EXISTENCE AND LOCAL UNIQUENESS OF SOLUTIONS TO
CONTACT PROBLEMS IN ELASTICITY WITH NONLINEAR
FRICION LAWS

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Abstract—In this paper, we show how the use of interface models greatly facilitates the mathematical
analysis of static friction phenomena. Such models have already found extensive arguments in their
support, from both the experimental and numerical sides. They are then not conveniently introduced
for purely mathematical purposes. Nevertheless, they also appear to provide a satisfactory substitute
to the classical approach in terms of a variational problem with unilateral constraint. Indeed, we
have been able to prove, with relatively simple arguments, the existence and uniqueness results
that the classical theory has been unable to generate despite numerous attempts. In addition, in
the case with no friction, the solution obtained through the classical theory is shown to be recovered
in the limiting case of an infinite normal stiffness, thus giving one more justification of the validity
of such models.

1. INTRODUCTION

The classical theory of contact in elastostatics with Coulomb's law of friction in force on
the contact surface has not been a successful theory, from either the physical or the
mathematical point of view. Physically, it is known that standard contact laws provide a
crude and imperfect model of actual contact phenomena on dry metallic surfaces. On the
mathematical side, the significant difficulties inherent in proving the existence of solutions
for contact problems with friction are well-known. The question of existence of solutions
to the general Signorini problem with friction was put forth as an open problem by Duvaut
and Lions [5], although Nečas, Jarůsek and Haslinger [8,15] have recently managed to
state existence and uniqueness results under special hypotheses. Duvaut [4] pointed out
that a mollification of the contact pressure would provide sufficient regularity for the
establishment of existence of solutions to Signorini-type problems, and this led to several
studies of non-local friction laws (e.g. [3,18–20]). While physical arguments can be given
that support such laws, it is not known how adequate they are for modeling many common
frictional phenomena.

In a recent memoir [17], Oden and Martins presented extensive arguments in support
of phenomenological laws governing the contact interface of the form

\[ \sigma_n = - c_n a^m, \]
\[ \sigma_T = c_T a^m \tau \text{ (if sliding occurs).} \] (1.1)

on the contact surface, where \( \sigma_n \) and \( \sigma_T \) are normal and tangential stresses, \( \tau \) is a unit
tangent vector field parallel to the sliding velocity, \( c_n, c_T, m_n \) and \( m_T \) are material parameters
and \( a \) is the penetration approach of the contact surfaces. These laws assert that the contact
interface has a normal compliance characterized by a power law relation between the
normal stress and \( a \), the change in the distance between surfaces due to deformation, while
the frictional stresses are also defined by a power law in \( a \), which reduces to the usual
Coulomb's law of friction \( \sigma_T = \mu | \sigma_n | \tau \) if \( m_T = m_n \) and \( \mu = c_T / c_n \) is the coefficient of friction.
Arguments based on experimental, theoretical and computational results that the above
relations do, in fact, provide a reasonable model for a wide range of frictional phenomena are supplied in [17]. For representative physical experiments supporting the normal contact power law above, we refer to the paper by Back et al. [2].

A question that naturally arises at this point is whether or not the use of these new and more sound physical models of friction in a theory of contact problems in elastostatics will provide a mathematical framework that admits the establishment of existence and uniqueness results under appropriate hypotheses. We show in the present paper that the answer to this question is affirmative.

In the present work, we formulate a class of contact problems in elastostatics with nonlinear friction laws of the form (1.1) acting on the contact surface. We prove that solutions exist in appropriate Sobolev spaces under reasonable hypotheses on the regularity and smallness of the data, in particular the smallness of the coefficient of friction for prescribed external loads. These results are established through the implicit function theorem which also yields local uniqueness with no additional assumptions and, in some respects, are reminiscent of those in [3] and [4].

2. THE STEADY FRICTIONAL SLIDING OF A METALLIC BODY

We begin by considering a metallic body, the interior of which is an open bounded domain \( \Omega \) in \( \mathbb{R}^N \) \((N = 2 \text{ or } 3)\). Points (particles) in \( \Omega \) with cartesian coordinates \( x_i \), \( 1 \leq i \leq N \), relative to a fixed coordinate frame are denoted by \( x = (x_1, \ldots, x_N) \) and the volume measure by \( dx \). The smooth boundary \( \Gamma \) of \( \Omega \) (in practice, Lipschitz continuous in the sense of [14]) contains four disjoint open subsets \( \Gamma_C, \Gamma_D, \Gamma_E \) and \( \Gamma_F \) such that

\[
\Gamma = \bigcup \Gamma_\alpha,
\]

\[
\text{meas}(\Gamma_\alpha \setminus \Gamma_\beta) = 0.
\]

with \( \alpha \in \{C, D, E, F\}. \) The surface measure (cf. [14, pp. 119–123]) on \( \Gamma \) or any of its measurable subsets is denoted by \( ds \) and we shall set

\[
\Sigma = \text{int}(\Gamma \setminus \Gamma_D).
\]

We assume that the metallic body has a linearly elastic behavior characterized by the generalized Hooke's law

\[
\sigma_{ij}(u) = E_{ijkl}u_{k,l} \quad \text{in } \Omega,
\]

\[
1 \leq i, j, k, l \leq N.
\]

where \( u = (u_1, \ldots, u_N) \) is a sufficiently smooth field of displacements in \( \Omega \), \( E_{ijkl} \) are the usual elasticity coefficients of the material satisfying

\[
E_{ijkl} = E_{ijlk} = E_{jikl} = E_{klji}, \quad 1 \leq i, j, k, l \leq N
\]

(2.4)

and \( \sigma_{ij} \) are the cartesian components of the Cauchy stress tensor verifying

\[
\sigma_{ij} = \sigma_{ji}, \quad 1 \leq i, j \leq N.
\]

In (2.3) and throughout this work, standard indicial notation and the summation convention are employed (e.g. \( u_{k,l} = \partial^2 u_k / \partial x_l \)).

We suppose that body forces with components per unit volume \( b_i, 1 \leq i \leq N \), act in \( \Omega \). We also suppose that the body is fixed on \( \Gamma_D \) and tractions \( t \) are prescribed on \( \Gamma_F \). The body may be elastically supported on \( \Gamma_E \) by distributed springs of moduli \( K_{ij} \) satisfying the symmetry conditions

\[
K_{ij} = K_{ji}, \quad 1 \leq i, j \leq N.
\]

(2.5)

The undeformed configuration of these springs corresponds to the displacement \( \mathbf{U}^0 \) on \( \Gamma_E \). Finally, we assume that the body may come in contact with a rough foundation along a candidate contact surface \( \Gamma_C \). The foundation slides by the material contact surface with a velocity parallel to a prescribed unit vector field \( \mathbf{r} \) tangent to \( \Gamma_C \). The initial gap (in the reference configuration \( u = 0 \)) between the body and the foundation is denoted by \( g \geq 0 \).
Of course, since \( g \) is measured in the reference configuration, physical significance requires \( g \) to be "small" along \( \Gamma_C \). As mentioned earlier, \( c_n, c_T, m_n \) and \( m_T \) will denote material parameters of the contact interface. We shall denote by \( n = (n_1, \ldots, n_N) \) the outward normal unit vector along the boundary \( \Gamma \), by \( \sigma_n \) and \( \sigma_T \) the normal stress and the tangent vector stress on the boundary, whose values at a displacement \( u \) are, respectively

\[
\sigma_n(u) = \sigma_i(u) n_i n_j, \quad \sigma_T(u) = \sigma_i(u) n_j - \sigma_i(u) n_i, \quad 1 \leq i, j \leq N.
\]

Similarly, the displacement \( u \) on the boundary \( \Gamma \) will be decomposed into normal and tangential components \( u_n \) and \( u_T \) respectively, according to

\[
u_n = u \cdot n = u_i n_i, \quad u_T = u - u_n n \quad \text{(i.e. } u_{\Gamma T} = u_i - u_n n_i, 1 \leq i \leq N).\]

Assuming sufficient smoothness for all the functions involved, the steady sliding equilibrium position of the metallic body is characterized by the following system of equations:

1. **Equilibrium equations**

\[
\sigma_i(u) n_j + b_i = 0 \quad \text{in } \Omega, \quad 1 \leq i \leq N. \tag{2.6}
\]

where \( \sigma_i(u) \) is defined by the constitutive relation (2.3).

2. **Boundary conditions**

   (a) Prescribed displacements

\[
u_i = 0 \quad \text{on } \Gamma_D, \quad 1 \leq i \leq N. \tag{2.7}
\]

(b) Prescribed tractions

\[
\sigma_i(u) n_j = t_{ij} \quad \text{on } \Gamma_F, \quad 1 \leq i, j \leq N. \tag{2.8}
\]

(c) Linear elastic constraint

\[
\sigma_i(u) n_j = -K_{ij}(u_i - U_i^1) \quad \text{on } \Gamma_E, \quad 1 \leq i, j \leq N. \tag{2.9}
\]

(d) Normal contact constitutive equation

\[
\sigma_n(u) = -c_n[(u_n - g)_+]^m_n \quad \text{on } \Gamma_C. \tag{2.10}
\]

(e) Frictional contact condition

\[
\sigma_T(u) = c_T[(u_n - g)_+]^{m_T} \quad \text{on } \Gamma_C. \tag{2.11}
\]

3. **VARIATIONAL FORMULATION**

In order to establish a variational principle for the steady sliding equilibrium problem introduced in the previous section, we shall now make precise the assumed smoothness of the domain \( \Omega \) and of the functions involved. Additional properties of these functions, relevant for later purposes, will also be specified.

For a domain \( \Omega \) with a Lipschitz continuous boundary, it is well-known that the unit outer normal \( n \) is defined almost everywhere on \( \Gamma \) and is a measurable (and bounded) function (see e.g. [11.14]). In other words

\[
n_i \in L^\infty(\Gamma), \quad 1 \leq i \leq N, \sum_{i=1}^N n_i^2 = 1 \text{ a.e. on } \Gamma. \tag{3.1}
\]

We shall denote by \( H^1(\Omega) \) the usual Sobolev space. Restrictions to the boundary \( \Gamma \) of elements of \( H^1(\Omega) \) will always be understood in the sense of traces [thus. as elements of \( H^{1/2}(\Gamma) \)]. With this convention, we define the space \( V \) of admissible displacements by
\[ V = \{ v \in [H^1(\Omega)]^N : v = 0 \text{ a.e. on } \Gamma_D \}. \]  
(3.2)

As a closed subspace of \([H^1(\Omega)]^N\), the space \(V\) is a Hilbert space for the usual \([H^1(\Omega)]^N\)-inner product inducing the norm

\[ \|v\|_V = \left[ \int_\Omega (v_i v_i + v_{ij} e_{ij}) \, dx \right]^{1/2}. \]  
(3.3)

The trace operator maps the space \(V\) linearly and continuously onto the subspace \(W\) of \([H^{1/2}(\Gamma)]^N\) defined by

\[ W = \{ \xi \in [H^{1/2}(\Gamma)]^N : \xi = 0 \text{ a.e. on } \Gamma_D \}. \]  
(3.4)

As a closed subspace of \([H^{1/2}(\Gamma)]^N\), the space \(W\) is a Hilbert space. A possible choice for the Hilbertian norm on \(W\) (which amounts to identify \(W\) with the orthogonal of \([H^1_0(\Omega)]^N\) in \(V\)) is

\[ \|\xi\|_W = \inf \{ \|v\|_V : v = \xi \text{ on } \Gamma \}. \]  
(3.5)

The (topological) dual spaces of \(V\) and \(W\) will be denoted by \(V'\) and \(W'\) respectively. The notations \(\langle \cdot, \cdot \rangle_V\) and \(\langle \cdot, \cdot \rangle_W\) will be used for the corresponding duality pairings.

From the Sobolev's embedding theorems, the space \(H^{1/2}(\Gamma)\) is continuously embedded in \(L^q(\Gamma)\) for \(1 \leq q < +\infty\) if \(N = 2\) and \(1 \leq q \leq 4\) if \(N = 3\) (see e.g. [1, 11]). As a result, for these values of \(q\), the space \(W\) is continuously embedded in the subspace of \([L^q(\Gamma)]^N\) functions vanishing almost everywhere on \(\Gamma_D\), canonically isomorphic to the space \([L^q(\Sigma)]^N\) [cf. (2.2) for the definition of \(\Sigma\)]. The topological dual of which is \([L^{q^*}(\Sigma)]^N\) where \(q^*\) denotes the H"older conjugate of \(q\) [i.e. \(q^* = q/(q - 1)\)]. It is also well-known for the above values of \(q\) that the space \(H^{1/2}(\Gamma)\) is dense in \(L^q(\Gamma)\). Similarly, the space \(W\) is dense in \([L^q(\Sigma)]^N\); a similar result obtained using a system of local charts is mentioned in [7, Lemma 1.4, pp. 212–213] when \(q = 2\).

From (3.1) and for every \(1 \leq p \leq +\infty\), we shall also use the following decomposition of an element \(\xi \in [L^p(\Gamma)]^N\) (resp. \([L^p(\Sigma)]^N\))

\[ \xi = \xi_T + \xi_n. \]  
(3.6)

with

\[ \xi_n = \xi \cdot n = \xi_i n_i \in L^p(\Gamma) \]  
(3.7)

\[ \xi_T = \xi - \xi_n n \in [L^p(\Gamma)]^N \]  
(3.8)

Note that

\[ \xi_T \cdot n = 0 \text{ a.e. on } \Gamma \text{ (resp. } \Sigma). \]  
(3.9)

So as to allow the greatest generality in our later purposes, we shall somewhat specify the value of \(q\) by setting

\[ q = \begin{cases} \text{any convenient real number} > 2 & \text{if } N = 2, \\ 4 & \text{if } N = 3. \end{cases} \]  
(3.10)

We now turn to listing the assumptions that will henceforth be in force on the data of the problem. First, we assume that the elasticity coefficients \(E_{ijkl}\) verify

\[ E_{ijkl} \in L^\infty(\Omega) \]  
(3.11)

and that the uniform ellipticity condition

\[ E_{ijkl}(x) A_{kl} A_{ij} \geq \varepsilon_0 A_{ij} \]  
(3.12)

holds for almost all \(x \in \Omega\) and every \(N \times N\) symmetric array \(A_{ij}\), with some constant \(\varepsilon_0 > 0\). Analogous conditions are imposed on the springs moduli \(K_{ij}\), namely

\[ K_{ij} \in L^\infty(\Gamma_D). \]  
(3.13)
for almost all \( x \in \Gamma_E \) and every vector \( a = (a_1, \ldots, a_N) \) where \( \alpha_E > 0 \) is a constant. As for the displacement \( U^E \) (cf. Section 2) we assume

\[
U^E \in [L^N/(\Gamma_E)]^N.
\]  

(3.15)

The prescribed tractions \( t \) on \( \Gamma_F \) verify

\[
t \in [L^N/(\Gamma_F)]^N.
\]  

(3.16)

The exponents \( m_n, m_T \in \mathbb{R} \) and \( m_T \in \mathbb{R} \) are submitted to the conditions\(^\dagger\)

\[
1 < m_n, m_T < +\infty \text{ if } N = 2; \quad 1 < m_n, m_T \leq 3 \text{ if } N = 3.
\]  

(3.17)

Note from (3.10) that this assumption can be rewritten as

\[
1 < m_n, \quad m_T \leq q - 1.
\]  

(3.18)

after increasing the value of \( q \) when \( N = 2 \) if necessary. This modification does not affect (3.15) or (3.16) since \( q^* \) decreases as \( q \) increases. The vector field \( \tau \) tangent to \( \Gamma \) along \( \Gamma_C \) is taken unitary, i.e.

\[
\tau \in L^\infty(\Gamma_C), \quad \sum_{i=1}^{N} \tau_i^2 = 1 \text{ a.e. on } \Gamma_C,
\]

(3.19)

\[
\tau \cdot n = 0 \text{ a.e. on } \Gamma_C.
\]  

(3.20)

For consistency in our assumptions, we assume that the initial gap \( g \) verifies

\[
g \in L^q(\Gamma_C),
\]

(3.21)

although \( g \) is always continuous in practice, so that condition (3.19) is not affected by possibly increasing \( q \) as above if necessary. The material parameters \( c_n \) and \( c_T \) characterizing the normal and tangential response along \( \Gamma_C \) are supposed to verify

\[
c_n, c_T \in L^\infty(\Gamma_C)
\]

(3.22)

and, indeed, they are constant in most applications. Finally, the body forces \( b \) satisfy the condition

\[
b \in [L^2(\Omega)]^N.
\]

(3.23)

At this stage, we introduce

(i) The continuous bilinear from \( a(\cdot, \cdot) \) over \( V \) defined by

\[
a(u, v) = a_0(u, v) + a_E(u, v).
\]

(3.24)

where

\[
a_0(u, v) = \text{virtual work produced by the action of the stresses } \sigma_{ij}(u)
\]

on the strains \( \varepsilon_{ij}(v) \)

\[
= \int_{\Omega} \sigma_{ij}(u)\varepsilon_{ij}(v)\,dx = \int_{\Omega} E_{ijkl}u_{k,i}v_{l,j}\,dx.
\]

(3.25)

\[
a_E(u, v) = \text{virtual work produced by the deformation of the linear springs on } \Gamma_E
\]

\[
= \int_{\Gamma_E} K_{ij}u_{j}\rho_{i}\,ds.
\]

(3.26)

(ii) The nonlinear operators \( P \) and \( \Phi \) from \( V \) to \( V' \)

\(^\dagger\) Very little restrictive in the applications (see [17] for details).
\[ \langle P(u), v \rangle_v = \text{virtual work produced by the normal stresses on the \ displacement } v \]
\[ = \int_{T_c} c_n[(u_n - g)_+]^{m} v_n \, ds. \quad (3.25) \]

\[ \langle \Phi(u), v \rangle = \text{virtual work produced by the friction stresses on the \ displacement } v \]
\[ = -\int_{T_c} c_f[(u_n - g)_+]^{m} v \cdot v \, ds. \quad (3.26) \]

(iii) The external forces \( f \in V' \) (body forces \( b \), prescribed tractions \( t \), initial deflection \( U^0 \) of the linear springs)
\[ \langle f, v \rangle_v = \int_{\Omega} b \cdot v \, dx + \int_{\Gamma_T} tv \, ds + \int_{\Gamma_E} K_{ij} U^0_j v_i \, ds. \quad (3.27) \]

Variational statement. With the above definitions and notations established, we can now state a weak formulation for the steady sliding problem (2.6)–(2.11):

Find a function \( u \in V \) such that
\[ a(u, v) + \langle P(u), v \rangle_v + \langle \Phi(u), v \rangle_v = \langle f, v \rangle_v \quad (3.28) \]
for every \( v \in V \).

4. RELATIONSHIP BETWEEN THE CLASSICAL AND THE VARIATIONAL FORMULATIONS

We now show in what sense the solutions of the variational problem (3.28) can be interpreted for the solutions to problem (2.6)–(2.11). Our approach here is quite standard and relies on a generalized Green's formula, given below in Lemma 4.1. In what follows, \( \mathcal{D}(\Omega) \) denotes the space of indefinitely differentiable functions with compact support in \( \Omega \) equipped with the usual inductive limit topology and by \( \mathcal{D}'(\Omega) \) its topological dual, the space of distributions over \( \Omega \). We introduce the operator
\[ \text{div} \, \sigma : V \to [\mathcal{D}'(\Omega)]^N \]
\[ [\text{div} \, \sigma(v)]_l = \sigma_{ij}(v)_j = (E_{ijkl}u_{kl})_j, \quad (4.1) \]
with \( E_{ijkl} \) satisfying the symmetry conditions (2.4) and (3.11)–(3.12). Let \( V_* \) denote the subspace of \( V \) defined by
\[ V_* = \{ v \in V : \text{div} \, \sigma(v) \in [L^2(\Omega)]^N \}. \quad (4.2) \]

Lemma 4.1 (generalized Green's formula): There is a unique linear continuous mapping
\[ \pi : V_* \to W' \]
verifying
\[ [\pi(u)]_l = \sigma_{ij}(u)n_j \quad \text{on } \Gamma. \quad (4.3) \]
for every \( u \in V_* \) such that \( \sigma_{ij}(u) = E_{ijkl}u_{kl} \in C^1(\Omega) \), \( 1 \leq i, j, k, l \leq N \), and
\[ a_0(u, v) + \int_{\Omega} [\text{div} \, \sigma(u)]_j \cdot v \, dx = \langle \pi(u), v \rangle_w. \quad (4.4) \]
for every pair \( (u, v) \in V_* \times V \).

Proof: The proof of this result follows standard arguments and is omitted. See e.g. [9, Theorems 5.8 and 5.9] for a similar statement.

On the basis of the above lemma, we can now prove
Theorem 4.1: An element \( u \in V \) is a solution of the variational problem (3.28) if and only if

(i) \( u \in V \) and

\[
\text{div} \, \sigma(u) + b = 0 \text{ a.e. in } \Omega.
\]

(ii) \( \pi(u) \in [L^{q^*}(\Sigma)]^N \) and [recall the notations (3.7)-(3.8)]

\[
\pi(u) = t \text{ a.e. on } \Gamma_F,
\]

\[
[\pi(u)]_t = -K_{ij}(u_j - U_j^f) \text{ a.e. on } \Gamma_F.
\]

\[
[\pi(u)]_n = -c_n[(u_n - g)_+]^q \text{ a.e. on } \Gamma_C.
\]

\[
[\pi(u)]_T = c_T[(u_n - g)_+]^{q_T} \text{ a.e. on } \Gamma_C.
\]

Proof. Let \( u \in V \) be a solution of the variational problem (3.28) and let \( v \in [\mathcal{D}(\Omega)]^N \subseteq V \). Since all the boundary terms vanish for such a choice of \( v \), one has

\[
a_0(u, v) = \int_\Omega b \cdot v \, dx.
\]

i.e.

\[
\int_\Omega \sigma(u) : \varepsilon_v \, dx = \int_\Omega b \cdot v \, dx.
\]

As the above relation holds for every \( v \in [\mathcal{D}(\Omega)]^N \), it is equivalent to

\[
- \text{div} \, \sigma(u) = b \quad \text{in } [\mathcal{D}'(\Omega)]^N,
\]

which proves (i). Next, as we have just seen that \( u \in V \) and for any given element \( v \in V \), the generalized Green’s formula of Lemma 4.1 and the definitions (3.22)-(3.27) yield,

\[
- \int_\Omega (\text{div} \, \sigma(u)) \cdot v \, dx + \langle \pi(u), v \rangle_W = \int_\Omega b \cdot v \, dx
\]

\[
+ \int_{\Gamma_F} t \cdot \varepsilon_v \, ds - \int_{\Gamma_F} K_{ij}(u_j - U_j^f)v_i \, ds
\]

\[
+ \int_{\Gamma_C} \{-c_n[(u_n - g)_+]^q \varepsilon_v + c_T[(u_n - g)_+]^{q_T} \tau \cdot v\} \, ds.
\]

With (4.5) and since the trace map is onto from \( V \) to \( W \) (cf. Section 3), this reduces to

\[
\langle \pi(u), \xi \rangle_W = \int_\Sigma \psi \cdot \xi \, ds
\]

for every \( \xi \in W \) where \( \psi \in [L^{q^*}(\Sigma)]^N \) is (uniquely) defined by

\[
t \quad \text{on } \Gamma_F,
\]

\[
\psi = [-K_{ij}(u_j - U_j^f)]_{1 \leq i \leq N} \quad \text{on } \Gamma_F,
\]

\[
-c_n[(u_n - g)_+]^q + c_T[(u_n - g)_+]^{q_T} \tau \quad \text{on } \Gamma_C.
\]

From the density of the space \( W \) in the space \([L^q(\Sigma)]^N\) (cf. Section 3), it follows that \( \pi(u) \) can be uniquely extended (by \( \psi \)) as a linear continuous form on \([L^q(\Sigma)]^N\). Hence \( \pi(u) = \psi \in [L^{q^*}(\Sigma)]^N \). Boundary conditions (4.6)-(4.9) are immediate from this result and the notations (3.7)-(3.8).

To prove that conditions (i) and (ii) of Theorem 4.1 imply \( u \in V \) and \( u \) is a solution to the variational problem (3.28), we need only to reverse the steps of the above proof.
5. EXISTENCE AND LOCAL UNIQUENESS OF STEADY SLIDING EQUILIBRIUM SOLUTIONS

In this section we study questions of existence and uniqueness of solutions to the variational problem (3.28). Results will be established either for arbitrary coefficient of friction and sufficiently small external forces† or for arbitrary external forces and sufficiently small coefficient of friction.

It will be convenient to rewrite the variational problem (3.28) as a functional equation between \( V \) and \( V' \). To do this, we introduce the operator \( A \in \mathcal{L}(V, V') \) defined by

\[
a(u, v) = \langle Au, v \rangle_v,
\]

for every pair \((u, v) \in V \times V\) so that the problem becomes: Find \( u \in V \) such that

\[
Au + P(u) + \Phi(u) = f.
\]

We now proceed to review a few mathematical concepts and related results. Given an open subset \( \omega \) of \( \mathbb{R}^d \), let us first recall that a mapping

\[
F : \omega \times \mathbb{R} \to \mathbb{R}
\]

is said to be a Carathéodory function if

(i) for every \( t \in \mathbb{R} \), the function \( F(\cdot, t) : \omega \to \mathbb{R} \) is measurable,

(ii) for almost all \( x \in \omega \), the function \( F(x, \cdot) : \mathbb{R} \to \mathbb{R} \) is continuous. With the Carathéodory function \( F \), we associate the operator \( \tilde{F} \) from the space of measurable real-valued functions on \( \omega \) into itself by

\[
\tilde{F}(y)(x) = F[x, y(x)],
\]

for almost all \( x \in \omega \) (the measurability of \( \tilde{F}(y) \) for a measurable \( y \) is not an obvious result; see e.g. [6, Proposition 1.1, p. 218]). The operator \( \tilde{F} \) is usually referred to as the Nemytskii operator associated with \( F \). A key property, established by Krasnoselskii [10], is that the operator \( \tilde{F} \) acts from \( L^p(\omega) \) into \( L^r(\omega) \), \( 1 \leq p, r \leq +\infty \) if and only if there is a function \( \pi \in L^r(\omega) \) and a constant \( \beta > 0 \) such that, for almost all \( x \in \omega \) and every \( t \in \mathbb{R} \)

\[
|F(x, t)| \leq \pi(x) + \beta|t|^\beta,
\]

if so, \( \tilde{F} \) is then automatically continuous from \( L^p(\omega) \) into \( L^r(\omega) \). Differentiability of the Nemytskii operator can also be expressed in terms of the mapping \( F \) under additional hypotheses. We shall make use of the following result (cf. Rabier [21, Theorem 3.2]).

**Proposition 5.1:** Let \( F : \omega \times \mathbb{R} \to \mathbb{R} \) be a Carathéodory function and suppose that the function \( F(x, \cdot) \) is continuously differentiable for almost all \( x \in \omega \). Suppose further that for some \( 2 < p < +\infty \) there is a function \( \pi \in L^{p/2}\pi(\omega) \) and a constant \( \beta > 0 \) such that for almost all \( x \in \omega \) and every \( t \in \mathbb{R} \)

\[
|F_t(x, t)| \leq \pi(x) + \beta|t|^\beta
\]

where we have set \( F_t = \partial F/\partial t \). Then \( F_t \) is a Carathéodory function and \( \tilde{F}_t \in C^0[L^p(\omega), L^{p/2}\pi(\omega)] \). Besides, if the function \( F(\cdot, 0) \) belongs to the space \( L^{p*}(\omega) \), one has

\[
\tilde{F} \in C^1[L^p(\omega), L^{p*}(\omega)]
\]

and the derivative of \( \tilde{F} \) is given by

\[
D\tilde{F}(y) : z = \tilde{F}_t(y)z \in L^p(\omega)
\]

for every pair \((y, z) \in [L^p(\omega)]^2\).

With an appropriate modification on the growth of \( F_t \), the result stated in Proposition 5.1 remains true for \( p = +\infty \) (cf. [21]) but it is known to be always false when \( p = 2 \), except if \( F(x, t) \) is linear in \( t \) and \( F(x, 1) \in L^{p*}(\omega) \). In other words, restriction to spaces \( L^p(\omega) \) with \( p > 2 \) is indispensable to have differentiability properties of nonlinear Nemytskii's operators. The previous definitions and properties easily carry on to the case when the open set \( \omega \) is replaced by the Lipschitz continuous boundary \( \Gamma \) (in the sense of [14]) of

† This condition can however be partly released; see Remark 5.1.
an open subset $\Omega$ of $\mathbb{R}^m$. Indeed, it suffices to use a system of local charts and an associated partition of the unity to reduce the problem on a collection of open subsets $\omega$ of $\mathbb{R}^m$. Note that by definition of the measure on $\Gamma$ and since $\Gamma$ is Lipschitz continuous, such a transformation introduces $L^m(\omega)$-weights only, which does not affect the value of the exponents $p$ and $r$. Taking $M = N - 1$ and $\Omega$ as in Sections 1–4, we deduce

**Proposition 5.2:** Let $m > 1$ be a given real number. Then, the mapping

$$y \in L^{m+1}(\Gamma) \to (y_+)^m \in L^{m+1}(\Gamma),$$

is of class $C^1$ and its derivative is given by

$$z \in L^{m+1}(\Gamma) \to m(y_+)^{m-1}z \in L^{m+1}(\Gamma).$$

**Proof:** For $(x, t) \in \Gamma \times \mathbb{R}$, let us set

$$F(x, t) = (t_+)^m,$$

where $t_+ = t$ for $t \geq 0$ and $t_+ = 0$ otherwise. It is obvious that $F$ is a Carathéodory function and that $F_t(x, t) = m(t_+)^{m-1}$ is continuous. The growth condition (5.3) is satisfied with $p = m + 1$, $x = 0$ and $\beta = m$. As $F(\bullet, 0) = 0 \in L^{m+1}(\Gamma)$, the result follows from the preceding comments and Proposition 5.1.

**Corollary 5.1:** Under the standing assumptions, the operators $P$ and $\Phi$ defined by (3.25) and (3.26) respectively are of class $C^1$ from $V$ to $V'$. Besides, for every pair $(u, h) \in V^2$,

$$\langle DP(u) \cdot h, v \rangle_V = \int_{\Gamma} m_n c_n[(u_+ - g)_+]^{m-1} h_n v_n ds. \quad (5.5)$$

for every $v \in V$ and

$$\langle D\Phi(u) \cdot h, v \rangle_V = \int_{\Gamma} m_\Gamma c_\Gamma[(u_+ - g)_+]^{m-1} h_\Gamma v_\Gamma ds. \quad (5.6)$$

for every $v \in V$.

**Proof:** Denoting again by $c_n$ and $g$ the extensions by 0 of $c_n$ and $g$ on $\Gamma \setminus \Gamma_C$ an equivalent definition of the operator $P$ (3.25) is

$$\langle P(u), v \rangle_V = \int_{\Gamma} c_n[(u_+ - g)_+]^{m-1} v_n ds. \quad (5.7)$$

for every pair $(u, v) \in V \times V$. In this form, it is clear that $P$ is composed with the following $C^1$ mappings:

(i) the trace $u \in V \to u|_\Gamma \in W$ [cf. (3.4)].

(ii) the linear continuous embedding $W \subset [L^q(\Gamma)]^N$ (cf. Section 3).

(iii) the linear continuous inner product with the normal vector $n \in [L^q(\Gamma)]^N$

$$u \in [L^q(\Gamma)]^N \mapsto u_n = u \cdot n \in L^q(\Gamma).$$

(iv) the translation [cf. (3.25)]

$$y \in L^q(\Gamma) \mapsto y - g \in L^q(\Gamma).$$

(v) the linear continuous embedding $L^q(\Gamma) \subset L^{m+1}(\Gamma)$ [cf. (3.17)]

(vi) the mapping [cf. (3.17) and Proposition 5.2 with $m = m_n$]

$$y \in L^{m+1}(\Gamma) \mapsto (y_+)^m \in L^{m+1}(\Gamma).$$

(vii) the linear continuous multiplication by the function $c_n \in L^{\infty}(\Gamma)$

$$y \in L^{m+1}(\Gamma) \mapsto c_n y \in L^{m+1}(\Gamma).$$

(viii) the continuous embedding $L^{m+1}(\Gamma) \subset L^{\infty}(\Gamma)$.

(ix) the transpose of (iii), (ii), and (i) above.

Formula (5.5) is then a simple application of the chain rule.
A similar method establishes the continuous differentiability of the operator $\Phi(3.26)$ after writing

$$
\langle D\Phi(u) \cdot \mathbf{h}, \mathbf{v} \rangle_V = \int_{\Omega} m_T c_T [(u_n - g)_+]^{m_T-1} h_n \mathbf{v} : \mathbf{v} \, ds,
$$

where $c_T$, $g$ and $\tau$ have been extended by 0 on $\Gamma \setminus \Gamma_C$. The steps (i)-(viii) are the same, except for changing $m_n$ into $m_T$ and $c_n$ into $c_T$. The step (ix) must be modified into “the transpose of the linear continuous inner product with the $[L^q(\Gamma)]^N$ vector field $\tau$:

$$
v \in [L^q(\Gamma)]^N \rightarrow \tau \cdot v \in L^q(\Gamma),
$$

and the transpose of (ii) and (i)".

We are now in position to prove our first existence and local uniqueness result.

**Theorem 5.1:** Assume that $\text{meas}(\Gamma_0) > 0$ or $\text{meas}(\Gamma_E) > 0$. Then, there are neighborhoods $U$ and $U'$ of the origin in $V$ and $V'$ respectively such that for every $f \in U'$ the functional equation (5.2) has one and only one solution $u \in U$.

**Proof:** If $\text{meas}(\Gamma_0) > 0$, it is well-known from Korn's inequality and by a simple contradiction argument that the bilinear continuous form $a_0(\cdot, \cdot)$ is also $V$-elliptic (see e.g. [9]) and, due to (3.14) and (3.24), the same is true with the bilinear continuous form $a(\cdot, \cdot)$ (3.22). If $\text{meas}(\Gamma_E) > 0$, the same arguments as above joined to (3.14) prove the $V$-ellipticity of $a(\cdot, \cdot)$. As a result, the operator $A$ (5.1) verifies

$$
A \in \text{Isom}(V, V').
$$

Let us denote by $G: V \rightarrow V'$ the mapping

$$
G(u) = Au + P(u) + \Phi(u).
$$

Since $g \geq 0$ on $\Gamma_C$, one has $G(0) = 0$. Besides, from Corollary 5.1, the mapping $G$ is continuously differentiable and (using $g \geq 0$ again) $DG(0) = A$. From (5.8) and the inverse mapping theorem, $G$ is a $C^1$-diffeomorphism between two neighborhoods of the origin in $V$ and $V'$ respectively and our assertion follows.

**Remark 5.1:** Existence and uniqueness for “small” right-hand sides $f \in V'$ can be partly released as follows: set

$$
V_0 = \{ v \in V; v_n = v \cdot n = 0 \text{ on } \Gamma_C \}.
$$

An immediate verification shows that $V_0$ is a closed subspace of $V$ so that

$$
V = V_0 \oplus V_0^\perp,
$$

where the orthogonal is taken in the sense of the equivalent inner product $a(\cdot, \cdot)$ (assuming of course $\text{meas}(\Gamma_0) > 0$ or $\text{meas}(\Gamma_E) > 0$ as in Theorem 5.1). For each $v \in V$, let us write

$$
v = v_0 + v_0^\perp,
$$

according to the decomposition (5.10). The variational problem (3.28) can then be rewritten as: Find $u = u_0 + u_0^\perp$ such that

$$
a(u_0, v_0) + a(u_0^\perp, v_0^\perp) + \langle P(u_0 + u_0^\perp), v_0 + v_0^\perp \rangle_V + \langle \Phi(u_0 + u_0^\perp), v_0 + v_0^\perp \rangle_V
$$

$$
= \langle f, v_0 \rangle_V + \langle f, v_0^\perp \rangle_V
$$

(5.11)

for every $v \in V$. Now, from the definition (5.9) of $V_0$ and that of the operators $P$ and $\Phi$ [cf. (3.25)-(3.26)] it is clear that

$$
P(u_0 + u_0^\perp) = P(u_0^\perp), \quad \Phi(u_0 + u_0^\perp) = \Phi(u_0^\perp).
$$

In addition, it is also immediate that $\langle P(u), v_0 \rangle_V = 0$ for every $u \in V$ and every $v_0 \in V_0$. It follows that (5.11) reduces to the system:

Find $u_0^\perp \in V_0^\perp$ such that

$$
\langle P(u_0 + u_0^\perp), v_0 + v_0^\perp \rangle_V + a(u_0 + u_0^\perp, v_0 + v_0^\perp)
$$

$$
= \langle f, v_0 + v_0^\perp \rangle_V
$$

(5.11)

for every $v \in V$. Now, from the definition (5.9) of $V_0$ and that of the operators $P$ and $\Phi$ [cf. (3.25)-(3.26)] it is clear that

$$
P(u_0 + u_0^\perp) = P(u_0^\perp), \quad \Phi(u_0 + u_0^\perp) = \Phi(u_0^\perp).
$$

In addition, it is also immediate that $\langle P(u), v_0 \rangle_V = 0$ for every $u \in V$ and every $v_0 \in V_0$. It follows that (5.11) reduces to the system:

Find $u_0^\perp \in V_0^\perp$ such that

$$
\langle P(u_0 + u_0^\perp), v_0 + v_0^\perp \rangle_V
$$
for every \( v_0 \in V_0 \). Equation (5.12) has the same form as the original problem but is posed on the space \( V_0^1 \) instead of \( V \). The same arguments as in Theorem 5.1 thus yield existence and uniqueness of \( u_0 \) for sufficiently small right-hand sides, namely provided that the action of \( f \) on the elements of \( V_0^1 \) (and not of the whole space \( V \)) is small enough. Once \( u_0 \) has been determined from (5.12), existence and uniqueness of \( u_0 \) solution to (5.13) follows with no further assumption since equation (5.13) is linear in \( u_0 \). □

The second part of our results consists in proving existence and local uniqueness of a solution to the variational problem (3.28) [or to the equivalent functional equation (5.2)] for arbitrary external forces \( f \in V' \) but sufficiently small coefficient of friction. To do this we need the following preliminary lemma

**Lemma 5.1**: Assume that \( \text{meas}(\Gamma_D) > 0 \) or \( \text{meas}(\Gamma_u) > 0 \). Then, for every \( f \in V' \) the equation

\[
Au + P(u) = f
\]

has a unique solution \( u \) and the operator \( A + D P(u) \) is an isomorphism of \( V \) to \( V' \).

**Proof**: We begin with the observation that, for every \( u \in V \), \( P(u) \in V' \) is the derivative at \( u \) of the functional (well-defined from the admissible values of \( m_n \) given in (3.17))

\[
u \in V \rightarrow \int_{\Gamma_c} \frac{c_n}{m_n + 1} [(u_n - g)_+]^{m_n+1} \, ds \geq 0.
\]

Using a system of local charts and an associated partition of the unity this follows from a well-known result by Krasnoselskii [10, Lemma 5.1, p. 70] when \( \Gamma_c \) is replaced by an open subset of \( \mathbb{R}^{N-1} \). From Corollary 5.1, one has

\[
\langle DP(u) \cdot v, v \rangle_V = \int_{\Gamma_c} m_n c_n [(u_n - g)_+]^{m_n+1} \, ds \geq 0.
\]

for every \( v \in V \). From (5.16) and the \( V \)-ellipticity of \( a(\cdot, \cdot) \) under either assumption \( \text{meas}(\Gamma_D) > 0 \) or \( \text{meas}(\Gamma_u) > 0 \) as recalled in Theorem 5.1, the continuous functional

\[
J(u) = \frac{1}{2} a(u, u) + \int_{\Gamma_c} \frac{c_n}{m_n + 1} [(u_n - g)_+]^{m_n+1} \, ds - \langle f, u \rangle_V
\]

is strictly convex and coercive. As solutions to equation (5.14) are precisely the critical points of the functional \( J \), existence and uniqueness follows. For every \( u \in V \), the \( V \)-ellipticity of the bilinear form \( a(\cdot, \cdot) \) also yields the \( V \)-ellipticity of the bilinear form (c.f. (5.16))

\[
(v, w) \in V \times V \rightarrow a(v, w) + \langle DP(u) \cdot v, w \rangle_V = \langle [A + D P(u)] \cdot v, w \rangle_V
\]

and hence,

\[
A + D P(u) \in \text{Isom}(V, V').
\]

**Theorem 5.2**: Assume that \( \text{meas}(\Gamma_D) > 0 \) or \( \text{meas}(\Gamma_u) > 0 \). For every \( f \in V' \) denote by \( u^0 \in V \) the unique solution of the equation

\[
Au^0 + P(u^0) = f
\]

(cf. Lemma 5.1). Then, there is a neighborhood \( \Delta \) of the origin in the space \( L^\infty(\Gamma_c) \) and there is a neighborhood \( U_0 \) of \( u^0 \) in \( V \) such that equation (5.2) has a unique solution \( u \in U_0 \) for every \( c_r \in \Delta \).

**Proof**: In this proof we shall make explicit the dependence of the operator \( \Phi \) on the coefficient \( c_r \) by setting \( \Phi = \Phi(c_r, u) \). For a given coefficient \( c_r \in L^\infty(\Gamma_c) \) we proved in Corollary 5.1 that \( \Phi \) is continuously differentiable with respect to \( u \in V \) with the formula
\[
\left\langle \frac{\partial \Phi}{\partial u}(c_T, u) \cdot h, v \right\rangle_V = \int_{\Gamma_C} m_T c_T (u_n^0 - g)^+ h_n^0 v \, ds
\]  
(5.19)

holding for every \((c_T, u, h, v) \in L^\infty(\Gamma_C) \times V^3\). The definition (3.26) and formula (5.19) make it clear that both mappings \(\Phi\) and \(\partial \Phi/\partial u\) are continuous with respect to the pair \((c_T, u) \in L^\infty(\Gamma_C) \times V\). Now define the operator \(G: L^\infty(\Gamma_C) \times V \to V'\) by

\[
G(c_T, u) = Au + P(u) + \Phi(c_T, u),
\]

for \((c_T, u) \in L^\infty(\Gamma_C) \times V\). As \(\Phi(0, \bullet) = 0\), equation (5.18) reads

\[
G(0, u^0) = f.
\]

On the other hand, as \((\partial \Phi/\partial u)(0, \bullet) = 0\) [cf. (5.19)] one has

\[
\frac{\partial G}{\partial u}(0, u^0) = A + DP(u^0).
\]  
(5.20)

From the above properties of \(\Phi\), both \(G\) and \(\partial G/\partial u\) are continuous with respect to the pair \((c_T, u) \in L^\infty(\Gamma_C) \times V\) and, from (5.20) and Lemma 5.1.

\[
\frac{\partial G}{\partial u}(0, u^0) \in \text{Isom}(V, V').
\]

With the so-called "strong version" of the implicit function theorem (see e.g. Lyusternik and Sobolev [13]) there are neighborhoods \(\Delta\) of the origin in \(L^\infty(\Gamma_C)\) and \(U_0\) of \(u^0\) in \(V\) such that the equation \(G(c_T, u) = f\) [which is nothing but equation (5.2)] has exactly one solution \(u \in V\) for every given \(c_T \in \Delta\) and the proof is complete. \(\blacksquare\)

6. THE PROBLEM WITH NO FRICTION: RELATIONSHIP TO SIGNORINI'S PROBLEM

We shall complete this paper by showing how the problem with no friction (i.e. \(c_T = 0\)) relates to the classical Signorini's problem:

Minimize

\[
\frac{1}{2} a(v, v) - \langle f, v \rangle_V
\]  
(6.1)

over the closed convex set

\(K = \{v \in V; v_n \leq 0 \text{ on } \Gamma_C\}\).  
(6.2)

It is known (see e.g. [5]) that this problem has a unique solution \(v^0\) verifying (formally at least)

\[
\begin{align*}
\text{div} \sigma(u^0) + b & = 0 \quad \text{in } \Omega, \\
\pi(u^0) & = t \quad \text{on } \Gamma_F, \\
\pi(u^0)_n & = -K_i (u^0_f - U^F_i) \quad \text{on } \Gamma_F, \\
u^0_n & \leq 0 \quad \text{on } \Gamma_C, \\
[u^0]_{n} & \leq 0, \quad [\pi(u^0)]_n u^0_n = 0 \quad \text{on } \Gamma_C, \\
[\pi(u^0)]_T & = 0 \quad \text{on } \Gamma_C.
\end{align*}
\]  
(6.3)-(6.8)

In this case, we have assumed for simplicity that \(g = 0\).

From Lemma 5.1. the problem with no friction and power-like normal response has a unique solution, characterized as a minimizer of the functional

\[
v \in V \to J(v) = \frac{1}{2} a(v, v) + \int_{\Gamma_C} \frac{c_n}{m_n + 1} [(v_n)_+^+ m_n + 1] ds - \langle f, v \rangle_V.
\]  
(6.9)
For $\lambda \geq 1$, denote by $u(\lambda)$ the solution with $\lambda c_n$ replacing $c_n$, i.e. $u(\lambda)$ minimizes the functional

$$
J_\lambda(v) = \frac{1}{2} a(v, v) + \lambda \int_{\Gamma_c} \frac{c_n}{m_n + 1} [(v_n)_+]^{m_n + 1} ds - \langle f, v \rangle_{\mathcal{V}}.
$$

(6.10)

An equivalent characterization of $u(\lambda)$ is obtained by saying that $u(\lambda)$ is the (unique) critical point of the functional $J_\lambda$ (6.10), namely,

$$
a(u(\lambda), v) + \lambda \int_{\Gamma_c} c_n [(u_n(\lambda))_+]^{m_n} v_n ds = \langle f, v \rangle_{\mathcal{V}}
$$

(6.11)

for every $v \in \mathcal{V}$.

Theorem 6.1: Assume $c_n > 0$ on $\Gamma_c$. Then

$$
\lim_{\lambda \to +\infty} \|u(\lambda) - u^0\|_{\mathcal{V}} = 0.
$$

Proof: It can be easily shown that the functional

$$
\forall v \in \mathcal{V} \to p(v) = \int_{\Gamma_c} \frac{c_n}{m_n + 1} [(v_n)_+]^{m_n + 1} ds
$$

is an exterior penalty functional for the constraint set $K$ (6.2). In fact $p$ is weakly lower semicontinuous (since it is convex and differentiable) and satisfies the conditions: $p(v) \geq 0 \\forall v \in \mathcal{V}$ and $p(v) = 0$ if and only if $v \in K$ (since $c_n > 0$). It follows from the coercivity and weak lower semicontinuity of the quadratic form $a(v, v)$ and from well-known results (see e.g. [16]), that there exists a subsequence of the family of solutions $(u(\lambda))_{\lambda \geq 1}$ converging weakly in $\mathcal{V}$, as $\lambda \to +\infty$, to a minimizer of the functional (6.1) on $K$. Since the minimizer of (6.1) on $K$ is unique it follows by a standard argument that the whole sequence $(u(\lambda))_{\lambda \geq 1}$ converges weakly to the unique solution $u^0$ of the Signorini's problem. which is characterized by the variational inequality

$$
(u^0, v - u^0) \geq \langle f, v - u^0 \rangle_{\mathcal{V}} \forall v \in K
$$

(6.12)

To prove strong convergence it suffices to show that

$$
\lim_{\lambda \to +\infty} \|u(\lambda)\|_{\mathcal{V}} = \|u^0\|_{\mathcal{V}}.
$$

Actually, since the coerciveness of the bilinear form $a(\bullet, \bullet)$ over the space $\mathcal{V}$ implies that $a(\bullet, \bullet)$ is an equivalent inner product over $\mathcal{V}$, we may limit ourselves to proving

$$
\lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda)) = a(u^0, u^0).
$$

(6.13)

From the weak lower semicontinuity of the quadratic form $a(v, v)$ and the weak convergence of $u(\lambda)$ to $u$ one has

$$
a(u^0, u^0) \geq \lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda)).
$$

(6.14)

On the other hand, letting $v = u(\lambda)$ in (6.11) and observing that $(u_n)_+^{m_n + 1} \geq 0$ it follows that

$$
a(u(\lambda), u(\lambda)) \leq \langle f, u(\lambda) \rangle_{\mathcal{V}}.
$$

which, together with (6.12) at $v = 2u^0$ implies

$$
a(u(\lambda), u(\lambda)) \leq \langle f, u(\lambda) \rangle_{\mathcal{V}} + a(u^0, u^0) - \langle f, u^0 \rangle_{\mathcal{V}}.
$$

Hence

$$
\lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda)) \leq a(u^0, u^0)
$$

(6.15)

Relations (6.14) and (6.15) show that $\lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda))$ exists and coincides with the common value $\lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda)) = \lim_{\lambda \to +\infty} a(u(\lambda), u(\lambda))$. The desired equality (6.13) is now immediate.
and the proof is complete. ■

From Theorem 6.1 and when $c_\sigma > 0$ on $\Gamma_C$ (the only physically acceptable assumption!) the solution $u^\theta$ of Signorini's problem (6.3)–(6.8) then appears as the limiting case of an infinite normal stiffness on $\Gamma_C$. Of course, such an hypothesis is only an approximation of the physical reality which can a priori be made in the aim of simplifying the model. But it is well known that this simplification does not allow in general to consider friction phenomena: by the means of a powerlike normal response, we have shown that friction problems could be taken into account while the solution to the problem with no frictions is "close" to the solution of Signorini’s problem provided that the normal stiffness is "large enough", a physical reality for highly polished surfaces [17].

REFERENCES


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