New Approaches to Error Estimation and Adaptivity for the Navier-Stokes Equations

J.T. Oden and S. Prudhomme

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ABSTRACT

We present a new approach to a posteriori estimate numerical errors in finite element approximations of the time-dependent Navier-Stokes equations and a new strategy to automatically control the numerical error within some preset tolerances. The error estimates are derived from two residuals, which are compared, in their respective measures, to two vector components of the error in the velocity variable. This approach is circumscribed to the investigation of numerical errors due to the spatial discretization only. The performance of the methodology is demonstrated, among others, for two-dimensional channel flows past a cylinder in the periodic regime.

INTRODUCTION

The error estimation method considered here belongs to the family of Error Residual Methods. These methods were originally developed for linear elliptic problems [1], and were later extended to the Stokes [2, 3, 4, 5], and steady-state Navier-Stokes problem [2, 6, 7]. In the present approach, we introduce two residuals, one deriving from the momentum equation and the other from the continuity equation. These residuals constitute the sources of errors in the numerical approximations due to the spatial discretization, and, as such, are related to the actual errors in some appropriate measures. The objective in error control is to contain the estimated errors within some preset tolerances. This is accomplished by reducing the effects of the source terms, i.e. the residuals, as soon as the errors exceed the prescribed tolerances. One advantage of our method is the possibility to prescribe two different tolerances, one for each of the two residuals, in order to adequately control their respective effects. The method relies on fast global iterative techniques [5] which have been developed to deliver accurate approximations of the norm of the residual in the momentum equation.

PRELIMINARIES AND NOTATION

Let Ω denote an open bounded domain in $\mathbb{R}^n$, $n = 2$ or 3, with boundary $\partial\Omega$. The flow of a viscous incompressible fluid in $\Omega$ is modeled by the Navier-Stokes equations, given here in non-dimensionalized form,

$$
\partial_t u + (u \cdot \nabla)u - \text{Re}^{-1} \Delta u + \nabla p = f \quad \text{in } \Omega \times (0, T) \\
\nabla \cdot u = 0 \quad \text{in } \Omega \times (0, T)
$$

(1)

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with boundary condition \( u(x, t) = g(x, t) \), for all \( x \in \partial \Omega \) and \( t \in (0, T) \), and initial condition \( u(x, 0) = u_0(x) \), for all \( x \in \Omega \). Here \( u = u(x, t) \) and \( p = p(x, t) \) are respectively the velocity vector and the pressure scalar at point \( x \in \Omega \) and at time \( t \in [0, T) \), \( \text{Re} \) is the Reynolds number, \( f = f(x, t) \) is a prescribed body force and \( u_0 = u_0(x) \) a prescribed initial velocity field that satisfies the continuity equation \( \nabla \cdot u_0 = 0 \).

For the sake of simplicity in the mathematical development, we consider only homogeneous boundary conditions \( g = 0 \) on \( \partial \Omega \). We then introduce the trial spaces of velocities \( V \) and pressures \( Q \) with associated norms:

\[
V = H^1_0(\Omega) = (H^1_0(\Omega))^n, \quad \|v\|_1^2 = \int_{\Omega} \nabla v : \nabla v \, dx,
\]

\[
Q = \{q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0\}, \quad \|q\|_0^2 = \int_{\Omega} q^2 \, dx.
\]

We also introduce the bilinear forms \( a \) and \( b \), as well as the trilinear form \( c \), such that for all \( u, v, w \in V \) and for all \( q \in Q \):

\[
a(u, v) = \text{Re}^{-1} \int_{\Omega} \nabla u : \nabla v \, dx.
\]

\[
b(v, q) = -\int_{\Omega} q \nabla \cdot v \, dx.
\]

\[
c(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w \, dx.
\]

The forms are all continuous on their respective spaces of definition. Moreover, the bilinear form \( b \) is known to satisfy the standard LBB condition \([8], \) and in particular, there exists a constant \( \beta > 0 \) such that:

\[
\sup_{v \in V \setminus \{0\}} \frac{|b(v, q)|}{|v|_1} \geq \beta \|q\|_0, \quad \forall q \in Q. \tag{2}
\]

Solutions of the Navier-Stokes problem are given by the pair of functions \((u, p) \in V \times Q\), for all \( t \in [0, T) \), which satisfy:

\[
(\partial_t u, v) + c(u, u, v) + a(u, v) + b(u, p) = (f, v), \quad \forall v \in V,
\]

\[
b(u, q) = 0, \quad \forall q \in Q,
\]

\[
u = u_0, \quad \text{at} \ t = 0.
\]

The above problem is then approximated using \( h-p \) finite element spaces \( V^h \subset V \) and \( Q^h \subset Q \) \([9]\) and any numerical time marching scheme we wish. Next, we define and study numerical errors in finite element approximations \((u_h, p_h) \in V^h \times Q^h\), for \( t \in [0, T) \), of the Navier-Stokes equations.

**ERROR ESTIMATION**

The numerical error is defined as the pair \((e, E) = (u - u_h, p - p_h) \in V \times Q\) for each time \( t \in [0, T) \). Replacing \( u \) and \( p \) respectively by \((u_h + e)\) and \((p_h + E)\) in (3), the error \((e, E)\), due to the discretization in space (we suppose that the error due to the discretization in time is relatively
small), is shown to satisfy the following time evolution equation and constraint:

\[
(\partial_t e, v) + c(e, u_h, v) + c(u_h, e, v) + c(e, e, v) + a(e, v) + b(v, E) = R_h^m(v), \quad \forall v \in V, \\

b(e, q) = R_h^c(q), \quad \forall q \in Q, 
\]

where the residuals \( R_h^m \) and \( R_h^c \) are the linear functionals

\[
R_h^m(v) = (f, v) - (\partial_t u_h, v) - c(u_h, u_h, v) - a(u_h, v) - b(v, p_h), \\
R_h^c(q) = -b(u_h, q).
\]

The objective in global error estimation is to relate, inexpensively but accurately, the residuals to the errors in some relevant global measures. We establish here the relationship between the quantity \( |e|_1 \) to the norms of the residuals \( R_h^m \) and \( R_h^c \), defined as

\[
||R_h^m||_* = \sup_{v \in V \setminus \{0\}} \frac{|R_h^m(v)|}{|v|_1}, \\
||R_h^c||_* = \sup_{q \in Q \setminus \{0\}} \frac{|R_h^c(q)|}{||q||_0}. 
\]

The key point in our approach is to decompose the error \( e \) into two unique vectors \( e_d \in J \) and \( e_\perp \in J^\perp \), \( e = e_d + e_\perp \), where the space \( J \) is the subspace of \( V \) which contains all the divergence-free functions of \( V \), whereas \( J^\perp \) is the orthogonal complement of \( J \) with respect to the inner product \( \langle \nabla \cdot, \nabla \cdot \rangle \) (see [8]). The norm of \( e \) is then given by \( |e|_1 = |e_d|_1 + |e_\perp|_1^2 \).

**Theorem 1** Let \( e_\perp \in J^\perp \) be the error component in the numerical velocity \( u_h \). Then, for all \( t \in (0, T) \),

\[
\beta |e_\perp(t)|_1 \leq ||R_h^c||_* \leq \sqrt{n} |e_\perp(t)|_1. 
\]

**Proof:** We refer to [5].

Such a result shows that \( ||R_h^c||_* \) provides a reasonable estimate of \( |e_\perp|_1 \). In order to evaluate the quantity \( |e_d|_1 \), we assume that \( e_\perp \) is maintained so as to be negligible with respect to \( e_d \). Replacing \( e \) and \( v \) by \( e_d \) in the time evolution (4), we obtain:

\[
\frac{1}{2} \frac{d}{dt} ||e_d||^2 = -c(e_d, u_h, e_d) - Re^{-1} |e_d|_1^2 + R_h^m(e_d), 
\]

where \( ||\cdot|| \) denote the \( L^2(\Omega) \) norm in \( V \). Applying Kolmogorov's scaling theory [10], which conjectures the existence of a dissipation length scale below which viscosity dominates the dynamics, we are allowed to neglect the inertial term with respect to the viscous term, so that:

\[
\frac{1}{2} \frac{d}{dt} ||e_d||^2 \approx -Re^{-1} |e_d|_1^2 + R_h^m(e_d). 
\]

When the error "increases", i.e. \( d||e_d||^2/dt \geq 0 \), we can further get the bound:

\[
|e_d|_1 \leq Re ||R_h^m||_*.
\]

Therefore, the cost in evaluating such estimates amounts to calculating the norm of \( R_h^m \) and \( R_h^c \).
In [5], it is shown that the computation of \( \|R_h^n\| \) is exact and cheap, as for all \( t \in [0, T) \),

\[
\|R_h^n\| = \|\nabla \cdot u_h(t)\|_0.
\]  

On the other hand, the calculation of \( \|R_h^m\| \) is more demanding. However, low-cost iterative techniques have been developed in [5] to obtain approximate functions \( \psi_h \) in the space of bubble functions \( W^h, V^h \subseteq V^h \subseteq V \), such that

\[
\sqrt{(1 - \beta^2)(1 - \gamma^2)} \|R_h^m\| \leq \|\psi_h\|_1 \leq \|R_h^m\|.  
\]  

We refer to [11] for the definitions of the constants \( \beta \) and \( \gamma \).

**ADAPTIVE CONTROL OF THE ERROR**

The objective in *adaptive control* is to contain the error within some preset tolerances by reducing the local effects of the sources of errors \( R_h^n \) and \( R_h^m \). For time-dependent simulations, the initial mesh is constructed making sure that the error in the discretization of the initial condition \( u_0 \) is small. Then we expect the numerical errors (residuals) to grow in time where the mesh needs to be refined.

Let \( N_e \) be the number of elements in the finite element mesh. We therefore decompose the norm of the residuals into elementwise contributions \( \eta_{c,K} \) and \( \eta_{m,K} \) for each element \( \Omega_K, K = 1, \ldots, N_e \):

\[
\|R_h^n\|^2 = \sum_{K=1}^{N_e} \eta_{c,K}^2 = \sum_{K=1}^{N_e} \int_{\Omega_K} |\nabla \cdot u_h|^2 \, dx
\]

\[
\|R_h^m\|^2 \approx \sum_{K=1}^{N_e} \eta_{m,K}^2 = \sum_{K=1}^{N_e} \int_{\Omega_K} \nabla \psi_h : \nabla \psi_h \, dx.
\]

The ultimate goal is to control the global relative errors

\[
E_c = \frac{|e_{1,1}|}{|u_h|_1}, \quad E_m = \frac{|e_d|}{|u_h|_1},
\]

within some user-prescribed tolerances \( C^c \) and \( C^m \) such as:

\[
E_c \approx \|R_h^n\| / |u_h|_1 \leq C^c, \quad E_m \leq \|R_h^m\| / |u_h|_1 \leq C^m.
\]

In [12], we devised a strategy where the targeted tolerances \( C^c \) and \( C^m \) were used at the element level to guarantee the global relative errors to be smaller than these tolerances. The drawback of this method was to overcontrol the numerical errors with regard to the targeted tolerances. Alternatively, we propose here to refine a given percentage of those elements for which the quantities \( \eta_{c,K} \) and \( \eta_{m,K} \) contribute the most to the global residuals only when the relative global errors \( E_c \) and \( E_m \) exceed the tolerances \( C^c \) and \( C^m \). The main advantage in our approach is the possibility to select two distinct values for the tolerances \( C^c \) and \( C^m \) in order to control the errors \( e_{c,1} \) and \( e_d \) accordingly. As a matter of fact, \( C^c \) needs to be smaller than \( C^m \), firstly to carefully control the component \( e_{c,1} \), as it is responsible for generating *undesirable numerical instabilities* in the flows, and, secondly, to obtain better estimates of \( e_d \). On the other hand, the perturbation \( e_d \), comparable to perturbations in experimental flows, requires less precision in its control.
NUMERICAL EXPERIMENTS

The strategy for error estimation and control is illustrated in the case of the two-dimensional channel flow past a cylinder. It is well-known that such a flow develops into a periodic vortex shedding as it becomes unstable to unsymmetric perturbations past a critical value of the Reynolds number, which is defined here as $Re = U_c d / \nu$ where $d$ is the diameter of the cylinder and $U_c$ the maximal velocity of the parabolic profile at the inflow. We select $Re = 100$ in the present experiment. The flow domain $\Omega$ is initially discretized into the 160-element mesh shown in Figure 1, on which the initial state $u_0$ has been chosen as the Navier-Stokes solution computed at $Re = 1$.

![Figure 1: Geometry and initial mesh.](image)

![Figure 2: Evolution of the adapted mesh (close-up views around the cylinder) at (a) $t = 5$ (b) $t = 25$ (c) $t = 50$ (d) $t = 200$.](image)

The numerical solution is advanced in time using the Adams-Bashforth Cranck-Nicolson scheme (ABCN). The dimensionless timestep is fixed to the value $\Delta t = 0.01$ so that the errors due to the time discretization are kept small. We show in Figure 2 close-up views of the adapted mesh.
around the cylinder at various times of the simulation. We remark, however, that no adaptation was performed in the time period $[100, 200]$ as the flow had reached the permanent periodic regime. The tolerances $G_c$ and $G_m$ are chosen as functions of time in order to allow for larger numerical errors during the transients. The estimated relative global errors $E_c$ and $E_m$ are eventually reduced to about 5% and 10.5% respectively.

**SUMMARY AND CONCLUSIONS**

We have presented a fully automatic adaptive strategy for the control of the numerical error for the time-dependent Navier-Stokes equations. It has been experimented for the simulation of a channel flow past a cylinder in the periodic regime, for which a numerical solution has been controlled within some preset error tolerances. The error estimation approach is based on the computation of the norm of two residuals, which are related to two components of the velocity vector. We emphasize that it is applicable to flows at any given Reynolds number and work is currently in progress to apply the adaptive strategy for the simulation of various flow regimes.

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**References**


