PARADIGMATIC ERROR CALCULATIONS FOR ADAPTIVE
FINITE ELEMENT APPROXIMATIONS OF
CONVECTION DOMINATED FLOWS

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Abstract: New error estimation techniques are described for two-dimensional convection-dominated flows. Examples of applications to linear convection problems are given. The methods are based on discontinuous finite-element techniques. Results show that conventional interpolation methods may lead to gross underestimates of actual error and be inadequate for use in adaptive methods.

1 INTRODUCTION

Adaptive finite element methods may have a great impact on computational fluid dynamics; they give credence to the concept of optimal CFD algorithms which can deliver the best possible numerical results for the least computational effort, and they have proved to be very effective in many complex flow simulations (see, e.g., [5, 6, 13, 14, 15] or the survey [12]).

The success of adaptive methods hinges on the availability of a reliable error estimator — a technique for using a computed solution to obtain an a-posteriori estimate of error in each element in an appropriate norm. When the estimated error exceeds a preassigned tolerance, the mesh is refined (an h-refinement) and/or the polynomial degree is increased (a p-enrichment). Thus, adaptive methods not only provide for a systematic reduction of local error but also can yield an estimate of the error itself, which can be a very useful indication of the overall reliability of the calculation.

To date, several methods of error analysis have been proposed for linear elliptic problems [1, 2, 3] and some techniques for parabolic problems and incompressible Navier–Stokes equations have been tested [5, 10, 16], but the overwhelming majority of adaptive codes concerned with solving practical flow problems employ simple interpolation results (e.g., [6, 8, 9, 14, 15]). The behavior of error in convection-dominated flows has apparently not been addressed in the literature.

In the present paper, we describe some preliminary results on the estimation of error in finite element approximations of linear convection problems in two dimensions. The
study is part of a broader study of error estimation for general flow simulations that will appear in a forthcoming paper. The principal issues of interest uncovered or developed in the present study are:

- The standard interpolation methods [6, 8, 9, 14, 15] may be completely inadequate for error prediction in many situations in convection-dominated flows. They can yield estimates many orders-of-magnitude less than the actual error and, thus, cannot be used to drive an effective adaptive scheme.

- The discontinuous finite element method of Lesaint and Raviart [7] can be used as a basis for an effective error estimation scheme; such a scheme is developed in this note.

- The error predictor developed here can be used effectively in problems with varying smoothness in initial data, including problems with quite sharp gradients.

2 ERROR ESTIMATION FOR A MODEL HYPERBOLIC PROBLEM

For clarity of presentation, we shall confine our attention to the model convection problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} + a \cdot \nabla u &= f & \text{in } \Omega \subset \mathbb{R}^2 \\
u &= g & \text{on } \partial \Omega^- \\
u(x, y, 0) &= u_0(x, y) & \text{in } \Omega
\end{align*}
\]

(1)

Here \( u = u(x, y, t) \) is a scalar-valued function on \( \Omega \times (0, \infty) \), \( a = a(x, y) \) is a given background velocity field, \( f = f(x, y, t) \) a source term, \( u_0 \) the initial data, and \( g \) the inflow data on the inflow boundary \( \partial \Omega^- = \{(x, y) \in \partial \Omega : a \cdot n \leq 0 \text{ a.e.}\} \), and we assume a Lipschitz boundary \( \partial \Omega = \partial \Omega^- \cup \partial \Omega^+ \).

A weak form of (1), valid over the space-time domain

\[ D = \Omega \times (0, T) \]

is characterized as the following problem:

Find \( u \in W \) such that

\[
\int_D (u_t + a \cdot \nabla u - f)v \, d\Omega dt + \int_0^T \int_{\partial \Omega^-} [a \cdot n](u - g)v \, ds dt \]

\[ + \int_\Omega (u(\cdot, 0^+) - u_0(\cdot))v(\cdot, 0) \, d\Omega = 0 \quad \forall v \in W \]

(2)

wherein \( W \) is an appropriate space of admissible test functions, e.g., \( W = \{ v = v(x, y, t) \in L^\infty(H^1(\Omega), (0,T)) \} \).

It is possible to develop a "local" form of (2) valid for subdomains of \( D \). Let \( T_h \) denote a partition of \( \Omega \) into convex, almost disjoint subdomains \( K \) (eventually \( K \) will denote a finite element) and consider a partition of the time interval into time steps,

\[ 0 = t_0 < t_1 < \cdots < t_n < \cdots < t_N = T; \Omega = \bigcup_{K \in T_h} K \]
Then, over the slice $D_n = K \times (t_n, t_{n+1})$, the solution of (2) satisfies

$$
\int_{D_n} (u_1 + a \cdot \nabla u - f) v \, d\Omega dt + \int_{t_n}^{t_{n+1}} \int_{\partial K^-} |a \cdot n_K| (u^+ - u^-) v \, ds dt
$$

$$
+ \int_{K} (u(\cdot, t_{n}^+) - u(\cdot, t_n)) v(\cdot, t_{n}^+) \, d\Omega = 0 \quad \forall v \in W_K
$$

(3)

Here $\partial K^-$ is the inflow boundary to $K(\partial K^- = \{(x, y) \in \partial K; a \cdot n_K < 0 a.e.)$ and $n_K$ is the unit exterior normal to the subdomain boundary $\partial K$, and $W_K$ is a space of test functions with domain $D_n = K \times (t_n, t_{n+1})$.

Now suppose that a finite element (or finite difference) approximation $u^h$ to (2) (or (1)) has been obtained by some numerical scheme. Then it is clear that the approximation error $e^h = u - u^h$ satisfies the equality

$$
\int_{D_n} (e_1^h + a \cdot \nabla e^h) v \, d\Omega dt + \int_{t_n}^{t_{n+1}} \int_{\partial K^-} |a \cdot n| (u^+ - u^-) v \, ds dt
$$

$$
+ \int_{K} (u(\cdot, t_{n}^+) - u(\cdot, t_n)) v(\cdot, t_{n}^+) \, d\Omega = \int_{D_n} r^h v \, d\Omega dt \quad \forall v \in W_K
$$

(4)

where $r^h = f - u^h - a \cdot \nabla u^h$ is the local element residual. The test functions $v$ in (4) may possess jumps at $\partial K$.

The plan is now to construct an approximate solution $E^h$ of (4) over each space-time element. Thus, if $E^h = E^h(x, y, t)$ is an approximation of the error on $D$ and $E^h_K$ is its restriction over $K$, we write

$$
E^h_K = \sum_{i,j} E^h_{ij} \phi^i(t) \psi^j(x, y)
$$

with $\phi^i, \psi^j$ polynomial shape functions in $t$ and $(x, y)$, we introduce this function into (4) in place of $e^h$, and replace $v$ by $v^h = \phi^i(t) \psi^j(x, y)$ to obtain a system of local equations for $E^h_K$. With the approximation to the error thus determined, the local error can be computed in any norm.

### 3 THE STEADY-STATE CASE

In the steady-state case, (4) reduces to the local discontinuous finite element approximation of Lesaint and Raviart [7]. The discrete approximation of the local equation for error reduces to

$$
\int_K a \cdot \nabla E^h v^h \, d\Omega + \int_{\partial K^-} |a \cdot n_K|(E^h_+ - E^h_-) v^h \, ds = < r^h, v^h > \quad \forall v^h \in W^h_K
$$

(5)

where now

$$
< r^h, v^h > = \int_K (f - a \cdot \nabla u^h) v^h \, d\Omega + \int_{\partial K^-} |a \cdot n_K|(u^h_+ - u^h_-) v^h \, ds
$$

(6)

$$
E^h = \sum_{i} E^h_i \psi_i(x, y)
$$

(7)

and $W^h_K$ is a space of piecewise polynomials defined on $K$ which are generally discontinuous across $\partial K$.

The solution of (5) element-by-element is straightforward. We assign an ordering of the elements in $\Omega^h$, following [7], so as to trace the propagation of error along stream lines throughout the mesh. We compute the approximate inflow condition $u^h|_{\partial \Omega^-} = g^h$ and the corresponding inflow error $e^h|_{\partial \Omega^-} = g - g^h$. This defines $E^h$ on all elements intersecting the inflow boundary $\partial \Omega^-$. Element residuals in the inflow boundary elements are then computed and (5) is solved for each of these elements. With $E^h$ then determined for the
next set of adjoining elements, the solution can be advanced to these elements. Repeating
this process advances the calculation of $E^h$ throughout the mesh.

4 SAMPLE RESULTS

The approximate error $E^h$ computed using the above procedure is generally computed
using polynomial shape functions $v^h = \phi_j$ of higher order than those used to calculate $u^h$,
but this may be unnecessary.

It is interesting to compare error estimates computed in this way with those obtained
using interpolation estimates. The so-called interpolation estimates are based on finite
element interpolation theory; since

$$\| u - \hat{u}^h \|_{m,q,K} \leq \| u - u^h \|_{m,q,K} + \| u^h - \hat{u}^h \|_{m,q,K},$$

where $\hat{u}^h$ is the local interpolant of the solution $u$ and $\| \cdot \|_{m,q,K}$ denotes the $W^{m,q}(K)$-
Sobolev norm, and since (see Ciarlet [4] or Oden and Carey [11]) for quasi uniform refine-
ments,

$$\| u - \hat{u}^h \|_{m,q,K} \leq Ch_K^{\frac{1}{2}+\frac{1}{m+1}+\frac{1}{r}} \| u \|_{p+1,r,K},$$

$C$ = constant independent of $u$ or the mesh size $h_K$

$$\| \hat{u}^h \|_{p+1,r,K} = W^{p+1,r}(K) \text{ seminorm}$$

$$1 \leq q, r \leq \infty$$

then, if $\| u^h - \hat{u}^h \|_{m,q,K}$ is of higher order in $h_K$ than the interpolation bound, the local
error in the $W^{m,q}(K)$ norm is bounded by the local error indicator,

$$\theta_K = Ch_K^{\frac{1}{2}+\frac{1}{m+1}+\frac{1}{r}} \| u^h \|_{p+1,r,K},$$

provided $\| u^h \|_{p+1,r,K} = \| u \|_{p+1,r,K} + O(h_K^p)$. For example, for $m = 0$, $q = r = 2$, $p = 1$,
this reduces to the familiar estimate [8, 9, 15],

$$\| e^h \|_{L^2(K)} \leq \theta_K = Ch_K^{\frac{1}{2}} \| u^h \|_{2,2,K}$$

The constant $C$ can be estimated in a straightforward calculation of the “master” element
$K$ used to generate the mesh.

Let us now compare the quality of this crude error estimator with that derived in the
preceding sections.

We let $\alpha = (2, 2)$, $\Omega = [0, 64]^2$ and by imposing different types of inflow boundary
conditions $g$ or right-hand side forcing functions $f$, we obtain steady-state solutions with
varying degree of smoothness.

Example 1. Smooth Sinusoidal Solution

We first consider the case in which the steady-state solution is a smooth sinusoidal
wave, namely:

$$u(x, y) = \sin \left( \frac{\pi y}{32} \right)$$

with $\bar{y} = y - x$

We let the inflow conditions $g$ coincide with the exact solution and let $f = 0$. An approximate
steady-state solution was obtained on a uniform $32 \times 32$ grid by integrating a set of
initial data using a one-step finite-element Lax–Wendroff algorithm. The steady-state error
was estimated using interpolation-type estimates, using both bilinear and biquadratic
local approximations of the equation of the error respectively. We use the notations:
\[ DG(1) = \text{bilinear error estimator} \]
\[ DG(2) = \text{biquadratic error estimator} \]

In order to compare the various estimates we constructed cumulative curves of the form
\[ \varepsilon(M) = \sum_{i=1}^{M} \eta_{n_i}^2, \quad M = 1, 2, \ldots, \text{NELEM} \]
where \( \eta_{n_i}^2 \) stands for the square of the \( L^2 \)-norm of the error or its estimator computed over the \( n_i \)th element. Here \( \{n_i\} \) denotes the special ordering which is used to solve the error equation with the discontinuous Galerkin. Figure 1 gives the cumulative curves for the square of the \( L^2 \)-norm of the error and its various estimators for the problem with smooth sinusoidal solution. We note that while the interpolation-type estimate appears to provide a sharp bound for the exact error the \( DG(2) \)-approximation of the error practically coincides with the exact error.

**Example 2. Almost-Discontinuous Solution**

We next construct a smooth solution with a sharp transition layer by choosing the boundary conditions to correspond to the steady-state solution:
\[
\begin{aligned}
&u(x, y) = \sin \left( \frac{\pi y}{64} \right) \tanh \left( \frac{1}{8} (y - 32) \right) \\
&\bar{y} = \begin{cases} 
  y - x, & \text{if } y - x \geq 0 \\
  y - x + 64, & \text{if } y - x < 0
\end{cases}
\end{aligned}
\]

Figure 2 gives the cumulative error curves for the square of the \( L^2 \)-norm of the error and the various estimators. We note that the \( DG \)-approximations are very close to the exact error (the \( DG(2) \)-approximation coincides with the exact error) but the interpolation-error estimate differs from the exact error by several orders of magnitude. Figure 3 presents the finite-element grid, the contours of the error and the contours of the \( DG(1) \)-approximation of the error. The contours of the \( DG(2) \)-approximation of the error are indistinguishable from those of the exact error.

**Example 3. "Discontinuous" Solution**

In order to show the collapse of the interpolation-type estimates for non-smooth solution, we consider the following solution:
\[
\begin{aligned}
&u(x, y) = \begin{cases} 
  1, & \bar{y} \leq 8 \\
  1 - \frac{\bar{y} - 8}{4}, & 8 \leq \bar{y} \leq 12 \\
  0, & \bar{y} \geq 12
\end{cases} \\
&\bar{y} = y - x
\end{aligned}
\]

Figure 4 contains the adaptive finite-element grid and the contours of the steady-state numerical solution. Figure 5 shows the contours of the exact error and the \( DG(2) \)-approximation. Figure 6 gives the contours of the \( DG(1) \)-approximation of the error and the interpolation error contours. Note that the contours of the interpolation-bound are several orders of magnitude less than those of the exact error or the \( DG \)-approximations.
Figure 1. Smooth Sinusoidal Solution. Cumulative curves for the square of the $L^2$-norm of the exact error and its estimates.

Figure 2. Almost Discontinuous Solution. Cumulative curves for the square of the $L^2$-norm of the exact error and its estimates.
Figure 3. Almost Discontinuous Solution: (a) Finite-element grid; (b) contours of the exact error and (c) contours of the $DG (1)$-approximation of the error.
Figure 4. Discontinuous Solution: (a) Adaptive finite-element grid and (b) contours of the steady-state numerical solution.
Figure 5. Discontinuous Solution. (a) Contours of the exact error and (b) its $DG(2)$-approximation.
Figure 6. Discontinuous Solution. (a) Contours of the $DG(1)$-approximation of the error and (b) contours of the interpolation-type error estimate.
Moreover, the interpolation–error bound decreases in the streamline–direction, which is opposite to the behavior of the exact error. Figure 7 presents the cumulative error curves for the steady–state solutions; again the DG-approximations are very close with the exact error while the values of the interpolation–bound are too small (in the figure the interpolation–bound curve coincides with the x-axis).

The magnitude of the squares of the global $L^2$-error norms and the corresponding estimates are given in the table below.

<table>
<thead>
<tr>
<th>Solution</th>
<th>Exact Error $(global L^2$-error$^2)$</th>
<th>Interpolation-type estimate</th>
<th>DG(1) error estimate</th>
<th>DG(2) error estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Smooth Sinusoidal Solution</td>
<td>0.119612</td>
<td>0.120957</td>
<td>0.112623</td>
<td>0.119596</td>
</tr>
<tr>
<td>Almost–Discontinuous Solution</td>
<td>0.315754</td>
<td>0.011202</td>
<td>0.302882</td>
<td>0.315670</td>
</tr>
<tr>
<td>&quot;Discontinuous&quot; Solution</td>
<td>4.446456</td>
<td>0.007325</td>
<td>3.861574</td>
<td>4.404879</td>
</tr>
</tbody>
</table>

Example 4. Steady-Convection of a Cosine–Hill

We also consider the non–homogeneous linear hyperbolic equation:

$$u_t + a \cdot \nabla u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial \Omega$$
$$u(\cdot, 0) = U(\cdot)$$

Again we let $\Omega = [0, 64]^2, \ a = (2, 2)$ and we chose $f = a \cdot \nabla U$ where $U$ is a "cosine–hill" given by:

$$U(x, y) = \begin{cases} A \sin \left( \frac{\pi r}{2R} \right) & , \ r \leq R \\ 0 & , \ r > R \end{cases}$$

Here $r$ is the polar radius from the center of the cosine–hill,

$$r = \sqrt{(x-x_c)^2 + (y-y_c)^2}$$

and we let $x_c = y_c = 32, R = 16$ and $A = 100$. (The initial grid was taken to be a uniform $8 \times 8$ grid while the interpolant of the exact solution on the initial grid was used as the initial condition for the calculation.) Figure 8 shows the time–evolution of the exact error, its $DG(2)$-approximation and the interpolation–type estimate as the solution is integrated forward in time. Figure 9 gives 2D and 3D views of the final solution and grid. Note that the grid was adapted using the error indicators based on the $DG(2)$-approximation of the error. Finally, Figure 10 presents a comparison of the contours of the exact and approximate error function at the final time.
Figure 7. Discontinuous Solution. Cumulative curves for the square of the $L^2$-norm of the exact error and its estimates.

Figure 8. Steady-Convection of a Cosine-Hill. Time-evolution of the square of the $L^2$-norm of the error, its $DG(2)$-approximation and of the interpolation-type error bound.
Figure 9. Steady-Convection of a Cosine-Hill. (a) Finite-element grid, contours and (b) 3D view of the approximate solution at the final time.
Figure 10. Steady-Convection of a Cosine-Hill. (a) Contours of the exact and (b) the $DG(2)$-approximation of the error at the final time step.
5 CONCLUSIONS

In this note, a simple procedure for a-posteriori error analysis is given which can be used effectively for estimating local errors in finite element approximations of convection-dominated flows. The method is based on a discontinuous Galerkin approximation of the evolution of error over individual elements.

The results of this study also show that popular interpolation methods can produce estimates of error in adaptive finite element computations that are grossly in error. However, the accumulated error over a mesh obtained using the crude interpolation estimates can be estimated quite accurately with the discontinuous Galerkin technique developed here. This suggests that these methods may be useful as post-processing techniques that can enhance adaptive results generated using a fast but low accurate error estimator. Results on nonlinear problems and Navier–Stokes equations are to be reported in a lengthier companion paper.

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