A NOTE ON APPLICATIONS OF ADAPTIVE FINITE
ELEMENTS TO ELASTOHYDRODYNAMIC LUBRICATION
PROBLEMS

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SUMMARY

Adaptive methods are applied to improve the finite element solutions for highly nonlinear elastohydrodynamic lubrication problem. A refinement scheme is developed which is based on the idea of equidistribution of local errors. Numerical experiments on the line contact problem show that the adaptive method is very effective in obtaining good approximations. Adaptive methods for point contact problems with light loads are also examined.

INTRODUCTION

The solutions of many elastohydrodynamic lubrication problems exhibit large gradients in the pressure distribution. For improving the quality of approximations of the pressure, more nodes can be placed in the region where the pressures varies rapidly. Alternatively, one can use higher order elements in that region. However, the structure of such regions may vary case by case and the use of very fine uniform meshes is not feasible as it may require excessive computer storage and time. This is particularly true in the case of point contact problems. To obtain good approximations in that case, the use of adaptive finite element methods to automatically produce a suitable refinement is thus an attractive alternative.

Such schemes have been applied to many engineering problems in recent years (see, for example, References 1-3). The basic ideas and schemes of the adaptive method can be found in References 4 and 5, and the recent advances and applications are summarized in Reference 6. Here we shall apply h-methods, which involve adaptive mesh refinement, to the elastohydrodynamic lubrication problem.

PRELIMINARIES

For classical elastohydrodynamic lubrication problems, we have Reynolds' equation

\[ \begin{align*}
-\nabla \cdot (h^3 e^{-\alpha p} \nabla p) + 12\mu_0 \partial (uh_1)/\partial x &= -12\mu_0 \partial (uh_2)/\partial x, \\
p > 0 &\text{ in } \Omega_1 \\
p = 0 &\text{ in } \Omega_0, p_{\text{on}} = 0, \Omega = \Omega_1 \cup \Omega_0
\end{align*} \tag{1} \]

where (referring to Figure 1)

\[ h = h_1(p) + h_2 \tag{3} \]

\[ \begin{align*}
h_1(p) &= 2/\pi E_f \int p(\xi) \ln (x, -\xi/x - \xi) d\xi \text{ (line contact)} \\
&= 2/\pi E_f \int p(\xi)/r(x, \xi) d\xi \text{ (point contact)} \\
h_2 &= h_0 + h_z \\
h_z &= R - \sqrt{(R^2 - x^2)} \text{ (line contact); } R - \sqrt{(R^2 - x^2 - y^2)} \text{ (point contact)} \tag{4}
\end{align*} \]

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Here $h_0$ is the reference thickness. One may also use the minimum film thickness $h_m$ in the formulation, where

$$h_1(p) = h_0(p) - S(h)$$

$$h_2 = h_m + h_z$$

$$S(h) = \min (h_0(p) + h_z)$$

$h_1(p)$ is the change in film thickness due to elastic deformation and $E'$ is the effective elastic modulus; $\mu_0$ is the viscosity of the lubricant under atmospheric pressure, $\alpha$ is the exponential parameter for the viscosity, and $\Omega$ is the rolling velocity.

There is an undetermined boundary $\partial\Omega_1 = \Omega_1 \cup \Omega_n$, where $p=0$. Thus (1)–(4) characterize a highly nonlinear free boundary problem. We shall take $p \geq 0$ in $\Omega$ as a constraint and consider the fact that our Reynolds' equation is valid only in the contact region $\Omega_1$. We define the operator in Reynolds' equation as

$$A: p \mapsto -\nabla \cdot (h^3(p)e^{-\alpha p} \nabla p) + 12\mu_0 \partial(\mu_0 h_2(p)) / \partial x$$

and we define the function

$$f = -12\mu_0 \partial(\mu_0 h_2) / \partial x$$

A weak statement for the problem in the form of a variational inequality is as follows:

(P): Find $p \in K = \{ q \in V | q \geq 0 \text{ a.e. in } \Omega \}$ such that

$$\langle A(p) - f, q - p \rangle_{V \times V'} \geq 0 \quad \forall q \in K$$
Here we use the Sobolev space \( V = H^1_0(\Omega) \) as an appropriate domain for operator \( A : V \rightarrow V' = H^{-1}(\Omega) \), the dual space of \( V \), and denote by \( \langle \cdot , \cdot \rangle \) the duality on \( V \times V' \). \( K \) is the constraint set.

It is easy to show the following (see References 7 and 8).

**Lemma 1.** If \( p \) is a solution of the variational inequality (7), then \( p \) is a solution of (1)-(4) in a distributional sense (a so-called weak solution).

To further the study and to implement an effective numerical scheme, we introduce a penalty term to regularize the variational inequality (7). Penalty methods have been successful in dealing with certain classes of contact problems and other kind of free boundary problems. For a general reference, see Oden and Kikuchi.\(^9\)

Let \( p^- = \min (p, 0) = (p - |p|)/2 \). Introduce a penalty operator \( \Phi : H^1_0(\Omega) \rightarrow H^{-1} \) as

\[
\Phi(p) = p^-/\varepsilon \quad \text{with} \quad \varepsilon > 0
\]

We then construct a penalty problem,

\[
(P_\varepsilon) : \text{For} \ \varepsilon > 0, \text{find} \ p_\varepsilon \in V = H^1_0(\Omega) \text{such that} \nabla A(p_\varepsilon, q) + \langle \Phi(p_\varepsilon), q \rangle = (f, q) \ \forall q \in V
\]

The validity of the penalty method is confirmed by the following theorem (for the details of the proof, see References 7 and 10).

**Theorem 2.** Denote by \( p_\varepsilon \) the solutions of (9) corresponding to \( \varepsilon > 0 \). As \( \varepsilon \rightarrow 0 \), there exists a subsequence of \( \{p_\varepsilon\} \) which converges weakly to some \( p \) in \( V \), where \( p \) is a solution of the variational inequality (7), and, therefore, also a weak solution of (1)-(4). Moreover, this sequence also converges to \( p \) strongly in \( V \).

**Strategy for Adaptive Refinement**

In a previous work\(^{11}\), we derived an a \textit{priori} error estimate for the finite element solutions for the penalized problem (9) (for details, see Reference 7 or the forthcoming paper, Reference 8).

\[
\|\text{error}\|_{\text{rel}} \leq Ch^k[p_\varepsilon]|_{\text{rel}}^{k+1}
\]

We shall use the number \( h^k[p|_{\text{rel}}^{k+1} \) as an error indicator for each element and pick the elements with larger indicators for refinement. The idea is to attempt to distribute the error evenly throughout the mesh.

For Lagrangian finite elements, the functions interpolated with shape functions are continuous, but may have jumps in the derivatives across the inter-element boundaries.

For one-dimensional linear elements (the \( k=1 \) case), the first derivatives of the shape functions are step functions. When we calculate the second-order semi-norm, we obtain \( \delta \)-functions. Thus,

\[
(|p|_2)^2 = \int (d^2p/dx^2)^2 dx = \Sigma_i (|dp/dx|_i)^2
\]

For two-dimensional problems, the rectangular bilinear elements have the same feature, except that the jumps on the boundaries \( x=x^i \) (or \( y=y^i \)) are still functions of \( y \) (or \( x \), respectively), so

\[
(|p|_2)^2 = \int (d^2p/dx^2)^2 + (d^2p/dy^2)^2) d\Omega
= \Sigma_i \int (|dp/dx|_i)^2 dy + \Sigma_i \int (|dp/dy|_i)^2 dx
\]

For simplicity, we use rectangular elements with shape functions of a fixed polynomial degree (generally \( k = 1 \) or 2) for the two-dimensional point contact problem. When we refine an element into four elements, if the neighbouring element is not to be refined, it will have extra nodes on its boundary. To maintain consistency, these new nodes should be constrained until the neighbouring element is refined. Meanwhile, we employ the one-node rule, which means that on one side of a linear element at most one constrained node (a pair for quadratic elements) is allowed. Thus, when a second node on the side is to be constrained, the neighbouring element should be refined instead.
APPLICATIONS TO LINE CONTACT PROBLEMS

Figure 2 shows the pressure distributions solved on a series of adaptively refined linear element meshes starting with a uniform 10-element mesh, for a light load case ($H_0=0.0, W=0.304E-5$). It is shown that the solutions from the fourth (28 elements) and the fifth (33 elements) refined meshes are very close. In fact, the relative difference of pressure $\|p_4-p_5\|/\|p_5\|$ is 5.0 percent in the $H^1$-norm and 0.75 percent in the $L^2$-norm. The relative difference of load is $0.40\times10^{-3}$. For the film thickness, it is $0.96\times10^{-3}$ in the $H^1$-norm and $0.41\times10^{-3}$ in the $L^2$-norm.

Figure 3 shows the adaptive solutions with quadratic element, starting with a uniform 5-element mesh. The relative difference of pressure between the solutions from the fourth (17 elements) and the fifth (20 elements) refined meshes is 2.0 percent in the $H^1$-norm and 0.45 percent in the $L^2$-norm.

It is interesting to make a comparison with the solutions from uniform fine meshes. Figure 4(a) shows the comparison between the solutions from the fifth refined linear element mesh (33 elements) and a uniform fine mesh (120 elements). The comparison for solutions with quadratic elements is shown in Figure 4(b). Figure 4(c) compares the linear and quadratic adaptive solutions. It is observed that the adaptive solutions and those from fine uniform meshes are very close, and the adaptive solutions with linear elements and quadratic elements are also very close. These results thus show that the adaptive method is quite efficient and reliable in obtaining an accurate approximation.

A second example is for a heavy load case ($H_0=-0.16E-3, W=0.233E-4$). Figure 5 shows the pressure distributions solved on a series of adaptive-refined linear element meshes, starting with a 10-element uniform mesh. The computation for quadratic elements is shown in Figure 6. In the process, the scheme at first appeared to be divergent at the load level; the first two refinements are performed at a lower loading level. Then we increased the load again and continued the refinement procedures.
Figure 3. Pressure distributions (one-dimensional adaptive quadratic elements, \( H_a = 0.0 \))

Figure 4. Comparisons with fine uniform meshes (one-dimensional, \( H_a = 0.0 \))
Figure 5. Pressure distributions (one-dimensional adaptive linear elements, $H_i = -0.16E-3$)

Figure 6. Pressure distributions (two-dimensional adaptive quadratic elements, $H_i = -0.16E-3$)
Figure 7 shows a comparison of the adaptive solutions with the fine uniform mesh solutions. Figure 8 shows the film shape near the contact centre.

These experiments and the comparisons show that adaptive methods can be very effective for these types of nonlinear problems.

APPLICATION TO POINT CONTACT PROBLEM

For a light load case \((H_0=0.5E-5, W=0.680E-8)\), computed with bilinear elements, after three refinements the adaptive solutions become quite close to each other. The pressure profiles, at \(y=0\), of the adaptive solutions are shown in Figure 9. Figure 10 exhibits the sixth refined mesh generated by the adaptive method.

Figure 11 shows the result for biquadratic elements. Also, after several refinements, the solutions become very close to each other. Figure 12 shows the sixth refinement mesh patterns.

The numerical results for light-load cases are encouraging and suggest that further studies of adaptive techniques for these types of problems may be warranted.
Figure 9. Pressure profiles of adaptive solutions at $y=0$ (two-dimensional bilinear elements, $H_0=0.5E-5$)

Figure 10. Refinement pattern for Figure 9.
Figure 11. Pressure profiles of adaptive solutions at \( y=0 \) (two-dimensional bi-quadratic elements, \( H_0=0.5E-5 \))

Figure 12. Refinement pattern for Figure 11
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