A Conservative DGM for Convection-Diffusion and Navier-Stokes Problems

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A Conservative DGM for Convection-Diffusion and Navier-Stokes Problems

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Abstract. An *hp*−adaptive conservative Discontinuous Galerkin Method for the solution of convection-diffusion problems is reviewed. A distinctive feature of this method is the treatment of diffusion terms with a new variational formulation. This new variational formulation is *not* based on mixed formulations, thus having the advantage of not using flux vairables or *extended stencils and/or global matrices' bandwidth* when the flux variables are statically condensed at element level.

The variational formulation for diffusion terms produces a compact, locally conservative, higher-order accurate, and stable solver. The method supports *h*−, *p*−, and *hp*−approximations and can be applied to any type of domain discretization, including non-matching meshes. *A priori* error estimates and numerical experiments indicate that the method is robust and capable of delivering high accuracy.

1 Introduction

In this paper we review a discontinuous Galerkin method for the solution of second order convection-diffusion problems which is based on the classical discontinuous Galerkin approximation for convection terms [9,14,23,24] and on the technique developed in [2,5,31] for diffusion terms. This new methodology supports *h-, p-, and hp*-version approximations and can produce highly accurate solutions.

This discontinuous Galerkin method produces truly compact approximations, only the *nearest* neighbors' degrees of freedom interact with the degrees of freedom of any element. A different approach to the treatment of diffusion terms is to cast the second order problem as a first order system using a mixed formulation. This has been done for the solution of the Navier-Stokes equations by Bassi and Rebay [3,4], and by Lomtev, Quillen and Karniadakis in [25–28], and Warburton, Lomtev, Kirby and Karniadakis in [33]. A generalization of the mixed approach was proposed by Cockburn and Shu under the name of Local Discontinuous Galerkin method [13,18].

A disadvantage of using mixed methods is that for a problem in $\mathbb{R}^d$, for each variable subject to a second-order differential operator, d more variables and equations have to be introduced to obtain a first-order system of equations. In practical applications, however, it is customary to use static condensation of the diffusion fluxes at element level. This strategy removes

the diffusion related unknowns from the global problem, but produces non-compact approximations, the degrees of freedom of a given element have interaction with *non-nearest* neighbors' degrees of freedom. Related properties of the discrete systems of equations and associated bandwith are not favorable for the mixed approach.

The DGM reviewed in this paper has the following features:

- the method does not use auxiliary variables such as those used in mixed formulations;
- the method is robust, stable, and exhibits elementwise conservative approximations;
- the method still produces *block diagonal* global mass matrices with *uncoupled* blocks;
- the parameters affecting the rate of convergence and the limitations of this method are well established;
- the method is very well suited for adaptive control of error, and can deliver very high orders of accuracy when the exact solution is smooth;
- the cost of solution and implementation is low; and
- implementations of this method on multiprocessor machines can deliver high levels of parallel efficiency.

A detailed overview of the development and use of discontinuous Galerkin methods is presented by Cockburn, Karniadakis and Shu [15]. What follows is a short description of the literature related to our review paper.

The solution of second-order partial differential equations with discontinuous basis functions dates back the the early 1970's, Nitsche [29] introduced the concept of replacing the Lagrange multipliers used in hybrid formulations with averaged normal fluxes at the boundaries, and added stabilization terms to produce optimal convergence rates. A similar approach was used by Percell and Wheeler [32]. A different approach was the Global Element Method (GEM) of Delves and Hall [19], applications of this method were presented by Hendry and Delves in [21]. The GEM consists of the classical hybrid formulation for a Poisson problem with the Lagrange multiplier eliminated in terms of the dependent variables; namely, the Lagrange multiplier is replaced by the average flux across interelement boundaries, without the addition of penalty terms. A major disadvantage of the GEM is that the matrix associated with space discretizations of diffusion operators is indefinite, therefore the method can not be used to solve efficiently time dependent diffusion problems; and being indefinite, the linear systems associated with steady state problems needs special iterative schemes. A study of the eigenvalue spectrum associated with a number of discontinuous techniques for diffusion problems is presented by Hughes *et al.* [22]. In the same paper Hughes *et al.* present new developments in conservation properties of continuous and discontinuous FEMs. The interior penalty formulations presented by Wheeler *et al.* in [32] utilize the bilinear form of the GEM augmented with a penalty term which includes the jumps of the solution across elements. The disadvantages of the

last approach include the dependence of stability and convergence rates on the penalty parameter, the loss of the conservation property at element level, and a bad conditioning of the matrices. The DGM for diffusion operators reviewed in this paper is a modification of the GEM, which is free from the deficiencies that affect the GEM. More details on these formulations, and the relative merits of each one are presented in [5,31].

This paper is structured as follows: Section 2 introduces a model scalar convection-diffusion problem with the associated notation. Section 3 presents the associated discontinuous Galerkin approximation with *a priori* error estimation and Section 4 reviews the application of the method to the Navier-Stokes equations. Finally, numerical experiments are discussed in Section 5 and conclusions are collected in Section 6.

2 Scalar convection-diffusion problem

Let Ω be an open bounded Lipschitz domain in $\mathbb{R}^d$. We consider a model second-order convection diffusion problem characterized by the following scalar partial differential equation and boundary conditions

$$\boxed{-\nabla\cdot(\boldsymbol{A}\nabla u)+\nabla\cdot(\boldsymbol{\beta}\,u)+\sigma u=S \quad \text{in } \Omega\subset\mathbb{R}^d} \tag{1}$$

$$\boxed{\begin{array}{rl} u=f & \text{on } \Gamma_{\mathrm{D}} \\ (\boldsymbol{A}\nabla u)\cdot\boldsymbol{n}=g & \text{on } \Gamma_{\mathrm{N}} \end{array}} \tag{2}$$

where $\boldsymbol{\beta}\in(L^{\infty}(\Omega))^{d}$ is the mass flux vector, $\sigma\in L^{\infty}(\Omega)$, $\sigma\geq 0$ a.e. in Ω, $S\in L^{2}(\Omega)$, and $\boldsymbol{A}\in(L^{\infty}(\Omega))^{d\times d}$ is a diffusivity matrix characterized as follows:

$$\begin{gathered} \boldsymbol{A}(\boldsymbol{x})=\boldsymbol{A}^{T}(\boldsymbol{x}), \\ \alpha_1\boldsymbol{a}^{T}\boldsymbol{a}\geq\boldsymbol{a}^{T}\boldsymbol{A}(\boldsymbol{x})\boldsymbol{a}\geq\alpha_0\boldsymbol{a}^{T}\boldsymbol{a}, \quad \alpha_1\geq\alpha_0>0, \quad \forall\boldsymbol{a}\in\mathbb{R}^{d}, \end{gathered} \tag{3}$$

a.e. in Ω.

The boundary $\partial\Omega$ consists of disjoint parts, Γ_{D} on which Dirichlet conditions are imposed, and Γ_{N} on which Neumann conditions are imposed: $\Gamma_{\mathrm{D}}\cap\Gamma_{\mathrm{N}}=\emptyset$, $\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}=\partial\Omega$, and *meas* $\Gamma_{\mathrm{D}}>0$. The inflow Γ_{-} and outflow Γ_{+} parts of the boundary are defined as follows:

$$\Gamma_{\mathrm{D}}\supseteq\Gamma_{-}=\{\boldsymbol{x}\in\partial\Omega\mid(\boldsymbol{\beta}\cdot\boldsymbol{n})(\boldsymbol{x})<0 \text{ a.e.}\}, \quad \Gamma_{+}=\partial\Omega\setminus\Gamma_{-}.$$

2.1 Regular partitions

Let us consider regular partitions of Ω [12,30]. Let $\mathcal{P}=\{\mathcal{P}_h(\Omega)\}_{h>0}$ be a family of regular partitions of $\Omega\subset\mathbb{R}^{d}$ into $N\doteq N(\mathcal{P}_h)$ subdomains Ω_e such

that for $\mathcal{P}_h \in \mathcal{P}$,

$$\overline{\Omega} = \bigcup_{e=1}^{N(\mathcal{P}_h)} \overline{\Omega}_e, \quad \text{and } \Omega_e \cap \Omega_f = \emptyset \quad \text{for } e \neq f.$$

The *interelement boundary* is defined as follows:

$$\Gamma_{\text{int}} = \bigcup_{\Omega_f, \Omega_e \in \mathcal{P}_h} (\partial\Omega_f \cap \partial\Omega_e). \tag{4}$$

On Γ_{int}, we define $\boldsymbol{n} = \boldsymbol{n}_e$ on $(\partial\Omega_e \cap \partial\Omega_f) \subset \Gamma_{\text{int}}$ for indices e, f such that $e > f$.

For $v|_{\Omega_e} \in H^{3/2+\epsilon}(\Omega_e)$ and $v|_{\Omega_f} \in H^{3/2+\epsilon}(\Omega_f)$, we introduce the *jump* operator $[.]$ defined on $\Gamma_{ef} = \overline{\Omega}_e \cap \overline{\Omega}_f \neq \emptyset$ as follows:

$$[v] = (\gamma_0 v)|_{\partial\Omega_e \cap \Gamma_{ef}} - (\gamma_0 v)|_{\partial\Omega_f \cap \Gamma_{ef}}, \quad e > f, \tag{5}$$

and the *average* operator $\langle . \rangle$ for the normal flux is defined for $(\boldsymbol{A}\boldsymbol{\nabla} v) \cdot \boldsymbol{n} \in L^2(\Gamma_{ef})$, as

$$\langle (\boldsymbol{A}\boldsymbol{\nabla} v) \cdot \boldsymbol{n} \rangle = \frac{1}{2} \left(((\boldsymbol{A}\boldsymbol{\nabla} v) \cdot \boldsymbol{n})|_{\partial\Omega_e \cap \Gamma_{ef}} + ((\boldsymbol{A}\boldsymbol{\nabla} v) \cdot \boldsymbol{n})|_{\partial\Omega_f \cap \Gamma_{ef}} \right), \; e > f \tag{6}$$

where $\boldsymbol{A}$ is the diffusivity. Note that $\boldsymbol{n}$ represents the outward normal of the element with higher index.

3 Discontinuous Galerkin approximation

The discontinuous Galerkin formulation for convection-diffusion problems is built as an extension of the classical discontinuous Galerkin method for hyperbolic problems [16,17,23,24], with the diffusion operators treated as in [5,7,31]. We review definitions and formulations presented in [7].

3.1 Weak formulation

Let $W(\mathcal{P}_h)$ be the Hilbert space on the partition $\mathcal{P}_h$ defined as the completion of $H^{3/2+\epsilon}(\mathcal{P}_h)$ under the norm $\|.\|_W$ defined as follows (induced by (12) below)

$$\|u\|_W^2 = \|u\|_V^2 + \|u\|_\beta^2 + \left\| u\sigma^{\frac{1}{2}} \right\|_{0,\Omega}^2, \tag{7}$$

$$\|v\|_V^2 = \sum_{\Omega_e \in \mathcal{P}_h} \int_{\Omega_e} \boldsymbol{\nabla} v \cdot \boldsymbol{A}\, \boldsymbol{\nabla} v \, \mathrm{dx} + |v|_{0,\Gamma_{\mathcal{P}_h}}^2, \tag{8}$$

$$\begin{aligned} \|u\|_\beta^2 = {} & \left| u|\boldsymbol{\beta}|^{\frac{1}{2}} \right|_{0,\Omega}^2 + \sum_{\Omega_e \in \mathcal{P}_h} \left| \boldsymbol{\nabla} u \cdot \boldsymbol{\beta}/|\boldsymbol{\beta}|^{\frac{1}{2}} \right|_{0,\Omega_e}^2 + \left| u\,|\boldsymbol{\beta} \cdot \boldsymbol{n}|^{\frac{1}{2}} \right|_{0,\Gamma_+}^2 \\ & + \left| h^\alpha u^-\,|\boldsymbol{\beta} \cdot \boldsymbol{n}|^{\frac{1}{2}} \right|_{0,\Gamma_{\text{int}}}^2 + \left| h^{-\alpha}[u]\,|\boldsymbol{\beta} \cdot \boldsymbol{n}|^{\frac{1}{2}} \right|_{0,\Gamma_{\text{int}}}^2, \end{aligned} \tag{9}$$

$$|v|^2_{0,\Gamma_{\mathcal{P}_h}} = |h^{-\alpha}\, v|^2_{0,\Gamma_D} + |h^{\alpha}\ (\boldsymbol{A}\boldsymbol{\nabla} v)\cdot\boldsymbol{n}|^2_{0,\Gamma_D} + \ |h^{-\alpha}\,[v]|^2_{0,\Gamma_{\rm int}} + |h^{\alpha}\,\langle(\boldsymbol{A}\boldsymbol{\nabla} v)\cdot\boldsymbol{n}\rangle|^2_{0,\Gamma_{\rm int}}, \tag{10}$$

and

$$|v|^2_{0,\Gamma} = \int_\Gamma v^2\ \mathrm{ds}, \quad \text{for} \quad \Gamma \in \{\Gamma_D, \Gamma_N, \Gamma_{\rm int}\}\,.$$

The terms $h^{\pm\alpha}$, with $\alpha = 1/2$, are introduced to minimize the mesh-dependence of the norm. In (10), the value of h is $h_e/(2\alpha_1)$ on Γ_D, and the average $(h_e + h_f)/(2\alpha_1)$ on that part of $\Gamma_{\rm int}$ shared by two generic elements Ω_e and Ω_f, and the constant α_1 is defined in (3). In (9), however, h is $h_e/2$ on Γ_D, and the average $(h_e + h_f)/2$ on $\partial\Omega_e \cap \partial\Omega_f$.

A consistent formulation of problem (1-2) is the following variational statement:

$$\boxed{\begin{array}{l}\text{Find } u \in W(\mathcal{P}_h) \text{ such that} \\ \qquad\qquad B(u,v) = L(v) \quad \forall\, v \in W(\mathcal{P}_h)\end{array}} \tag{11}$$

where

$$B(u,v) = \sum_{\Omega_e\in\mathcal{P}_h} \left\{ \int_{\Omega_e} \left[\boldsymbol{\nabla} v\cdot\ \boldsymbol{A}\ \boldsymbol{\nabla} u - (\boldsymbol{\nabla} v\cdot\boldsymbol{\beta})\ u + v\sigma u\right]\ \mathrm{dx} + \int_{\partial\Omega_e\setminus\Gamma_-} v u^-\,(\boldsymbol{\beta}\cdot\boldsymbol{n}_e)\ \mathrm{ds}\right\} + \int_{\Gamma_D} \left((\boldsymbol{A}\boldsymbol{\nabla} v)\cdot\boldsymbol{n}\, u - v\,(\boldsymbol{A}\boldsymbol{\nabla} u)\cdot\boldsymbol{n}\right)\ \mathrm{ds} + \int_{\Gamma_{\rm int}} \left(\langle(\boldsymbol{A}\boldsymbol{\nabla} v)\cdot\boldsymbol{n}\rangle[u] - \langle(\boldsymbol{A}\boldsymbol{\nabla} u)\cdot\boldsymbol{n}\rangle[v]\right)\ \mathrm{ds}, \tag{12}$$

$$u^{\pm} = \lim_{\epsilon\to 0} u\,(\boldsymbol{x} \pm \epsilon\boldsymbol{\beta}), \quad \text{for } \boldsymbol{x} \in \Gamma_{\rm int},$$

and

$$L(v) = \sum_{\Omega_e\in\mathcal{P}_h} \int_{\Omega_e} vS\ \mathrm{dx} + \int_{\Gamma_D} (\boldsymbol{A}\boldsymbol{\nabla} v)\cdot\boldsymbol{n}\, f\ \mathrm{ds} + \int_{\Gamma_N} v\, g\ \mathrm{ds} - \int_{\Gamma_-} vf\,(\boldsymbol{\beta}\cdot\boldsymbol{n})\ \mathrm{ds}. \tag{13}$$

Remark: Note that $H^1_0(\Omega) \subset W(\mathcal{P}_h)$. Indeed, for $u, v \in H^1_0(\Omega)$, the bilinear and linear forms $B(u,v)$ and $L(v)$ reduce to those of the continuous Galerkin formulation, which is known to be unstable for not well resolved

convection-dominated problems. The use of discontinuous basis functions in combination with (12)-(13), however, produces a method with superior stability properties. It is proven in [7] that the formulation presented is globally and locally (elementwise) conservative. Section 4.2 in this paper includes a proof of the conservation property for the Navier-Stokes equations. □

3.2 Polynomial approximations on partitions

We review a well-known local approximation property of polynomial finite element approximations (see [1]). Let $\hat{\Omega}$ be a regular master element in $\mathbb{R}^d$, and let $\{F_{\Omega_e}\}$ be a family of invertible maps from $\hat{\Omega}$ onto Ω_e. For every element $\Omega_e \in \mathcal{P}_h$, the finite-dimensional space of real-valued shape functions $\hat{P} \subset H^m\left(\hat{\Omega}\right)$ is the space $P_{p_e}\left(\hat{\Omega}\right)$ of polynomials of degree $\leq p_e$ defined on $\hat{\Omega}$. Then we define

$$P_{p_e}(\Omega_e) = \left\{\psi \mid \psi = \hat{\psi} \circ F_{\Omega_e}^{-1},\ \hat{\psi} \in \hat{P} = P_{p_e}\left(\hat{\Omega}\right)\right\}. \tag{14}$$

Using the spaces $P_{p_e}(\Omega_e)$, we can define

$$W_p(\mathcal{P}_h) = \prod_{e=1}^{N(\mathcal{P}_h)} P_{p_e}(\Omega_e), \tag{15}$$

$N(\mathcal{P}_h)$ being the number of elements in $\mathcal{P}_h$.

The approximation properties of $W_p(\mathcal{P}_h)$ will be estimated using standard local approximation estimates (see [1]). Let $u \in H^s(\Omega_e)$; there exist a constant C depending on s and on the angle condition of Ω_e, but independent of u, $h_e = diam(\Omega_e)$, and p_e, and a polynomial u_p of degree p_e, such that for any $0 \leq r \leq s$ the following estimate hold:

$$\|u - u_p\|_{r,\Omega_e} \leq C\frac{h_e^{\mu-r}}{p_e^{s-r}}\|u\|_{s,\Omega_e}, \qquad s \geq 0, \tag{16}$$

where $\|.\|_{r,\Omega_e}$ denotes the usual Sobolev norm, and $\mu = min(p_e + 1, s)$.

3.3 Discontinuous Galerkin approximation

The variational formulation of our discontinuous Galerkin method (11) will be used as a basis to construct approximations to the exact solution in a finite dimensional space. The variational formulation in the space $W_p(\mathcal{P}_h)$ is the following:

$$\boxed{\begin{array}{l}\text{Find } u_{DG} \in W_p(\mathcal{P}_h) \text{ such that} \\ \qquad\qquad B(u_{DG}, v_h) = L(v_h)\ \forall\, v_h \in W_p(\mathcal{P}_h)\end{array}} \tag{17}$$

where $B(.,.)$ and $L(.)$ are defined in (12) and (13), respectively.

Note that all the properties of the discontinuous Galerkin method (11) also hold for the finite dimensional approximation (17); namely, solutions are elementwise conservative, mass matrices are block diagonal, and the space of dicontinuous functions provides the basis to obtain solutions with potentially good stability properties.

Stability is one of the most important characteristics of a method for the solution of convection-diffusion problems. The following section addresses this issue and provides *a priori* error estimation to solutions of (17).

3.4 *A priori* error estimation

***A priori* error estimates for pure diffusion cases:** We now review results presented in [31]. Assuming that

$$\inf_{\substack{u \in V_h \\ \|u\|_{V_h} = 1}} \sup_{\substack{v \in V_h \\ \|v\|_{V_h} \leq 1}} |B(u,v)| \geq \gamma_h \approx O\left(p_{\max}^{-\kappa}\right), \tag{18}$$

where $\|.\|_{V_h}$ is the norm defined in (8), and $\kappa \geq 0$, an estimate of the global rates of convergence of the DGM (13) with $\beta \equiv 0$ and $\sigma \equiv 0$ is given by the following theorem:

Theorem 1. *Let the solution $u \in H^s(\mathcal{P}_h(\Omega))$, with $s > 3/2$, and assume that the value of the inf-sup parameter is $\gamma_h = C_p\, p_{\max}^{-\kappa}$ with $\kappa \geq 0$. If the approximation estimate (16) hold for the spaces $W_p(\mathcal{P}_h)$, then the error of the approximate solution u_{DG} can be bounded as follows:*

$$\|u - u_{DG}\|_V^2 \leq C p_{\max}^{2\kappa} \sum_{\Omega_e \in \mathcal{P}_h} \left(\frac{h_e^{\mu_e - 1 - \epsilon}}{p_e^{s-3/2-\epsilon}} \|u\|_{s,\Omega_e} \right)^2, \tag{19}$$

where $\mu_e = min(p_e + 1, s)$, $\epsilon \longrightarrow 0^+$, and the constant C depends on s and on the angle condition of Ω_e, but it is independent of u, h_e and p_e.

Remark: The error estimate (19) is a bound for the worst possible case, including all possible data. For a wide range of data, however, the error estimate (19) may be pessimistic, and the actual rate of convergence can be larger than that suggested by the above bound.

The value of the parameter κ depends on p_e and on d. For $d = 1$, $\kappa = 0$ regardless of p_e, as long as $p_e \geq 2$, whereas for $d > 1$ the value depends on the mesh regularity; numerical evaluation of the inf-sup condition suggests that for $p_e \geq 2$ the exponent $\kappa \approx 1.0 - 1.5$. □

Riviere and Wheeler [10,11] have presented the following results related to the discontinuous Galerkin method reviewed in this paper:

Theorem 2. *Let the solution $u \in H^s(\mathcal{P}_h(\Omega))$, with $s > 3/2$, then the error of the approximate solution u_{DG} can be bounded as follows:*

$$\left(|u-u_{DG}|_1^2 + \left\|\sigma^{1/2}(u-u_{DG})\right\|_1^2\right)^{1/2} \leq C\frac{h^{\mu-1}}{p^{s-3/2}}\left(\sum_{\Omega_e\in\mathcal{P}_h}\|u\|_{s,\Omega_e}^2\right)^{1/2} \quad (20)$$

where $\mu = min(p+1, s)$, and the constant C depends on α_1, and $\|\sigma\|_\infty$. This estimate was proven along with some other estimates related to the Non-Symmetric Interior Penalty Galerkin (NIPG) method of Wheeler and Riviere.

A priori **error estimates for convection-diffusion cases:** In this section we review results presented in [7]. The norm $\|.\|_{W_1}$ used in the error estimate is the following:

$$\|u\|_{W_1}^2 = \|u\|_{V_1}^2 + \|u\|_{\beta_1}^2, \quad (21)$$

$$\|u\|_{V_1}^2 = |u|_{0,\mathcal{P}_h}^2 + |h^\alpha\, u|_{0,\Gamma_D\cup\Gamma_N}^2 + \left|h^\delta\alpha_1^{-1}\ (\boldsymbol{A}\nabla u)\cdot n\right|_{0,\Gamma_D}^2$$
$$+ |h^\alpha\,[u]|_{0,\Gamma_{int}}^2 + \left|h^\delta\alpha_1^{-1}\,\langle(\boldsymbol{A}\nabla u)\cdot n\rangle\right|_{0,\Gamma_{int}}^2 + |h^\alpha\,\langle u\rangle|_{0,\Gamma_{int}}^2, \quad (22)$$

$$|u|_{0,\mathcal{P}_h}^2 = \sum_{\Omega_e\in\mathcal{P}_h}\int_{\Omega_e} u^2\,\mathrm{dx},\ |v|_{0,\Gamma}^2 = \int_\Gamma v^2\,\mathrm{ds},\ \text{for}\ \Gamma\in\{\Gamma_D,\Gamma_N,\Gamma_{int}\},$$

and

$$\|u\|_{\beta_1}^2 = \left|h^\alpha u^+\right|_{0,\Gamma_-}^2 + |h^\alpha[u]|_{0,\Gamma_{int}}^2 + \|hu_\beta\|_{0,\Omega}^2, \quad (23)$$

with $\alpha = 1/2$, $\delta = 3/2$, and $u_\beta = |\beta|^{-1}(\nabla u\cdot\beta)$ when $|\beta| > 0$, otherwise $u_\beta = 0$. In $\|.\|_{V_j}$ and $\|.\|_{\beta_j}$ the scaling parameter h is $h_e/2$ on $\partial\Omega_e\cap\partial\Omega$, and the average $(h_e+h_f)/2$ on $\partial\Omega_e\cap\partial\Omega_f$.

Solutions to convection-diffusion problems can exhibit features that range from those of diffusion dominated problems to those of pure convection problems. The error of diffusion dominated problems is better measured in the H^1-norm because the associated physics depends on the solution gradient, such as heat transfer, viscous stresses, etc; whereas the error in convection dominated transport is better measured in the L^2-norm, because the underlying physics depends more on the solution values than on the solution gradient.

The range $[0,1)$ of local Peclet numbers (P_e) represents a situation in which diffusion effects are dominant, and for which the W-norm converges to the V-norm as $P_e \longrightarrow 0$. The analysis of stability in the V-norm for diffusion

dominated problems was presented in [5,31], where optimal h-convergence rates are presented.

For high P_e, where convection is important, the following *a priori* error estimate applies (see [7]). We assume that the reaction coefficient $\sigma = 0$, which is the worst case scenario from the point of view of stability.

Theorem 3. *Let the solution to (11) be* $u \in H^s(\mathcal{P}_h(\Omega))$, *with* $s > 3/2$, *and assume that there exists* $\kappa \geq 0$ *and* $C_p > 0$ *such that*

$$\inf_{\substack{u \in W_1 \\ \|u\|_{W_1} = 1}} \sup_{\substack{v \in W_1 \\ \|v\|_{W_1} \leq 1}} |B(u,v)| \geq C_p \, p_{\max}^{-\kappa}, \tag{24}$$

where $p_{\max} = \max_e(p_e)$. *If the approximation estimate (16) holds for the spaces* $W_p(\mathcal{P}_h)$, *then the error of the approximate solution* u_{DG} *is bounded as follows:*

$$\|u - u_{DG}\|_{W_1}^2 \leq C p_{\max}^{2\kappa} \sum_{\Omega_e \in \mathcal{P}_h} \left(\frac{h_e^{\mu_e - \epsilon}}{p_e^{s-3/2-\epsilon}} \|u\|_{s,\Omega_e} \right)^2, \tag{25}$$

where $\mu_e = min(p_e + 1, s)$, $\epsilon \longrightarrow 0^+$, *and the constant* C *depends on* s *and on the angle condition of the element, but it is independent of* u, h_e *and* p_e.

A proof of this result can be found in [7]. **Remark:** Numerical experiments presented in [7] indicate that $\kappa < 1.5$. □

4 Navier-Stokes problems

First we review a model problem and related notations in preparation for analyzing the discontinuous Galerkin formulation.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$.The governing equations for the conservation of mass, momentum, and energy can be written in vector form as follows:

$$\boxed{\begin{aligned} &\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}_i}{\partial x_i} = \frac{\partial \boldsymbol{F}_i^v}{\partial x_i} + \boldsymbol{S}, \quad \text{in } \Omega \\ &\boldsymbol{U}(x,0) = \boldsymbol{U}_o(x), \quad \text{at } t = 0 \end{aligned}} \tag{26}$$

where repeated indices are summed throughout their range, $\boldsymbol{U} = (u_1, ..., u_m) = \boldsymbol{U}(x,t) \in \mathbb{R}^m$ is a vector of conservation variables with $m = d + 1$ (momentum and mass conservation) or $d + 2$ when the energy equation is included, $\boldsymbol{F}_i(\boldsymbol{U}) = (f_{1i}, ..., f_{mi}) \in \mathbb{R}^m$ and $\boldsymbol{F}_i^v(\boldsymbol{U}) = (f_{1i}^v, ..., f_{mi}^v) \in \mathbb{R}^m$ are the inviscid and diffusive flux vectors associated with the i$-th$ space coordinate, and $\boldsymbol{S}$ represents the body forces in the momentum equations and a source of heat (e.g. heat source due to viscous dissipation) in the energy equation. The system of equations (26) is accompanied by appropriate boundary conditions for each problem.

4.1 Inviscid and viscous flux vectors

The inviscid flux vectors $\boldsymbol{F}_i$ are homogeneous functions of degree one in the conservation variables $\boldsymbol{U}$; therefore the fluxes can be written as $\boldsymbol{F}_i = \boldsymbol{A}_i(\boldsymbol{U})\,\boldsymbol{U}$, where $\boldsymbol{A}_i(\boldsymbol{U})$ is the Jacobian matrix.

Let $\boldsymbol{F}_n(\boldsymbol{U})$ be the normal flux at any point on a boundary $\partial\Omega$ with outward normal $\boldsymbol{n}$; then

$$\boldsymbol{F}_n = \boldsymbol{F}_i\, n_i \qquad i \in [1, ..., d],$$

$$\boldsymbol{A}_n(\boldsymbol{U}) = \frac{\partial \boldsymbol{F}_n(\boldsymbol{U})}{\partial \boldsymbol{U}} = \boldsymbol{A}_i(\boldsymbol{U})\, n_i, \qquad \boldsymbol{A}_n(\boldsymbol{U}) \in I\!R^m \times I\!R^m.$$

The flux vector $\boldsymbol{F}_n(\boldsymbol{U})$ can be split into inflow and outflow components $\boldsymbol{F}_n^+$ and $\boldsymbol{F}_n^-$; for example

$$\boldsymbol{F}_n^{\pm}(\boldsymbol{U}) = \boldsymbol{R}\,\boldsymbol{\Lambda}^{\pm}\boldsymbol{R}^{-1}\,\boldsymbol{U}, \qquad \boldsymbol{\Lambda}^{\pm} = \frac{1}{2}\left(\boldsymbol{\Lambda} \pm |\boldsymbol{\Lambda}|\right),$$

where $\boldsymbol{\Lambda}$ is the diagonal matrix of eigenvalues of $\boldsymbol{A}_n$, and the columns of the matrix $\boldsymbol{R}$ are the corresponding eigenvectors. From a physical point of view, $\boldsymbol{F}_n^+$ and $\boldsymbol{F}_n^-$ represent the fluxes of mass, momentum, and energy leaving $(+)$ and entering $(-)$ the domain through $\partial\Omega$.

Given that the approximation of field variables may be discontinuous across internal surfaces in Ω or across $\partial\Omega$, let us define

$$\boldsymbol{U}^{\pm} = \lim_{\epsilon \to 0^+} \boldsymbol{U}(\boldsymbol{x} \pm \epsilon\, \boldsymbol{n}),$$

where $\boldsymbol{x}$ is a point at a boundary which can be real (e.g. bounding walls) or artificial (e.g. interelement, far-field). With this notation, $\boldsymbol{F}_n^+(\boldsymbol{U}^-)$ is the flux in the direction $\boldsymbol{n}$, and $\boldsymbol{F}_n^-(\boldsymbol{U}^+)$ is that in the opposite direction.

The projection of the viscous flux vectors $\boldsymbol{F}_i^v$ onto the normal $\boldsymbol{n}$ to a boundary is a linear functional of $\boldsymbol{U}$, and will be written in the following alternative forms

$$\boldsymbol{F}_i^v\, n_i = \boldsymbol{F}_n^v = \boldsymbol{D}_n\, \boldsymbol{U},$$

where for Newtonian flows the matrix $\boldsymbol{D}_n$ is a linear differential operator.

4.2 Space discretization with broken spaces

Let $\boldsymbol{V}(\mathcal{P}_h)$ be a broken space of admissible vectors of conservation variables $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{x})$ having the necessary regularity conditions.

For a given initial data $\boldsymbol{U}_o$, and appropriate boundary conditions, the *space discretization* using the discontinuous Galerkin method can be stated

as follows:

$$
\begin{aligned}
&\text{Given } \boldsymbol{U}_o = \boldsymbol{U}_o(\boldsymbol{x}), \text{ for } t \in (0,T), \text{ find} \\
&\boldsymbol{U}(.,t) \in \boldsymbol{V}(\mathcal{P}_h) \times H^1(0,T) \text{ such that } \boldsymbol{U}(\boldsymbol{x},0) = \boldsymbol{U}_o(\boldsymbol{x}), \text{ and} \\
&\int_\Omega \boldsymbol{W}^T \frac{\partial \boldsymbol{U}}{\partial t}\, \mathrm{dx} + \sum_{\Omega_e \in \mathcal{P}_h} \int_{\partial\Omega_e} \boldsymbol{W}^T \left(\boldsymbol{F}^+_{n_e}(\boldsymbol{U}^-) + \boldsymbol{F}^-_{n_e}(\boldsymbol{U}^+)\right) \mathrm{ds} \\
&\qquad + \int_{\Gamma_{\mathrm{int}}} \left(\left\langle \boldsymbol{W}^T \boldsymbol{D}_n^T \right\rangle [\boldsymbol{U}] - \left[\boldsymbol{W}^T\right] \langle \boldsymbol{F}^v_n \rangle \right) \mathrm{ds} \\
&+ \int_{\Gamma_{\mathrm{D}}} \left(\boldsymbol{W}^T \boldsymbol{D}_n^T \boldsymbol{U} - \boldsymbol{W}^T \boldsymbol{F}^v_n \right) \mathrm{ds} + \sum_{\Omega_e \in \mathcal{P}_h} \int_{\partial\Omega_e} \frac{\partial \boldsymbol{W}^T}{\partial x_i} \left(\boldsymbol{F}^v_i - \boldsymbol{F}_i\right) \mathrm{dx} \\
&\qquad = \int_\Omega \boldsymbol{W}^T \boldsymbol{S}\, \mathrm{dx} + \int_{\Gamma_{\mathrm{D}}} \boldsymbol{W}^T \boldsymbol{D}_n^T \hat{\boldsymbol{U}}\, \mathrm{ds} + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{W}^T \hat{\boldsymbol{F}}^v_n\, \mathrm{ds} \\
&\qquad \forall\, \boldsymbol{W} \in \boldsymbol{V}(\mathcal{P}_h)
\end{aligned}
\tag{27}
$$

here

$$
\boldsymbol{F}_{n_e}(\boldsymbol{U}) = \boldsymbol{F}_i(\boldsymbol{U})\, n_{e_i}, \qquad \boldsymbol{U}^\pm = \lim_{\epsilon \to 0^+} \boldsymbol{U}(\boldsymbol{x} \pm \epsilon\, \boldsymbol{n}_e),
$$

$$
\boldsymbol{F}^v_n(\boldsymbol{U}) = \boldsymbol{F}^v_i(\boldsymbol{U})\, n_i, \qquad \boldsymbol{F}^v_n(\boldsymbol{U}) = \boldsymbol{D}_n \boldsymbol{U}.
$$

$\boldsymbol{F}^\pm_n$ are known in closed form for the usual flux vector and flux difference splittings (see [5] and references therein).

It is important to observe that (27) reduces to the classical weak Galerkin approximation if we restrict $\boldsymbol{V}(\mathcal{P}_h)$ to a space of continuous functions.

We now prove that (27) renders a *conservative* formulation. To show that (27) is *globally conservative,* let us pick a test function $\boldsymbol{W} = (v_1, ..., v_m)$ such that

$$
v_i(\boldsymbol{x}) = 1, \quad i = 1, ..., m \quad \forall \boldsymbol{x} \in \Omega,
$$

by definition $\boldsymbol{W} \in \boldsymbol{V}(\mathcal{P}_h)$. Substituting $\boldsymbol{W}$ in (27), we get

$$
\begin{aligned}
&\int_\Omega \frac{\partial \boldsymbol{U}}{\partial t}\, \mathrm{dx} + \sum_{\Omega_e \in \mathcal{P}_h} \int_{\partial\Omega_e} \left(\boldsymbol{F}^+_{n_e}(\boldsymbol{U}^-) + \boldsymbol{F}^-_{n_e}(\boldsymbol{U}^+)\right) \mathrm{ds} - \int_{\Gamma_{\mathrm{D}}} \boldsymbol{F}^v_n\, \mathrm{ds} \\
&\qquad = \int_\Omega \boldsymbol{S}\, \mathrm{dx} + \int_{\Gamma_{\mathrm{N}}} \hat{\boldsymbol{F}}^v_n\, \mathrm{ds}.
\end{aligned}
\tag{28}
$$

For any pair of adjoining elements (Ω_e, Ω_f), the following identities hold:

$$
\boldsymbol{F}^+_{n_e}(\boldsymbol{U}^-) = -\boldsymbol{F}^-_{n_f}(\boldsymbol{U}^+), \quad \text{and} \quad \boldsymbol{F}^-_{n_e}(\boldsymbol{U}^+) = -\boldsymbol{F}^+_{n_f}(\boldsymbol{U}^-),
$$

substituting the above identities in (28), we obtain

$$\int_\Omega \frac{\partial U}{\partial t}\,\mathrm{dx} + \int_{\partial\Omega} \left(\boldsymbol{F}_n^+(\boldsymbol{U}^-) + \boldsymbol{F}_n^-(\boldsymbol{U}^+)\right)\,\mathrm{ds} = \int_\Omega \boldsymbol{S}\,\mathrm{dx} + \int_{\Gamma_\mathrm{N}} \hat{\boldsymbol{F}}_n^v\,\mathrm{ds} + \int_{\Gamma_\mathrm{D}} \boldsymbol{F}_n^v\,\mathrm{ds}, \tag{29}$$

which shows that the formulation is globally conservative.

To show that the formulation is also *locally conservative,* we select a generic weighting function

$$\boldsymbol{W} = (v_1, ..., v_m) \in \boldsymbol{V}(\mathcal{P}_h) \quad \text{such that } v_i(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \Omega_e \\ 0 & \boldsymbol{x} \notin \Omega_e \end{cases} \qquad i = 1, ..., m,$$

and substituting $\boldsymbol{W}$ in (27), we get

$$\int_{\Omega_e} \frac{\partial \boldsymbol{U}}{\partial t}\,\mathrm{dx} + \int_{\partial\Omega_e} \left(\boldsymbol{F}_{n_e}^+(\boldsymbol{U}^-) + \boldsymbol{F}_{n_e}^-(\boldsymbol{U}^+)\right)\,\mathrm{ds} = \int_{\Omega_e} \boldsymbol{S}\,\mathrm{dx} + \int_{\partial\Omega_e\cap\Gamma_\mathrm{N}} \hat{\boldsymbol{F}}_n^v + \int_{\partial\Omega_e\cap\Gamma_\mathrm{D}} \boldsymbol{F}_n^v\,\mathrm{ds} + \int_{\partial\Omega_e\cap\Gamma_\mathrm{int}} \langle \boldsymbol{F}_n^v\rangle\,\mathrm{ds}, \tag{30}$$

which represents the conservation equations at *element* level when the interelement viscous forces are taken as the average $\langle \boldsymbol{F}_n^v\rangle$.

Remark: To insure local conservation at element level, the interelement viscous flux $F^v(\boldsymbol{U}^-, \boldsymbol{U}^+)$ shoud have the following property:

$$\text{if} \quad \frac{\partial \boldsymbol{U}^-}{\partial n} = \frac{\partial \boldsymbol{U}^+}{\partial n}, \quad \text{then} \quad F^v(\boldsymbol{U}^-, \boldsymbol{U}^+) = \boldsymbol{F}_n^v(\boldsymbol{U}^-) = \boldsymbol{F}_n^v(\boldsymbol{U}^+). \tag{31}$$

This property is verified by the interelement viscous flux $F^v(\boldsymbol{U}^-, \boldsymbol{U}^+) = \langle \boldsymbol{F}_n^v\rangle$ used in (27), see (30). Formulations that add *stabilizing* terms based on the jump of the solution $[\boldsymbol{U}] = \left(\boldsymbol{U}^+ - \boldsymbol{U}^-\right)$ do not possess property (31), because in general, approximate solutions have non-zero jump across interelement boundaries. □

5 Numerical tests

Numerical verification of convergence rates in the $\|.\|_V$ norm for diffusion problems and in the $\|.\|_{W_1}$ norm for convection diffusion problems have been presented in [31,7].

Next we present solutions to Hemker's problem, a convection-diffusion problem with a turning point in the middle of the domain:

$$\begin{cases} \alpha \dfrac{\partial^2 u}{\partial x^2} + x\dfrac{\partial u}{\partial x} = -\alpha\pi^2 \cos(\pi x) - \pi x \sin(\pi x) & \text{on } [0,1] \\ u(-1) = -2, \quad u(1) = 0 \end{cases} \tag{32}$$

for which the exact solution is

$$u(x) = \cos(\pi x) + \mathrm{erf}(x/\sqrt{2\alpha})/\mathrm{erf}(1/\sqrt{2\alpha}).$$

Figure 1 and Fig. 2 show the solutions to the above problem ($\alpha = 10^{-10}$ and $h = 1/10$) obtained with continuous and discontinuous Galerkin, respectively.

Numerical experiments show that when the best approximation to the exact solution in the L^2-norm is better represented using discontinuous functions, the DG method performs better than the continuous Galerkin method. This is in general true for solutions to convection dominated problems with boundary layers. If the quantity of interest is the gradient of the solution at the boundary layers, the DG method provides an excellent error indicator, namely, Dirichlet boundary conditions are not well approximated. This error indicator allows to capture steep gradients using adaptive refinement.

Solutions to the incompressible Navier-Stokes equations can be obtained using the artificial compressibility technique The test case selected is a popular benchmark for laminar viscous flows, the driven cavity problem described in [20] with *Rey* 7500.

A solution to this problem is obtained with a mesh of quadratic elements which is equivalent (in number of degrees of freedom) to a mesh of 60x60 linear elements (used for plotting results) as shown in Fig. 3, this figure also shows the pressure distribution on the background. Note that the pressure range shown is much smaller than the actual range, which is very wide because of the presence of singularities at the top corners of the cavity. The cutoff values $[p_{min}, p_{max}]$ applied to the range of pressure allow to observe small changes within the domain, excluding the areas adjacent to the top corners. Figure 4 shows the streamline pattern. This solution is very accurate, comparisons of velocity profiles through horizontal and vertical planes can be found in [6]. Summarizing, the numerical experiments confirm the stability and high accuracy that the method can deliver for the class of problems considered, even with the use of h refinements and p enrichments.

6 Conclusions

The salient properties of the discontinuous Galerkin formulation reviewed in this paper can be summarized as follows:

- diffusion terms are discretized with a variational formulation that is *not* based on mixed formulations, this is very advantageous because extra flux vectors (gradient of each scalar variable) are not required. When used, the flux vectors of mixed formulations increase considerably the bandwidth of the systems (when they are statically condensed at element level), and the formulations are not compact, the degrees of freedom of any given element interact with the degrees of freedom of non-nearest neighboors;

- the method is capable of solving convection-diffusion problems with an $hp-$approximation methodology. If the local regularity of the solution is high, the $p-$approximation can be used and the method delivers very high accuracy; otherwise the $h-$approximation can be used and the error is reduced by local refinement of the mesh;
- approximate solutions of unresolved flows (e.g. boundary layers) do not suffer from widespread oscillations, for these cases the treatment of interelement boundaries prevents the appearance and spreading of numerical oscillations;
- stability studies and numerical tests demonstrate cuantitatively and qualitatively the superiority of discontinuous over continuous Galerkin solutions for convection-diffusion problems;
- discrete representations are stable in the sense that the real part of the eigenvalues associated with the space discretization are strictly negative, a property that allows the use of time marching schemes for time-dependent problems and also allows to solve steady state problems with explicit schemes;
- an *a priori* error estimate for high Pe numbers indicates that the method delivers optimal $h-$convergence rate (accuracy); and
- contrasting other techniques that use artificial diffusion to improve the stability of continuous Galerkin approximations, the present discontinuous Galerkin method does not introduce such mesh-size dependent terms in the governing equations, which allows for approximation with unlimited order of accuracy; namely, the order of accuracy grows linearly with the order p of the basis functions.

The structure of this discontinuous Galerkin method, particularly the fact that the degrees of freedom of an individual element are coupled only with those of neighbors sharing a boundary makes this method easily parallelizable.

Stability analysis and *a priori* error estimates have been presented for the scalar case in [7,31]. Numerical evidence presented in [7,6,8,31] suggests that this discontinuous Galerkin formulation is highly reliable for obtaining numerical solutions to problems characterized by a wide range of fluid flow conditions. Remarkably, this formulation is stable even when the flow field is not well resolved, and does not produce the classical oscillations near sharp gradients (e.g. boundary layers) which are characteristic in classical H^1 approximations of under-resolved boundary layers.

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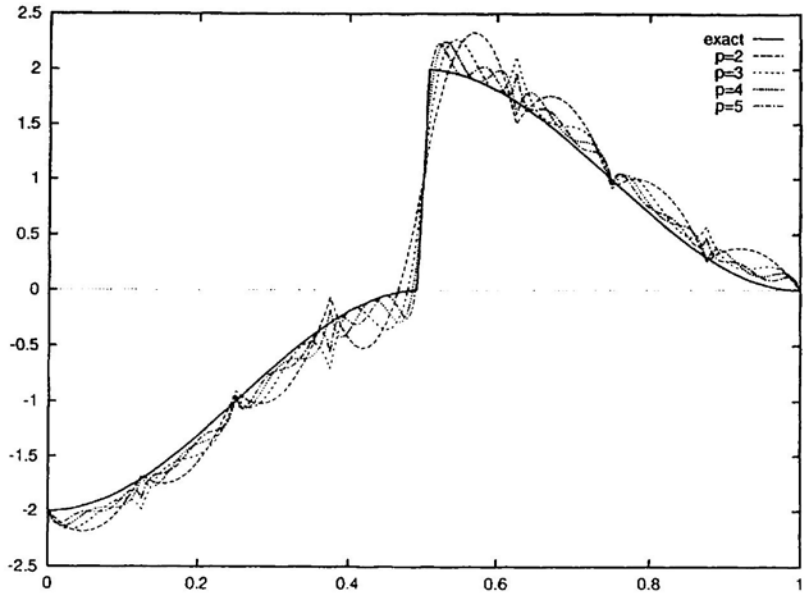

Fig. 1. Hemker problem: $\alpha = 10^{-10}$ and $h = 1/10$, Continuous Galerkin.

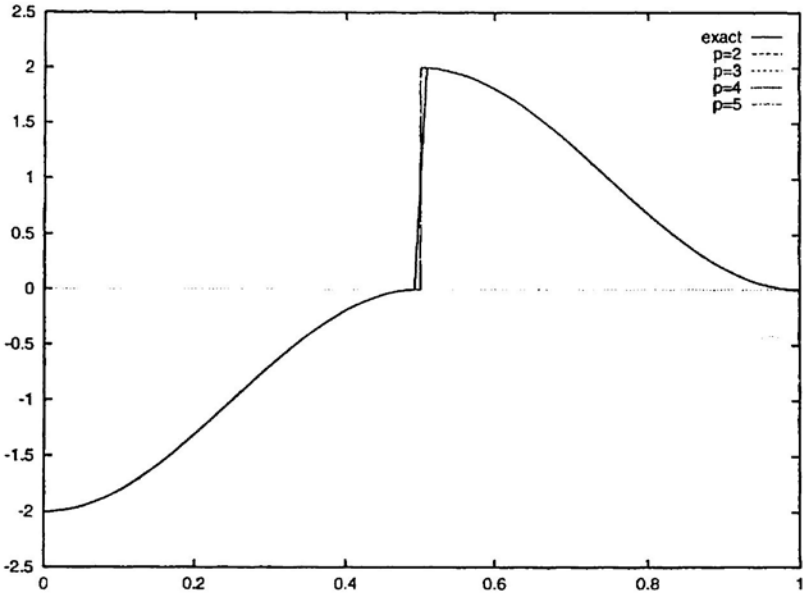

Fig. 2. Hemker problem: $\alpha = 10^{-10}$ and $h = 1/10$, Discontinuous Galerkin.