Local A Posteriori Error Estimators for Variational Inequalities

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Local a posteriori error estimators for finite element approximation of variational inequalities are derived. These are shown to provide upper bounds on the discretization error. Numerical examples are given illustrating the theoretical results. © 1993 John Wiley and Sons, Inc.

I. INTRODUCTION

In recent years there has been considerable interest in the development of efficient adaptive algorithms for the finite element approximation of boundary value problems. At the heart of any adaptive algorithm lies some method of assessing the accuracy of the current approximation. The decision of whether further refinement of the approximation is necessary is based on the estimate of the discretisation error. If further refinement is to be performed then the local estimate of the error is used as a guide as to how the refinement might be accomplished most efficiently.

Several error estimation techniques have been proposed and used successfully for the solution of boundary value problems. However, for nonlinear problems, the development of reliable schemes has been slow.

In the current work we consider a class of variational inequalities that can be used to describe the flow of an incompressible inviscid fluid through an unsaturated porous medium [1,2] or the unilateral contact of an elastic body [3,4]. We derive local a posteriori error estimator for assessing the error in a finite element approximation. These schemes represent a generalization of the element residual methods [5-7], which have proved to be among the most satisfactory schemes for linear problems. It is shown that the error estimators give upper bounds on the discretization error. Numerical examples are provided illustrating the theory and showing that the bounds are not at all pessimistic.

The article is organized as follows: In Sec. II, a model problem and its discretization are introduced. Some preliminary considerations on error estimation are made in Sec. III. The notation needed for the analysis is established in Sec. IV and in Sec. V a variational
principle for the error is developed which requires only local regularity. In Secs. VI and VII, the local problems for the error are dealt with and the main theorem giving upper bounds on the discretization error is proved. Three supporting numerical examples are given in Sec. VIII.

II. MODEL PROBLEM

Let $\Omega \subset \mathbb{R}^2$ be an open-bounded domain with smooth boundary $\partial \Omega$. For given $f$ and $\psi$ we consider the model problem:

find $u \in \mathcal{H}$ such that

$$a(u, v - u) \geq (f, v - u), \quad \forall \ v \in \mathcal{H},$$

where

$$\mathcal{H} = \{ v \in H^1(\Omega): v \geq \psi \text{ a.e. on } \Omega \text{ and } v = g \text{ a.e on } \partial \Omega \},$$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f, v) = \int_{\Omega} f(x)v(x)\, dx.$$  \hfill (2) \hfill (3)

We shall assume that $u$ and $\nabla u$ are continuous in the interior of the domain $\Omega$.

Let $\{\Omega_k\}$ be a partition of $\Omega$ into the union of a finite number of triangular and quadrilateral subdomains. Let $h_k$ denote the diameter of $\Omega_k$ and $h = \max h_k$. Let us denote the partitioning $\{\Omega_k\}$ by $\mathcal{T}^h$. The partitioning is assumed to satisfy the regularity conditions:

R1 $\bigcup_{k} \Omega_k = \bar{\Omega}$.

R2 $\Omega_k \cap \Omega_L$ is either empty or a common node or a common edge of $\Omega_k$ and $\Omega_L$.

R3 There exists a ball of radius $\rho h_k$ (where $\rho$ is a positive constant independent of $h_k$) with respect to which $\Omega_k$ is star shaped.

Let $V^h$ denote the space of functions continuous on the closure $\bar{\Omega}$ which are piecewise polynomial on each subdomain of $\mathcal{T}^h$. Let $\mathcal{H}^h = \mathcal{H} \cap V^h$, then the discretized version of (1) is

find $u_h \in \mathcal{H}^h$ such that

$$a(u_h, v_h - u_h) \geq (f, v_h - u_h), \quad \forall \ v_h \in \mathcal{H}^h.$$ \hfill (4)

It is known (see [1]) that if $\psi, f$, and $g$ are sufficiently smooth and $\psi \leq g$ on $\partial \Omega$, then there exists a unique solution of both (1) and (4). The problem in which we are interested is that of estimating numerically the discretization error $e = u - u_h$ measured in the energy norm defined by

$$\|v\|_e = \sqrt{a(v, v)}.$$ \hfill (5)

III. A POSTERIORI ERROR ANALYSIS

Let $\mathcal{W}^h$ denote the convex set

$$\mathcal{W}^h = \{ w: w = v - u_h, \text{ for some } v \in \mathcal{H} \}, \quad \text{for some } v \in \mathcal{H}.$$  \hfill (6)
or equally well
\[ \mathcal{W}^h = \{ w \in H^1(\Omega) : w + u_h \geq \psi \text{ a.e. in } \Omega \text{ and } w + u_h = 0 \text{ a.e. on } \partial \Omega \} . \] (7)

Obviously \( e \in \mathcal{W}^h \); in fact, from (1) it follows that the discretization error \( e \) is uniquely characterized to be the solution of the problem:

find \( e \in \mathcal{W}^h \) such that
\[ a(e, w - e) = (f, w - e) - a(u_h, w - e), \quad \forall \ w \in \mathcal{W}^h. \] (8)

The existence of a unique solution of (8) follows immediately from the existence and uniqueness of \( u \). In principle, (8) may be solved explicitly once \( u_h \) has been calculated. Obviously, this is an impractical proposition. In the present work, we shall use (8) as a vehicle in deriving an element residual-type method for estimating the error.

An alternative and equivalent form of (8) is

find \( e \in \mathcal{W}^h \) such that
\[ J(e) = J(w), \quad \forall \ w \in \mathcal{W}^h, \] (9)

where
\[ J(w) = \frac{1}{2} a(w, w) - (f, w) + a(u_h, w). \] (10)

Now, since \( \mathcal{K}^h \subset \mathcal{K} \), there follows from (1)
\[ \| e \|_{\mathcal{W}}^2 = a(e, e) \]
\[ = a(u, e) - a(u_h, e) \]
\[ \leq (f, e) - a(u_h, e) \]
\[ = - J(e) + \frac{1}{2} \| e \|_{\mathcal{W}}^2, \] (11)

and hence from (9) we obtain
\[ \inf_{w \in \mathcal{W}^h} J(w) = J(e) \leq - \frac{1}{2} \| e \|_{\mathcal{W}}^2. \] (12)

IV. NOTATION

Let \( m \in \mathbb{N} \); then the usual Sobolev space and norm are denoted by \( H^m(\Omega) \) and \( \| \cdot \|_{m, \Omega} \), respectively. In addition, we define the broken space \( H^m(\mathcal{T}^h) \) to be
\[ H^m(\mathcal{T}^h) = \{ v \in L^2(\Omega) : v|_{\Omega_K} \in H^m(\Omega_K), \quad \forall \ \Omega_K \in \mathcal{T}^h \} \] (13)
equipped with the norm
\[ \| v \|_{m, \mathcal{T}^h}^2 = \sum_{\Omega_K \in \mathcal{T}^h} \| v \|_{m, \Omega_K}^2. \] (14)

Let \( E \) denote the collection of arcs which form the edges of the elements \( \{ \Omega_K \} \) and let \( E_B \) denote the subset consisting of arcs lying on the boundary \( \partial \Omega \). The remaining arcs are denoted by \( E_I = E \setminus E_B \).

Given \( \Omega_K \in \mathcal{T}^h \), with each edge of \( \Omega_K \), we associate a unit normal vector \( n \). The direction of \( n \) is chosen so that \( n \) points outward from the subdomain with the largest index while on the boundary \( \partial \Omega \), it is the usual outward unit normal.
Suppose that $\Omega_L$ and $\Omega_R$ where $L > R$ intersect along a common edge $e$. Suppose $v \in H^2(\mathcal{T}^h)$. Let the restriction of $v$ to $\Omega_L$ be denoted by $v_L$. We define $[\cdot], \langle \cdot \rangle$ as follows:

$$[v] = \begin{cases} v_L - v_R & e \in E_I \\ v_L & e \in E_B, \end{cases}$$

$$\langle v \rangle = \begin{cases} \frac{1}{2} (v_L + v_R) & e \in E_I \\ \frac{1}{2} v_L & e \in E_B, \end{cases}$$

$$\left[ \frac{\partial v}{\partial n} \right] = \begin{cases} \frac{\partial v_L}{\partial n} - \frac{\partial v_R}{\partial n} & e \in E_I \\ \frac{\partial v_L}{\partial n} & e \in E_B, \end{cases}$$

$$\left\langle \frac{\partial v}{\partial n} \right\rangle = \begin{cases} \frac{1}{2} \left( \frac{\partial v_L}{\partial n} + \frac{\partial v_R}{\partial n} \right) & e \in E_I \\ \frac{1}{2} \frac{\partial v_L}{\partial n} & e \in E_B, \end{cases}$$

where the restrictions to the edges are taken in the sense of traces. For $v, w \in H^2(\mathcal{T}^h)$ we define

$$(v, w)_E = \sum_{e \in \mathcal{E}} \int_E v(s)w(s)ds$$

and $(\cdot, \cdot)_E$ is defined in a similar way. It is easily seen [5, 8] that if $v, w \in H^2(\mathcal{T}^h)$ then

$$\sum_{K \in \mathcal{S}_h} \left( \frac{\partial v_K}{\partial n_K}, w_K \right)_{\partial K} = \left( \left[ \frac{\partial v}{\partial n} \right], \langle w \rangle \right)_E + \left( \left\langle \frac{\partial v}{\partial n} \right\rangle, [w] \right)_E$$

where $n_K$ is the unit outward normal on $\partial K$. Let $\mathcal{W}^h(\mathcal{T}^h)$ and $\mathcal{W}_0^h(\mathcal{T}^h)$ be defined by

$$\mathcal{W}^h(\mathcal{T}^h) = \{ w \in H^1(\mathcal{T}^h): w + u_h \equiv \psi \text{ a.e. in } \Omega \text{ and } w + u_h = 0 \text{ a.e. on } \partial \Omega \}$$

and

$$\mathcal{W}_0^h(\mathcal{T}^h) = \{ w \in \mathcal{W}^h(\mathcal{T}^h): [w] = 0 \text{ a.e. on } E \}.$$ 

It follows that the following inclusions hold:

$$\mathcal{W}_0^h(\mathcal{T}^h) \subset \mathcal{W}^h \subset \mathcal{W}^h(\mathcal{T}^h).$$

Let $\mathcal{M}$ denote the space of continuous linear functionals $\mu$ on $\mathcal{W}^h$ defined by

$$\mu(w) = (\mu, [w])_E.$$

Observe that the operator $[\cdot]$ is continuous on $\mathcal{W}^h$ since the Trace Theorem [9] shows that the Trace Operator $\gamma: H^1(\Omega_K) \to H^{1/2}(\partial \Omega_K)$ is continuous. Later we choose $\mu = \langle du_h/\partial n \rangle$, which is obviously an element of $\mathcal{M}$, making a detailed characterization of $\mathcal{M}$ unnecessary.

V. RELAXED VARIATIONAL PRINCIPLE

Let $\mathcal{L}: \mathcal{W}^h(\mathcal{T}^h) \times \mathcal{M} \to \mathbb{R}$ denote the Lagrangian functional

$$\mathcal{L}(w, \mu) = J(w) - (\mu, [w])_E$$

(25)
where \( J \) is evaluated in an element by element sense. It is easily seen that

\[
\inf_{\mu \in \mathcal{M}^h} \sup_{\omega \in \mathcal{W}^h(\mathcal{T}^h)} \mathcal{L}(\omega, \mu) = \begin{cases} J(\omega) & \text{if } \omega \in \mathcal{W}^h_0(\mathcal{T}^h) \\ +\infty & \text{otherwise}. \end{cases}
\]

(26)

Therefore

\[
\inf_{\omega \in \mathcal{W}^h(\mathcal{T}^h)} \sup_{\mu \in \mathcal{M}^h} \mathcal{L}(\omega, \mu) = \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(\omega).
\]

(27)

By assumption \([u] = 0\) almost everywhere and so \([e] = 0\) almost everywhere. Therefore \( e \in \mathcal{W}^h_0(\mathcal{T}^h) \), and, in addition, using (12) and (23) gives

\[
\inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(\omega) = \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(e).
\]

(28)

Consequently

\[
\inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(\omega) = J(e) = \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(\omega).
\]

(29)

It now follows from (12), (27), and (29) that

\[
-\frac{1}{2} \| e \|_k^2 \geq J(e) = \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} J(\omega) = \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} \sup_{\mu \in \mathcal{M}^h} \mathcal{L}(\omega, \mu) \geq \sup_{\mu \in \mathcal{M}^h} \inf_{\omega \in \mathcal{W}^h(\mathcal{T}^h)} \mathcal{L}(\omega, \mu),
\]

(30)

and so we obtain

\[
-\frac{1}{2} \| e \|_k^2 \geq \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} \mathcal{L}(\omega, \mu), \quad \forall \mu \in \mathcal{M}^h.
\]

(31)

The significance of this result in contrast with (12) is that now \( \omega \) need only be an element of the broken space \( \mathcal{W}^h(\mathcal{T}^h) \). This means that feasible choices of \( \omega \) can be generated locally on each element, whereas the constraint \( \omega \in H^1(\Omega) \) prevented this in (12).

VI. LOCALIZATION

By construction, \( u_h \) is smooth on each subdomain \( \Omega_K \). Therefore

\[
\mu = \frac{\partial u_h}{\partial n} \in \mathcal{M}^h
\]

(32)

and from (31)

\[
-\frac{1}{2} \| e \|_k^2 \geq \inf_{\omega \in \mathcal{W}^h_0(\mathcal{T}^h)} \mathcal{L}(\omega, \left( \frac{\partial u_h}{\partial n} \right)).
\]

(33)

Suppose that \( \omega \in \mathcal{W}^h(\mathcal{T}^h) \). Applying Green's formula on each subdomain \( \Omega_K \) gives, with the aid of (20) (since \( u_h \mid_{\Omega_K} \) is smooth on \( \Omega_K \)),

\[
a(u_h, w) = \sum_{\Omega_K \in \mathcal{T}_h} \int_{\Omega_K} \nabla u_h \cdot \nabla w \, dx
\]

\[
= \sum_{\Omega_K \in \mathcal{T}_h} \left( -\int_{\Omega_K} \Delta u_h w \, dx + \oint_{\partial \Omega_K} \frac{\partial u_h}{\partial n_K} (s) w(s) \, ds \right)
\]

\[
= \sum_{\Omega_K \in \mathcal{T}_h} \left( -\int_{\Omega_K} \Delta u_h w \, dx \right) + \left( \left[ \frac{\partial u_h}{\partial n} \right], [w] \right)_E + \left( \left[ \frac{\partial u_h}{\partial n} \right], (w) \right)_E.
\]

(34)
VII. ANALYSIS OF LOCAL PROBLEMS

Consequently,

$$\mathcal{L}\left(w, \left(\frac{\partial u_h}{\partial n}\right)\right) = \sum_{\Omega_K \in \mathcal{T}^h} J_K(w),$$  \hspace{1cm} (35)

where $J_K: H^1(\Omega_K) \to \mathbb{R}$ is defined to be

$$J_K(w) = \frac{1}{2} \int_{\Omega_K} \nabla w \cdot \nabla w \, dx - \int_{\Omega_K} r_K(x)w(x)\, dx$$

$$+ \frac{1}{2} \int_{\partial \Omega_K} \left[ \frac{\partial u_h}{\partial n} \right](s)w(s)\, ds$$  \hspace{1cm} (36)

and

$$r_K(x) = \Delta u_h(x) + f(x), \quad \forall \ x \in \Omega_K$$  \hspace{1cm} (37)

is the residual on the subdomain $\Omega_K$. Summarizing these results gives:

**Theorem 1.** Let $J_K, \mathcal{W}^h(\mathcal{T}^h)$ be as above. Then

$$\|e_h\|_E^2 \leq -2 \inf_{w_h \in \mathcal{W}^h(\mathcal{T}^h)} \sum_{\Omega_K \in \mathcal{T}^h} J_K(w).$$  \hspace{1cm} (38)

The key feature of Theorem 1 is that in order to obtain a bound on the discretization error, one need only construct $w_K^* \in \mathcal{W}^h(\mathcal{T}^h)$. The set $\mathcal{W}^h(\mathcal{T}^h)$ does not impose any interelement restrictions on the choice of $w$. Moreover, by choosing $\mu$ suitably, the Lagrangian functional has decoupled into the sum of local functionals associated with each element. Together, these features allow one to deal with a series of local problems of the form:

find $w_K^* \in \mathcal{W}^h_K$ such that

$$J_K(w_K^*) \leq J_K(w_K), \quad \forall \ w_K \in \mathcal{W}^h_K$$  \hspace{1cm} (39)

where, with a slight abuse of notation, we define

$$\mathcal{W}^h_K = \{w_K \in H^1(\Omega_K): w_K + u_h \geq \psi \quad \text{a.e. on} \quad \Omega_K$$

and $w_K + u_h = 0 \quad \text{a.e. on} \quad \partial \Omega_K \cap \partial \Omega\}.$  \hspace{1cm} (40)

The main advantage associated with dealing with local problems is that the computational cost is negligible in comparison with the expense entailed in obtaining $u_h$. This is in contrast to the related ideas presented in [3, 10], in which it is necessary to solve a problem of comparable complexity to (4) to obtain an error estimator.

VII. ANALYSIS OF LOCAL PROBLEMS

We now focus attention on the local problem (39) associated with each subdomain $\Omega_K$. It is convenient to consider the following cases separately:

- $\Omega_K$: $\partial \Omega \cap \partial \Omega_K$ is nonempty. That is, the subdomain $\Omega_K$ lies on the boundary of $\Omega$. In this case the local problem (39) is always well posed owing to the condition

$$w_K: w_K + u_h = 0 \quad \text{a.e. on} \quad \partial \Omega \cap \partial \Omega_K.$$

Standard arguments [3] show that $J_K$ is continuous and coercive on $\mathcal{W}^h_K$, guaranteeing the existence of a unique solution $w_K^* \in \mathcal{W}^h_K$. 

\[ \Omega_K : \partial \Omega \cap \partial \Omega_K \text{ is empty. In this case the situation is less straightforward. In [11] it is shown that the local problem has a unique solution if and only if the condition} \]
\[ \int_{\Omega_K} r_k(x) dx - \frac{1}{2} \oint_{\partial \Omega_K} \left[ \frac{\partial u_h}{\partial n} \right] (s) ds < 0 \quad (41) \]

is satisfied. If the inequality is not strict then it is shown that there is a unique solution determined up to the addition of any sufficiently large constant. Let us assume for the time being that

\[ \int_{\Omega_K} r_k(x) dx - \frac{1}{2} \oint_{\partial \Omega_K} \left[ \frac{\partial u_h}{\partial n} \right] (s) ds \leq 0 \quad (42) \]

holds on all subdomains \( \Omega_K : \partial \Omega \cap \partial \Omega_K \text{ is empty. We shall discuss this assumption below.} \)

The following dual variational principle is proved in [11]:

**Theorem 2.** Let \( J_K, W^h_K \text{ and } w^*_K \in W^h_K \text{ be as above. Let} \)

\[ \mathcal{D}^h_K = \{ p_K \in L^2(\Omega_K) \times L^2(\Omega_K) : \int_{\Omega_K} \{ p_K \cdot \nabla v_K - r_k(x) v_K \} dx \]
\[ + \int_{\partial \Omega_K} \left\{ n_K \cdot p_K + \frac{1}{2} \left[ \frac{\partial u_h}{\partial n} \right] \right\} v_K ds \geq 0 \quad (43) \]

for all \( v_K \geq 0 \in H^1(\Omega_K) \text{ and define } G_K: \mathcal{D}^h_K \to \mathbb{R} \text{ to be} \)

\[ G_K(p_K) = -\frac{1}{2} \int_{\Omega_K} p_K \cdot p_K dx + \int_{\Omega_K} \left[ \nabla \cdot p_K + r_k(x) \right] (u_h - \psi) dx \]
\[ + \oint_{\partial \Omega_K} \left( \frac{1}{2} \left[ \frac{\partial u_h}{\partial n} \right] + n_K \cdot p_K \right) (u_h - \psi) ds. \quad (44) \]

Then \( \nabla w^*_K \in \mathcal{D}^h_K \) and

\[ J_K(w^*_K) = G_K(\nabla w^*_K) \geq G_K(p_K) \quad (45) \]

for all \( p_K \in \mathcal{D}^h_K \).

Using Theorem 1 and Theorem 2 we obtain

**Theorem 3.** Let \( \mathcal{D}^h_K \text{ and } G_K \text{ be as above. Then} \)

\[ \| \varepsilon \|_E^2 \leq -2 \sum_{\Omega_K \in \mathcal{T}} G_K(p_K) \quad (46) \]

for all \( p_K \in \mathcal{D}^h_K. \)

This result is potentially much more useful than Theorem 1 since it gives an upper bound for any choice of \( p \in \Pi_K \mathcal{D}^h_K \), whereas in Theorem 1 it is necessary to find the minimizers \( w^*_K \). Our procedure will be to approximate the optimal choice of \( p_K \) by calculating a finite element approximation of \( w^*_K \) using the primal problem, and then to take the gradient of our approximation to be the choice of \( p_K \). Of course, this method of choosing \( p_K \) does not guarantee that we have an element of the set \( \mathcal{D}^h_K \). However, if we solve the local problem with sufficient accuracy then we would obtain \( p_K = \nabla w^*_K \in \mathcal{D}^h_K \). In
our numerical examples, we shall find that even a fairly crude discretization of the primal
yields satisfactory results.

The theory depends on the assumption (42). In practice, this condition will, in general,
be violated on some subdomains. Therefore, when we implement the method, we shall
check to see if (42) holds by calculating

\[ \delta_K = \int_{\partial K} r_K(x) dx - \frac{1}{2} \int_{\partial K} \left[ \frac{\partial u_h}{\partial n} (s) \right] ds. \]  

(47)

If it is found that \( \delta_K > 0 \), then we modify the residual \( r_K(x) \) to \( r'_K(x) \), where

\[ r'_K(x) = r_K(x) - \frac{\delta_K}{\text{meas}(\Omega_K)}. \]  

(48)

This modified residual is then used in the functional \( J_K \), which will now satisfy the condi-
tion (42). In practice, one finds that \( \delta_K \) is an order of magnitude smaller than

\[ \int_{\partial K} \left[ \frac{\partial u_h}{\partial n} (s) \right] ds \]

and

\[ \int_{\partial K} r_K(x) dx. \]

Owing to the above analysis, we are at liberty to choose the term involving \( \delta_K \) to be as
we wish. The justification is provided by Theorem 3, which assures us an upper bound no
matter how we select \( p_K \). By selecting \( p_K \) to be the gradient of \( \phi_K \), where \( \phi_K \) solves the local
variational inequality with the fictitious force \( \delta_K \) appended, we hope to obtain a good esti-
mate of the error. The main result which we have obtained is the upper bound (In the dis-
cretization error. Our discussion does not attempt to quantify the effectiveness of the
resulting estimate, but the numerical examples in the following section give some indica-
tion of the quality. Equally well, we have not considered the local behavior of the error
estimate. In fact, there are no known results on the local behavior of the error estimate
even for the simplest model problems in one dimension.

VIII. NUMERICAL EXAMPLES

In order to illustrate the foregoing theory, we describe the results for three particular
problems. In each case the discretization scheme consisted of a piecewise affine approxi-
mation on uniform linear triangular elements.

The local problems (39) was approximated by further subdividing each subdomain into
uniform linear triangular subdomains and constructing a piecewise affine approximation
to the true solution of a local problem. In order to assess how many subdivisions are
needed to obtain a sufficiently accurate resolution of the local problem, results were ob-
tained for refinement of each subdomain into 1, 4, or 16 smaller triangles. The first case
corresponds to no refinement and is possible due to the local nature of the problems.

The quality of the error estimator is assessed by means of the effectivity index, defined
to be the ratio of the estimated to the true error. Theorem 3 predicts that the effectivity
index should be greater than unity if we solve the local problems exactly.
Example 1

\[ \psi = 0; \quad f(x) = 1; \quad \Omega := (0, \frac{1}{2}) \times (0, 1). \]

The true solution is 0 if \((x + 1)^2 + y^2 \geq 2\), otherwise

\[ u(x,y) = \frac{1}{4}[(x + 1)^2 + y^2] - \frac{1}{2} - \frac{1}{2} \ln \left[ (x + 1)^2 + y^2 \right]. \quad (49) \]

and \(g = u\) on \(\partial \Omega\). Table I shows the results for Example 1.

Example 2

\[ \psi = 0; \quad f = -1; \quad \Omega = (0, l) \times (0, h_1). \]

\[ g(x,y) = \begin{cases} \frac{1}{4}(h^2 - qx) & 0 \leq x \leq l, \quad y = 0 \\ \frac{1}{4}(h_1 - y)^2 & x = l, \quad 0 < y \leq h_1 \\ \frac{1}{4}(h_2 - y)^2 & x = 0, \quad 0 < y \leq h_1 \\ 0 & 0 < x < l, \quad y = h_1 \end{cases} \quad (50) \]

where \(h_1 = 10, h_2 = 2, l = 5, q = (h_1^2 - h_2^2)/(2l)\), and

\[ (h - y)_+ = \begin{cases} h - y & \text{if } h \geq y, \\ 0 & \text{otherwise}. \end{cases} \quad (51) \]

This problem arises in the modeling of the seepage of an incompressible, inviscid fluid through an unsaturated rectangular dam. The true solution of this problem is not readily accessible, so the effectivity of the error estimator is based on comparison with an approximate solution obtained using a relatively fine discretisation (of mesh size \(\frac{1}{2}\)). Table II and Table III show the effectivity indices and the accuracy of the approximate solutions, respectively.

**TABLE I.** Effectivity indices for Example 1.

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**TABLE II.** Effectivity indices for Example 2.

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<td>1.2265</td>
<td>1.2819</td>
<td>1.3024</td>
</tr>
</tbody>
</table>

**TABLE III.** Accuracy of solutions to Example 2.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Energy</th>
<th>Error</th>
<th>Accuracy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40.32</td>
<td>4.624</td>
<td>11.5</td>
</tr>
<tr>
<td>1/2</td>
<td>40.14</td>
<td>2.412</td>
<td>6.0</td>
</tr>
<tr>
<td>1/8</td>
<td>40.08</td>
<td>0.629</td>
<td>1.6</td>
</tr>
</tbody>
</table>
Example 1

\[ \psi = 0; \quad f(x) = 1; \quad \Omega = (0, \frac{1}{2}) \times (0, 1). \]

The true solution is \( 0 \) if \((x + 1)^2 + y^2 \geq 2\), otherwise

\[ u(x, y) = \frac{1}{4}((x + 1)^2 + y^2) - \frac{1}{2} \ln\left(\frac{1}{4}((x + 1)^2 + y^2)\right). \]  
(49)

and \( g = u \) on \( \partial \Omega \). Table I shows the results for Example 1.

Example 2

\[ \psi = 0; \quad f = -1; \quad \Omega = (0, 1) \times (0, 1). \]

\[ g(x, y) = \begin{cases} 
\frac{1}{2} h_1^2 - qx & 0 \leq x \leq l, \quad y = 0 \\
\frac{1}{2} (h_1 - y)^2 & x = l, \quad 0 < y \leq h_1 \\
\frac{1}{2} (h_2 - y)^2 & x = 0, \quad 0 < y \leq h_1 \\
0 & 0 < x < l, \quad y = h_1 
\end{cases} \]  
(50)

where \( h_1 = 10, \quad h_2 = 2, \quad l = 5, \quad q = (h_1^2 - h_2^2)/(2l) \), and

\[ (h - y)_+ = \begin{cases} 
h - y & \text{if } h \geq y, \\
0 & \text{otherwise.} 
\end{cases} \]  
(51)

This problem arises in the modeling of the seepage of an incompressible, inviscid fluid through an unsaturated rectangular dam. The true solution of this problem is not readily accessible, so the effectivity of the error estimator is based on comparison with an approximate solution obtained using a relatively fine discretisation (of mesh size \( \frac{1}{4} \)). Table II and Table III show the effectivity indices and the accuracy of the approximate solutions, respectively.

**TABLE I.** Effectivity indices for Example 1.

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Refinements</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
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<td>0.6841</td>
</tr>
<tr>
<td>1/8</td>
<td>1.1091</td>
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<tr>
<td>1/16</td>
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<td>1/32</td>
<td>1.3819</td>
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</tbody>
</table>

**TABLE II.** Effectivity indices for Example 2.

<table>
<thead>
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<th>Refinements</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>1</td>
<td>1.0207</td>
</tr>
<tr>
<td>1/2</td>
<td>1.2265</td>
</tr>
</tbody>
</table>

**TABLE III.** Accuracy of solutions to Example 2.

<table>
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<tr>
<th>Mesh size</th>
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<td>40.08</td>
<td>0.629</td>
<td>1.6</td>
</tr>
</tbody>
</table>
The numerical examples support the foregoing analyses, with effectivity indices of at least unity whenever the local problems are solved sufficiently accurately. Examining the results, one is led to conclude that subdividing each subdomain into four smaller subdomains gives satisfactory results. The numerical results indicate the upper bounds obtained are not all pessimistic.

The question of local behavior of the error estimators has not been discussed. In the numerical examples, it is found that the behavior of the local effectivity indices is rather erratic when the local problems are solved using no subdivisions. However, if further subdivisions are used, the behavior appears to stabilize.

### IX. CONCLUSIONS

The numerical examples support the foregoing analyses, with effectivity indices of at least unity whenever the local problems are solved sufficiently accurately. Examining the results, one is led to conclude that subdividing each subdomain into four smaller subdomains gives satisfactory results. The numerical results indicate the upper bounds obtained are not all pessimistic.

The question of local behavior of the error estimators has not been discussed. In the numerical examples, it is found that the behavior of the local effectivity indices is rather erratic when the local problems are solved using no subdivisions. However, if further subdivisions are used, the behavior appears to stabilize.

<table>
<thead>
<tr>
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<tr>
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<tr>
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<td>1.1089</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Mesh size</th>
<th>Energy</th>
<th>Error</th>
<th>Accuracy (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.49</td>
<td>2.853</td>
<td>18.4</td>
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<tr>
<td>1/2</td>
<td>15.85</td>
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<tr>
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<td>2.4</td>
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</tbody>
</table>

<table>
<thead>
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<th>Mesh size</th>
<th>Refinements</th>
<th>Accurate discharge</th>
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</thead>
<tbody>
<tr>
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<td>1.1279</td>
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<tr>
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<td>1.3327</td>
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</table>
divisions are performed then the results improve considerably; once again the case of subdivision into four triangles proves to be the most satisfactory.

References