac{\| \varepsilon \|}{\| u \|} \geq C,$$

where $C$ is a given tolerance. In our case, because the exact error and solution are not available, we use the error estimators $\eta_c$ and $\eta_m$ and the finite element solutions instead, so that the criteria for refinement at a given time $t^n$ become

$$\varepsilon_c = \frac{\eta_c}{\| u_h^m \|_1} \geq C_{c,\text{tol}},$$
where $C_{tol}^c$ and $C_{tol}^m$ define the error tolerances.

Therefore, whenever at least one of the two criteria is satisfied, the mesh should be refined according to the strategy proposed above. The first condition allows to control the error $e_i$, and a fortiori $e_d$ and $e_c$. The second one controls the error $e_m$. The main advantage in our approach is the possibility to select two distinct values for the tolerances $C_{tol}^c$ and $C_{tol}^m$ in order to control the respective errors $e_m$ and $e_c$ accordingly. Using a heuristic argument, it is desirable to choose $C_{tol}^c$ smaller than $C_{tol}^m$, in order to carefully control the component $e_i$, as it generates "unphysical" perturbations in the simulated flow. On the other hand, the error $e_m$, which is a divergence-free function, may be compared to small perturbations in experimental flows, and therefore requires less precision in its control.

Finally, in order to reduce the cost of the computations, we suggest to calculate the error estimates $\eta_c$ and $\eta_m$ only every $N$ timesteps, where $N$ may vary depending on the type of flows. Moreover, when one is interested in the long-term behavior of the flows, the tolerances $C_{tol}^c$ and $C_{tol}^m$ can be relaxed during the transition times so that larger errors are allowed.

### 6. Numerical experiments

We consider the simulation of channel flows past a cylinder to test the performances of the a posteriori error estimation and the error control strategy. Such flows develop into a periodic vortex shedding at low values of the Reynolds number, which is determined here as

$$Re = \frac{U_c D}{v},$$

where $D$ is the diameter of the cylinder, $U_c$ the maximal velocity of the parabolic profile at the inflow and $v$ the viscosity of the fluid. In the present experiments, we select $Re = 100$, which is greater than the critical Reynolds number $Re_{crit}$ at which vortex shedding appears. Moreover, the timestep is chosen as $\Delta t = 0.01$, so that the errors due to the time discretization are kept small.

#### 6.1. Error estimation

The quality of an error estimator is usually measured in terms of the effectivity index, which is the ratio of the error estimator to the exact error. In our case, the effectivity indices are

$$\lambda_m = \frac{\tilde{\eta}_m}{||e_m||}, \quad \lambda_i = \frac{\eta_m}{||e_m||}$$

for the estimators $\tilde{\eta}_m$ and $\eta_m$ defined in (27) and (26), respectively, and

$$\lambda_c = \frac{\eta_c}{|e_i|_1}$$
for the error estimator $\eta_e$ defined in (34). However, the component $e_\perp$ of $e_e$ is too expensive to compute. Therefore, we decide to use $|e_e|_1$ instead of $|e_\perp|_1$ in the expression of the effectivity index $\lambda_e$. Moreover, the exact errors $e_e$ and $e_m$ are not available for such a flow, so we compute some approximations of $e_e$ and $e_m$. Namely, we solve the problems (15) and (28) on the same finite element mesh, but using higher degree polynomial basis functions than the ones considered for the solution $(u_h, p_h)$. We note that these approximations of $e_e$ and $e_m$ are much more expensive to obtain than the estimators $\eta_e$ or $\eta_m$.

As a numerical application, we compute the effectivity indices with respect to the finite element solution $(u_h, p_h)$ obtained on the mesh shown in Fig. 1. The approximation $u_h$ is chosen continuous piecewise biquadratic while the approximation $p_h$ is continuous piecewise bilinear, so that the discrete LBB condition is satisfied. We carry out the simulation over the time range $[0, 200]$ during which the flow achieves the permanent periodic regime.

Plots of the effectivity indices are shown in Figs. 2 and 3. In all cases, the effectivity indices remain close to one during the whole simulation, which reveals the high quality of these residual-based estimators. In particular, we observe that $\lambda_m$ remains always smaller than $\tilde{\lambda}_m$ as it was expected. Nevertheless, $\lambda_m$ is closer to one than $\tilde{\lambda}_m$ is. Therefore, although $\eta_m$ is a less accurate approximation of $\|\mathcal{R}_h^n\|_*$, it is in this case a better estimate of $||e_m||$ than $\tilde{\eta}_m$ is, simply because $\|\mathcal{R}_h^n\|_*$ was shown to be an upper bound on $||e_m||$. 

---

Fig. 1. Mesh with $N_e = 417$ elements.

Fig. 2. Evolution of the effectivity indices $\tilde{\lambda}_m$ (Est. 1) and $\lambda_m$ (Est. 2).

Fig. 3. Evolution of the effectivity index $\lambda_e$. 

---
Next, we show in Figs. 4 and 5 the approximate relative errors $\hat{\epsilon}_m = \hat{\eta}_m/||\mathbf{u}_m||$, $\epsilon_m = \eta_m/||\mathbf{u}_m||$ and $\epsilon_c = \eta_c/||\mathbf{u}_c||$. We observe that $\epsilon_c$ is less than 9% over the whole simulation while $\epsilon_m \leq \hat{\epsilon}_m \leq 7.5\%$.

### 6.2. Error control

In the second set of experiments, we test the performance of the adaptive strategy. The flow domain $\Omega$ is now discretized into a coarse mesh of 160 elements shown in Fig. 6.

Here we utilize the estimate $\eta_m$ to approximate the norm of the residual $||\mathbf{r}_h||_\infty$. The mesh is then refined whenever the approximate relative errors $\epsilon_m$ and $\epsilon_c$ exceed the tolerances $C_{tol}^m = 5\%$ and $C_{tol}^c = 3.5\%$, except for the early transients, during which larger errors are allowed. We also take $C_{adp}^m = C_{adp}^c = 0.4$. We show in Fig. 7 the evolution of the approximate relative errors $\epsilon_m$ and $\epsilon_c$. We observe that these are eventually controlled within the prescribed tolerances after the early transients.

Fig. 8 displays close-up views of the adapted mesh around the cylinder at various times of the simulation. We notice that the mesh is highly refined upstream and in the wake of the cylinder as expected. About 10 mesh adaptations were performed until $t \approx 80$, time at which the flow reached the permanent periodic regime. The computed period for this flow is $\tau = 5.27$ and compares very well with the period $\tau = 5.28$ for a solution obtained on a very fine mesh. Moreover, we precise that
the adaptive control of the error is essential in this example as the finite element solution, if it were computed on the initial mesh without refinement, would have reached a steady-state regime instead of the periodic one.

We also display in Figs. 9 and 10 the contour lines of the velocity in the $x$-direction and of the pressure to show the quality of the solutions obtained on the adapted mesh.
7. Summary and conclusions

We have presented in this paper a methodology to compute a posteriori error estimates of the error in finite element solutions of the time-dependent Navier–Stokes equations. In particular, we have studied the numerical errors produced by the spatial discretization and supposed that the errors due to the time discretization are negligible. The error estimators we have proposed are based on the residuals $\mathcal{R}_h^m$ and $\mathcal{R}_h^t$ of the Navier–Stokes equations. We have shown that the norms of these residuals, namely $\|\mathcal{R}_h^m\|_*$ and $\|\mathcal{R}_h^t\|_*$, are comparable to specific norms of two components of the error in the velocity variable. We have then designed a fully automatic adaptive strategy for the control of these numerical errors, in which the finite element mesh is refined based on the information provided by the residuals $\mathcal{R}_h^m$ and $\mathcal{R}_h^t$. The theoretical results have been successfully validated for the case of a channel flow past a cylinder in the periodic regime, for which a numerical solution has been controlled within some preset error tolerances. We emphasize that the methodology is applicable to flows at any given Reynolds number and work is currently in progress to apply the adaptive strategy for the simulation of various flow regimes. The next issue would be to estimate the numerical errors with respect to the time-discretization in order to control the timestep.

Acknowledgements

The support of this work by the Office of Naval Research under contract N00014-95-1-0401 is gratefully acknowledged.

References


