A NOTE ON SOME MATHEMATICAL STUDIES ON
ELASTOHYDRODYNAMIC LUBRICATION

S. R. WU and J. T. ODEN
TICOM, University of Texas at Austin, U.S.A.

Abstract—The Reynolds-Hertz equations for elastohydrodynamic lubrication with an undetermined boundary of the contact region are formulated in the framework of a nonlinear variational inequality due to the constraint that the pressure must be nonnegative. The present study shows that the operator of the variational inequality is bounded, coercive, pseudomonotone and continuous. Then the existence of solutions to the variational inequality is proved and it is shown that the solutions are the weak solutions of the Reynolds-Hertz equations.

The variational inequality is regularized by a penalty method and the penalty method is proved to be convergent. A detailed study shows that there exists classical solutions of one-dimensional line contact problems. To show the potential of penalty method, an additional study shows the convergence of sequences of finite-dimensional approximations. A priori error estimates for finite element solutions are derived. Numerical experiments confirm the predicted rates of convergence for a range of applied loads.

1. INTRODUCTION

During the century since Reynolds' equation was derived, lubrication, particularly elastohydrodynamic lubrication, has been a very active field which has attracted a great number of researchers. This paper is presented in remembrance of the 100th anniversary of the publication of Reynolds equation.

While elastohydrodynamic lubrication theory has been available to tribologists for decades the mathematical theory of the governing equations and inequalities is in its very early stages of development. Capriz and Cimatti [1] summarized previous contributions to this subject. Cimatti [2,3] studied the Reynolds equation via a variational inequality and proved the existence of solutions to the hydrodynamic lubrication problems. Aden and Wu [4] studied the more general case of elastohydrodynamic lubrication characterized by a nonlinear variational inequality which featured a pseudomonotone operator, and proved the existence of weak solutions. Wu [5] further studied the mathematical properties of these problems using a penalty method and proved the regularity of one-dimensional solutions, i.e. the existence of classical solutions for line contact problems. This penalty method also provides a basis for generating numerical procedures.

Here we explore some essential qualitative properties of Reynolds equation and show the existence of solutions, regularity of one-dimensional solutions, the convergence of penalty method as well as the convergence of finite dimensional approximations. Based on our analyses we propose an a priori error estimate for finite element solutions, which is confirmed by numerical experiments. To avoid tedious proofs, the mathematical details have to be omitted when it is possible to refer to our earlier works (see [4-6]).

2. FUNDAMENTAL EQUATION

For classical elastohydrodynamic lubrication problems, the Reynolds-Hertz model of non-Newtonian flow through an elastic bearing is modeled by the following system of partial differential equations, inequality constraint and boundary conditions:

\[
\begin{align*}
-\nabla \cdot (h^3 e^{-ap} p) + 12\mu_0 \partial (uh_1)/\partial x &= -12\mu_0 \partial (uh_2)/\partial x, \\
p &> 0 \text{ in } \Omega, \\
p = 0 \text{ in } \Omega_0, & p|_{\partial \Omega} = 0, & \Omega = \Omega_1 \cup \Omega_0
\end{align*}
\]  

(2.1)
where (referring to Fig. 1)

\[ h = \bar{h}_1(p) + \bar{h}_2 \]  

\[
\begin{align*}
\bar{h}_1(p) &= \frac{2}{\pi E} \int_{-\infty}^{\infty} p(\xi) \ln(x_0 - \xi - x^2) d\xi \\
&\quad + \frac{2}{\pi E} \int_{0}^{\infty} p(\xi)/r(x, \xi) d\xi \\
\bar{h}_2 &= h_0 + h_z \\
h_z &= R - \sqrt{R^2 - x^2} \quad \text{(line contact)} \\
&= R - 2x^2 \quad \text{(point contact)} 
\end{align*}
\]  

here \( h_0 \) is the reference thickness. One may also use the minimum film thickness \( h_m \) in the formulation, where

\[
\begin{align*}
\bar{h}_1(p) &= h_1(p) - S(h) \\
\bar{h}_2 &= h_m + h_z \\
S(h) &= \min(h_1(p) + h_z)
\end{align*}
\]  

\( h_1(p) \) is the contribution of elastic deformation and \( E' \) is the effective elastic modulus. \( \mu_0 \) is the viscosity of the lubricant under atmospheric pressure, \( \alpha \) is the exponential parameter for the viscosity. \( u \) is the rolling velocity.

It is well known that this problem is a free boundary problem, with the interface, where the pressure recovers the ambient value, \( \partial \Omega = \Omega_1 \cap \Omega_0 \) undetermined. It is customary to
impose Reynolds condition $\nabla p = 0$ on $\partial \Omega_1$. Then (2.2) can be written

$$p|_{\partial \Omega_1} = 0 \quad \text{and} \quad \nabla p|_{\partial \Omega_1} = 0 \quad (2.2')$$

and we can consider (2.1) to hold only in the undetermined region $\Omega_1$.

### 3. VARIATIONAL INEQUALITY

Our analysis will be based on (2.1) and (2.2). Take $p \geq 0$ in $\Omega$ as a constraint and consider the fact that the Reynolds equation is valid only in the contact region $\Omega_1$. This situation can be characterized by a nonlinear variational inequality. First define the operator in Reynolds equation as

$$A(p) = - \nabla \cdot (h_1(p)e^{-sp}\nabla p) + 12\mu_0 \partial(u\tilde{h}_1(p)) / \partial x \quad (3.1)$$

and a function

$$f = - 12\mu_0 \partial(u\tilde{h}_2) / \partial x \quad (3.2)$$

Then (2.1) reduces to $Ap = f$ in $\Omega_1$. We may make a weak statement for the problem in the formulation of a variational inequality:

$$(P): \text{Find } p \in K = \{ q \in V | q \geq 0 \ \text{a.e. in } \Omega \} \text{ such that } \langle A(p) - f, q - p \rangle_V \cdot \nu \geq 0 \forall q \in K \quad (3.3)$$

Here we use the Sobolev space $V = H^1_0(\Omega)$ as an appropriate domain for operator $A: V \rightarrow V' = H^{-1}(\Omega, V')$ being the dual space of $V$, and denote by $\langle \cdot, \cdot \rangle$ the duality on $V \times V'$. $K$ is the constraint set.

It is easy to show the following (see [4, 6]).

**Lemma 3.1.** If $p$ is a solution of the variational inequality (3.3), then $p$ is a solution of (2.1) and (2.2) in a distributional sense. called a weak solution.

**Remark.** If $p$ is a solution of (2.1) and (2.2), from (3.3) we have $\int_{\Omega_0}(A(p) - f)q \, d\Omega \geq 0 \forall q \in K$. The fact that $q \geq 0$ results in $[A(p) - f] \geq 0$ in $\Omega_0$. It is easy to show that this requirement can be satisfied if in $\Omega_0$, $x > 0$ for line contact, and $x_M > 0$ (the largest value of $x$ in $\Omega_1$) for point contact. If so, the flow is possible; numerical solutions always exhibit this behavior.

### 4. FUNDAMENTAL PROPERTIES OF OPERATOR $A$

Starting with a Lemma on the property of the integral operator defined in (2.4) for evaluating $h_1(p)$, we study the behavior of operator $A$. Here we would like to list only the fundamental results and omit the lengthy details (see [4, 6]).

**Lemma 4.1.** For $h_1(p)$ defined in (2.4) and $0 < \delta < 1$, and $q = (2 - \delta)/(1 - \delta) > 2$, there exists a constant $C_0 > 0$ such that

$$\sup_{x \in \Omega} |h_1(p)| \leq C_0 \|p\|_{L^\delta} \quad \forall q \in V \quad (4.1)$$

**Theorem 4.2.** Operator $A$ is bounded.

**Theorem 4.3.** There exist constants $C_1$ and $C_2 > 0$ such that

$$\langle A(p), p \rangle \leq C_1 \|p\|^2_{L^\delta} - C_2 \|p\|_{V'} \quad \forall p \in V \quad (4.2)$$

Therefore operator $A$ is coercive, i.e.

$$\langle A(p), p \rangle / \|p\|_{V'} \rightarrow \infty \quad \text{as } \|p\|_{V'} \rightarrow \infty \quad (4.3)$$
Theorem 4.4. Operator $A$ is hemicontinuous, i.e.

$$\lim_{t \to 0} \langle A(p + t q), r \rangle = \langle A(p), r \rangle \quad \forall p, q, r \in V$$  \quad (4.4)$$

Lemma 4.5. Denote $B_\eta = \{ \|q\|_{H^1} \leq \eta \}$, a ball in $V$. There exists a constant $C_\eta > 0$ such that $\forall p, q \in V$

$$\langle A(p) - A(q), p - q \rangle \geq C_1 (\|p - q\|_V)^2 - C_\eta \|p - q\|_V \|p - q\|_V$$  \quad (4.5)$$

with the same $C_1$ that appeared in (4.2).

Theorem 4.6. Operator $A$ is pseudomonotone on any closed convex subset $K$ of $V$. That is, if $A$ and any sequence $\{u_n\} \in K$ satisfy property $P$:

$$u_n \to u \quad \text{weakly in } V \quad \text{and} \quad \lim_{n \to \infty} \langle A(u_n), u_n - u \rangle \leq 0$$  \quad (4.6)$$

then we have

$$\lim_{n \to \infty} \langle A(u_n), u_n - u \rangle \geq \langle A(u), u - v \rangle \quad \forall v \in K$$  \quad (4.7)$$

Theorem 4.7. Operator $A$ is continuous. For the ball $B_\eta$ there exists a constant $C_A(\eta) > 0$ such that

$$\|A(p) - A(q)\|_V \leq C_A \|p - q\|_V \quad \forall p, q \in B_\eta$$  \quad (4.8)$$

5. EXISTENCE OF SOLUTIONS

We cite here an important theorem in the theory of variational inequality. For details, see [7] or [8].

Theorem 5.1. Let $U$ be a separable and reflexive Banach space, $K$ be a nonempty, closed and convex subset of $U$, and $A: U \to U'$ be an operator on $K$.

If $A$ is bounded, pseudomonotone and (1) $K$ is bounded or (2) in case $K$ is unbounded, $A$ is coercive in the following sense: there exists a $u_0 \in K$ such that

$$\langle A(u), u - u_0 \rangle / \|u\|_U \to \infty \quad \text{as } \|u\|_U \to +\infty$$

Then for any $f \in U'$, there is solution $u \in K$ of the variational inequality

$$\langle A(u) - f, v - u \rangle \geq 0 \quad \forall v \in K$$

If $A$ is strictly monotone, the solution is unique.

Since the operator $A$ is bounded, pseudomonotone and coercive with $u_0 = 0$, we have directly the important conclusion:

Theorem 5.2. For the nonlinear free boundary problems in elastohydrodynamic lubrication, there exists at least one solution to the variational inequality (3.3), and it is a weak solution of Reynolds–Hertz equations (2.1)–(2.4).

Remark. By compactness properties of Sobolev spaces [refer to Adams [9]], there is a constant $C_q > 0$ such that $\|u\|_{L^\infty} \leq C_q \|u\|_{H^1}$ for any $u \in H^1$. Thus in (4.5), if $C_1$ is large enough that

$$C_1 - C_q C_q > 0$$  \quad (5.1)$$
then $\langle A(p) - A(q), p - q \rangle \geq (C_1 - C_q)C_n (\|\rho - q\|_{H^1})^2$, i.e. $A$ is strictly monotone. In this case the solution to (2.1) is unique.

In fact $C_1$ is determined by $h_m$, the minimum film thickness. In light load cases, $h_m$ may not be too small and one may expect (5.1) to hold. But, for heavier loads, we can conclude nothing on monotonicity from the present analysis.

For hydrodynamic lubrication without elasticity taken into account, as a special case, the $C_q$ term in (4.5) drops and the uniqueness is assured.

6. PENALTY METHOD

To further the study and to implement an effective numerical schemes, we introduce a penalty term to regularize the inequality constraint (2.1). The penalty method has been used successfully in dealing with certain classes of contact problems and other kinds of free boundary problems. For a general reference, see [10].

Let $p^- = \min(p, 0) = (p - |p|)/2$. Introduce a penalty operator $\Phi: H^1_0(\Omega) \to H^{-1}$ as

$$\Phi(q) = q^-/\varepsilon \quad \text{with } \varepsilon > 0 \quad (6.1)$$

We then construct a penalty problem

$$(P_\varepsilon): \text{For } \varepsilon > 0, \text{find } p_\varepsilon \in V = H^1_0(\Omega) \text{ such that }$$

$$\langle A(p_\varepsilon), q \rangle + \langle \Phi(p_\varepsilon), q \rangle = \langle f, q \rangle \quad \forall q \in V \quad (6.2)$$

It is straightforward to show that operator $\Phi$ is monotone and bounded. Thus we have Theorem 6.1. Operator $A + \Phi$ is bounded, coercive and pseudomonotone on $V = H^1_0(\Omega)$. Therefore the penalized problem (6.2) has at least one solution for each $\varepsilon > 0$.

A similar theorem on the existence of solutions to a nonlinear equation governed by a pseudomonotone operator of this type, not cited here, can be found in [7].

The significance and legitimacy of the penalty method, are established by the following convergence theorem:

Theorem 6.2. Denote by $p_\varepsilon$ the solutions of (6.2) corresponding to $\varepsilon > 0$. As $\varepsilon \to 0$, there exists a subsequence of $\{p_\varepsilon\}$ which converges weakly to some $p$ in $V$, where $p$ is just a solution of the variational inequality (3.3), also a weak solution of (2.1)-(2.4). Moreover it also converges to $p$ strongly in $V$.

7. REGULARITY OF ONE-DIMENSIONAL SOLUTIONS

With the convergence property, we are now able to study the behavior of the solutions via the penalty method. First we need a lemma for the integral operator in evaluating the elastic deformation.

Lemma 7.1. If $p(x) \in C^0[a, b]$, then

$$g(x) = \int_a^x p(\xi)\ln|\xi - x| d\xi \in C^0[a, b] \quad (7.1)$$

If $p \in C^1[a, b]$, then $g(x) \in C^1(a, b)$ and for $a < x < b$

$$g'(x) = \int_a^x p'(\xi)\ln|\xi - x| d\xi + p(a)\ln|a - x| - p(b)\ln|b - x| \quad (7.2)$$
Moreover, if \( p(a) = 0 \),
\[
g'(a + 0) = \int_a^b p'(\xi)\ln|\xi - x|\,d\xi - p(b)\ln|b - a| \quad (7.3)
\]
otherwise \( g'(a + 0) \) does not exist [a similar conclusion holds for \( g'(b - 0) \)].

From this Lemma, and from the compactness of the Sobolev space \( H^1(\Omega) \) in \( C^0(\Omega) \) for the one-dimensional case, we have the following results.

**Theorem 7.2.** The one-dimensional penalized problem (6.2) has solutions \( p_\varepsilon \in C^1[a, b] \) and \( p_\varepsilon \in C^2(a, b) \). The corresponding film thickness \( h(p_\varepsilon) \in C^1[a, b] \).

In fact we have stronger result:

**Theorem 7.3.** The one-dimensional penalized problem (6.2) has solutions \( \{p_\varepsilon\} \) forming a bounded set in \( H^2(a, b) \). Therefore (3.3) has solutions \( p \in C^1[a, b] \). Moreover \( p \) is a solution in \( C^2(a, b_1) \) in the classical sense, which satisfies Reynolds condition \( dp/dx|b_1 = 0 \) with \( b_1 \) as the free boundary of the contact region.

We have thus established that the solution is smooth and does not experience a singularity or a pressure spike as some authors conjectured in some cases.

Numerical experiments confirm these conclusions (see [5, 6]).

8. APPROXIMATIONS

For several decades, investigators have sought effective numerical procedures for elastohydrodynamic lubrication problems with only limited success. Moreover, studies of accuracy and convergence of numerical approximation do not appear to be available. We explore the convergence properties of finite-dimensional approximations of these classes of problems in the present section.

For (6.2) we consider a finite-dimensional penalized approximation in \( V_n \), an \( N \)-dimensional subspace of \( V \).

\((P^N_\varepsilon)\): Find \( p_\varepsilon^* \in V_n \) such that
\[
\langle A(p_\varepsilon^*), q^* \rangle + \langle \Phi(p_\varepsilon^*), q^* \rangle = \langle f, q^* \rangle \quad \forall q^* \in V_n \tag{8.1}
\]

Obviously, for the approximations we have orthogonality:
\[
\langle (A + \Phi)(p_\varepsilon^*), (A + \Phi)(q^*) \rangle = 0 \quad \forall q^* \in V_n \tag{8.2}
\]

Similarly, we may construct the finite-dimensional approximations for the variational inequality (3.3)

\((P^N)\): Find \( p^* \in K_n = K \cap V_n \), such that
\[
\langle A(p^*), f - q^* - p^* \rangle \geq 0 \quad \forall q^* \in K_n \tag{8.3}
\]

**Theorem 8.1.** Operators \( A \) and \( A + \Phi \) are bounded, coercive and pseudomonotone on the subsets \( K_n \) and subspaces \( V_n \); so (8.1) and (8.3) have solutions.

For the approximations we may establish the convergence relations states below. For details see the forthcoming paper by Wu and Oden [11].

**Theorem 8.2.** For the approximate solutions of (8.1), there exists a subsequence \( \{p_{\varepsilon_n}\} \) which converges to \( p_\varepsilon \in V \) as \( n \to \infty \), a solution of (6.2), and \( A(p_{\varepsilon_n}) \to f \) weakly in \( V' \).

For any \( n \), there exists a subsequence \( \{p_{\varepsilon_n}\} \) which converges to \( p^* \in V_n \) as \( \varepsilon \to 0 \), a solution of (8.3).

On the other hand, for the approximate solution of (8.3), there exists a subsequence \( \{p^*\} \) which converges to \( p \in V \) as \( n \to \infty \), a solution of (3.3).

In summary we have established the following convergence relations (in all the strong \( H^1 \) topology).
9. ERROR ESTIMATES FOR FINITE ELEMENT SOLUTIONS

In finite element method, we construct the approximate subspaces with piecewise polynomial shape functions. Based on the previous analysis, by the interpolation theorem (for further details, see the textbook [12], we develop the following as an a priori error estimate (the details can be found in the forthcoming paper by Wu and Oden [11]).

**Theorem 9.1.** If the penalty solution $p_\varepsilon \in H^r(\Omega), r \geq 1$, and the shape functions of finite elements contain $P_k$, the complete polynomials of degree $\leq k$, $k \geq 1$, then under the assumption (5.1): $C_1 - C_\varepsilon C_q > 0$, for the regular family of affine finite elements, there exists a constant $C > 0$, such that for the approximate solutions $p_\varepsilon^h \in V_h$, we have the error estimate

$$
\|p_\varepsilon - p_\varepsilon^h\|_{H^s} \leq C h^{\mu} \|p_\varepsilon\|_{H^r}, \quad \mu = \min(k, r - 1)
$$

(9.1)

where $h$ is the mesh size parameter, and $V_h$ denotes the approximation subspace, (here $h \to 0$ takes the role of $n \to \infty$ in Section 8).

**Theorem 9.2.** If the penalty solution $p_\varepsilon$ of (6.2) is smooth enough to be an element in $H^r, r \geq k + 1$, then under conditions in Theorem 9.1, we have the error estimates

$$
\|p_\varepsilon - p_\varepsilon^h\|_{H^s} \leq C_1 h^{k+1} \|p_\varepsilon\|_{H^r} \quad (k = 1), \text{linear elements)
$$

(9.2)

$$
\|p_\varepsilon - p_\varepsilon^h\|_{H^s} \leq C_2 h^{2k} \|p_\varepsilon\|_{H^r} \quad (k = 2), \text{quadratic elements)
$$

(9.3)

For the load $w = \int p \, d\Omega$, we have

$$
|w_\varepsilon - w_\varepsilon^h| \leq C \|p_\varepsilon - p_\varepsilon^h\|_{L^2}
$$

(9.4)

And for one-dimensional problems we can have for the film thickness

$$
\|h_\varepsilon - h_\varepsilon^h\|_{H^s} \leq \|p_\varepsilon - p_\varepsilon^h\|_{H^s} \quad (s = 0, 1)
$$

(9.5)

As we have shown previously, at least for the one-dimensional problem, we have $H^2$ solutions so (9.2), (9.4) and (9.5) are realistic.

Numerical experiments show that for light loads [in the case (5.1) holds], (9.2) and (9.3) hold for both one-dimensional and two-dimensional problems, as shown in Figs 2-5. Also one order higher convergence is observed when measured in the $L^2$-norm. Estimates (9.4) and (9.5) are also verified by the numerical experiments. For heavier loads, deterioration in the rate of convergence is observed, as shown in Fig. 6.

**Acknowledgement**—Support from the Air Force Office of Scientific Research (AFSC) under contract No. F49620-84-C-0024 is gratefully acknowledged.
Fig. 2. Error analysis (1-D linear elements, $h_0 = 0.2 \times 10^{-4}$) line contact ($w = 0.166 \times 10^{-5}$).

Fig. 3. Error analysis (1-D quadratic elements, $h_0 = 0$) line contact ($w = 0.305 \times 10^{-5}$).
A note on some mathematical studies on elastohydrodynamic lubrication

Fig. 4. Error analysis (2-D bilinear elements, \( h_0 = 0.1E-5 \)) point contact (\( w = 0.126E-4 \)).

Fig. 5. Error analysis (2-D biquadratic elements, \( h_0 = 0.9E-5 \)) point contact (\( w = 0.463E-8 \)).
Fig. 6. Error analysis (2-D bilinear elements, $h_0 = 0.5E-6$) point contact ($w = 0.142E-7$).

REFERENCES


(Received 9 September 1986)
THE PROBLEM OF TWO PLASTIC AND HETEROGENEOUS INCLUSIONS IN AN ANISOTROPIC MEDIUM

M. BERVEILLER
Laboratoire de physique et mécanique des matériaux, Faculté des Sciences, Ile du Saulcy, 57045, Metz Cedex, France

and

O. FASSI-FEHRI and A. HIHI
Laboratoire de mécanique et des matériaux, Faculté des Sciences, BP 1014, Rabat Agdal, Maroc

Abstract—We demonstrate an integral equation for the total local strain $\epsilon_T$ in an anisotropic heterogeneous medium with incompatible strain $\epsilon_P$ and which is at the same time submitted to an exterior field. The integral equation is solved in the case of an heterogeneous and plastic pair of inclusions, for which we calculate the average fields in each inclusion as well as the different parts of the elastic energy stocked in the medium.

The solution is applied to the case of two isotropic and spherical inclusions in an isotropic matrix loaded in shear. The results are compared with those deduced from a more approximate method based on Born's approximation of the integral equation. In appendix we give a numerical method for calculating the interaction tensors between anisotropic inclusions in an anisotropic medium as well as the analytic solution in the case of two spherical inclusions located in an isotropic medium.

1. INTRODUCTION

The natural or voluntary presence of elastic and (or) plastic inhomogeneities in a matrix often affects the mechanical behaviour of a heterogeneous medium in significant proportions. The composite materials, the metallic polycrystals and polyphase materials for example exhibit a behaviour which a specific part can be related to inhomogeneities of material.

Moreover, the formation and the growth of precipitates, of irradiation defects, the growth of the martensite on an austenite-martensite transformation are highly affected by internal stress and the interaction energy associated with the plastic incompatibilities and the elastic heterogeneities due to structural phenomena observed.

Eshelby [1] has developed a remarkable method for the survey of a plastic and (or) heterogeneous inclusion in an infinite medium, and for the local fields calculus and of the different energy forms (elastic energy, interaction energy ...) associated to the inclusion. This method very often has been used and extended to anisotropic materials [2-4], and to non-uniform plastic strains [5, 6].

The survey of a single inclusion in an infinite matrix therefore does not allow to take into account a given number of microstructural parameters when many inclusions interfere simultaneously.

It also applies to the heterogeneous materials (polycrystals, composites ...) where the self-consistent methods resulting from the solution of the problem of a single inclusion do not allow to take into account discrete interactions between the various constituents.
The same applies to the phenomenon of precipitate formation or phase change where the interactions between inclusions have an important effect on the microstructure. Hence a few given surveys on the problem of two inclusions have been proposed.

Sternberg and Sadowsky [7] have investigated the strain associated to a pair of rigid and spherical inclusions aligned in the direction of tensile test, whereas Chen and Acrivos [8] generalize the problem by the use of bispheric coordinates to the case of a uniform external but general field.

A more general approach, proposed by Moschovidis and Mura [9] consist in using polynomial expansion for the strains and in solving for an isotropic medium the general equations of the problem. An interesting conclusion is that the stress field inside inhomogeneities is practically uniform.

From this observation, we can then expand performing methods of calculus where directly we postulate that the strains are uniform in each inclusion. This has been done by Berveiller and Zaoui [10] for the problem of two plastic and spherical inclusions in an isotropic medium and by Johnson [11] for an heterogeneous pair of inclusions, who, using Born’s approximation for integral equation of the problem, has obtained an approximate solution only available for very weak heterogeneities and an isotropic matrix.

The purpose of our investigation here is to study the problem of the heterogeneous and plastic pair of inclusions in an anisotropic matrix—in the case which the assumption of uniform fields in each inclusion is realistic. In the first section we recall the integral equation proved by Berveiller and Zaoui [12] for an heterogeneous and plastic medium, and in the second section we apply it to the problem of plastic and heterogeneous two inclusions. The calculus of stress, strains and the elastic energy for the pair of inclusions in an anisotropic medium leads finally to the determination of a tensor of interaction similar to the Eshelby’s tensor.

In appendix is given a numerical method for calculating this tensor in the case of two-ellipsoidal inclusions in an anisotropic medium. For an isotropic medium and two spherical inclusions, we give an analytic solution for the interaction tensor.

Here, we only consider the case of isothermal and quasistatic elasticity at small strains. The summation convention over repeated indices is adopted, moreover, \( F_i \) means partial differentiation \( \partial F / \partial x_i \).

2. THE INTEGRAL EQUATION FOR AN HETEROGENEOUS AND PLASTIC MEDIUM

We consider an infinite heterogeneous medium for which elastic constants at a point \( r \) are \( c(r) \) and where a stress-free strain (plastic, thermal, or related to a phase change) \( \varepsilon^0(r) \) is fielded by (or results) an initial state without stress.

Stresses \( \Sigma \) are applied to the surface of the infinite medium, assuming, in the case of an homogeneous medium, they would give rise to a uniform strain \( \varepsilon^0 \).

The presence of stress free strains \( \varepsilon^0(r) \) which are incompatible, generates internal stress and elastic strains \( \varepsilon^e \) in the medium such that the total local strain \( \varepsilon^T(r) \) is compatible, i.e. derives from a displacement \( u^T(r) \).

We have then:

\[
\varepsilon^T_i(r) = \frac{1}{2}(u^T_{ij} + u^T_{ji}) = \varepsilon^p_i(r) + \varepsilon^e_i(r)
\]

(1)

the elastic part \( \varepsilon^e(r) \) of \( \varepsilon^T(r) \) is related to the stresses \( \sigma(r) \) by Hooke’s law.

\[
\sigma_{ij}(r) = c_{ijkl}(r)\varepsilon^e_{kl}(r) = c_{ijkl}(r)[\varepsilon^0_{kl}(r) - \varepsilon^T_{kl}(r)] = c_{ijkl}(r)[u^T_{kl}(r) - \varepsilon^T_{kl}(r)]
\]

(2)

the last equality is obtained accounting for the usual symmetry of the tensor \( c \)

\[
c_{ijkl} = c_{jikl} = c_{ijlk}
\]