ADAPTIVE FINITE ELEMENT METHODS FOR COMPLEX PROBLEMS IN SOLID AND FLUID MECHANICS

J. T. Oden and L. Demkowicz

Texas Institute for Computational Mechanics, The University of Texas at Austin, USA

ABSTRACT

This paper addresses the general topic of adaptive methods for automatically enhancing the quality of numerical solutions to linear and nonlinear boundary-value problems in solid and fluid mechanics, and reviews some of the recent work of the author and his collaborators on this subject.

KEYWORDS

Adaptive Finite Element Methods, A Posteriori Error Estimates.

INTRODUCTION

The basic objective of an adaptive finite element method is to improve the quality of an initial finite element approximation by automatically changing the model: refining the mesh, moving mesh nodal points, enriching the local order of approximation, etc. Thus, all adaptive methods must attempt to resolve two basic issues: 1) how is the quality of the approximate solution to be measured? and 2) how does one adapt the model to improve the quality of the approximation?

The first question is generally resolved by attempting to measure the local approximation error in some appropriate norm. The error, of course, is the difference between the exact solution $u$ and a finite element approximation $u_h$ of $u$ on a given mesh. Since $u$ is not known, the problem of assessing the quality of an approximation reduces to one of a-posteriori error estimation: the determination of estimates of the error using computed finite element solutions. A number of important papers on various schemes for a-posteriori error estimation has been contributed by Babuska and his collaborators (1978a, 1978b, 1985).

Once an estimate of the distribution of the error is available, the difficult question of how to best modify the model to improve accuracy arises. There are three general approaches:
h-Methods. Here the mesh is refined; the mesh size \( h \) is reduced and the number of elements in the mesh is increased in regions of large error.

p-Methods. Here the mesh is fixed but the degree \( p \) of the polynomial shape functions is increased over elements in which a high error is indicated.

Moving Mesh Methods. In these methods, the number of nodes and the type of finite element remains constant during the adaptive process and the nodal points are moved to regions of high error.

Of course, one can also employ combinations of these strategies. But the correct strategy for use of combined methods is apparently a delicate issue and one in which much additional study needs to be done.

We shall describe here two methods for error estimation and show how these can be implemented in each of the three adaptive schemes listed above.

A-PREORIORI ERROR ESTIMATES

We describe two classes of a-posteriori error estimation, one based on the computation of element residuals and the other based on interpolation error estimates. The former class of methods was introduced by Demkowicz, Oden, and Strouboulis (1984) and includes several of the results of Babuska and colleagues (1978a, 1978b, 1983), and the latter was first advocated by Diaz, Kikuchi, and Taylor (1983) and investigated by Demkowicz and Oden (1985a, 1985b, 1985c).

Residual Methods

Consider the abstract boundary-value problem, Find \( u \) in \( V \) such that

\[
<Au, v> = <f, v> \quad \text{for all } v \text{ in } V 
\] (1)

where

\[
A \quad \text{a (possibly nonlinear) operator from a reflexive Banach space of admissible functions } V \text{ into its dual } V^* \\
v \quad \text{an arbitrary test function in } V \\
f \quad \text{given data in } V^* \\
\langle \cdot, \cdot \rangle \quad \text{duality pairing on } V^* \times V
\]

This problem is equivalent to the abstract problem: \( Au = f \) in \( V^* \).

A Galerkin approximation of (1) consists of seeking a function \( u_h \) in a finite dimensional subspace \( V_h \) of \( V \) such that

\[
<A_{u_h}, v_h> = <f, v_h> \quad \text{for all } v_h \text{ in } V_h 
\] (2)

The residual \( r_h \) is the degree with which the approximation \( u_h \) fails to satisfy the original conditions on the solution:

\[
r_h = Au_h - f \neq 0 , \quad r_h \in V^* 
\] (3)

Since the residual belongs to the dual space \( V^* \) and not necessarily \( V \), its magnitude must be measured with respect to the norm \( \| \cdot \|_* \) on \( V^* \):
where $\| \cdot \|$ is the norm in $V$.

In some of the error estimators that we have developed, we use the following procedure to approximate the supremum in (4):

- The original finite element approximation $u_h$ is computed in a space $V_h = V_h^0$ of spanned by low-order (say, linear) piecewise polynomial shape functions, resulting in the residual $r_h^1$.

- The full space $V$ is approximated by a higher-order finite element space $V_h^p$, $p > 1$, spanned by piecewise polynomials of degree $p$.

An approximation of the residual $r_h^1$ is constructed according to

$$
\| r_h^1 \|_* \leq C \| v_0 - v_h^p \| + \sup_{\| v_h^p \| \leq 1} \langle r_h^1, v_h^p \rangle
$$

where $C$ is a constant, $v_0$ is an element of $V$ realizing the sup in (4) and $v_h^p$ is an arbitrary element of $V_h^p$. If $h$ is the mesh size (i.e. for a partition $T_h$ of elements $K$,

$$
h = \max_{K \in T_h} h_K, \quad h_K = \text{diameter} (K)
$$

we generally have

$$
\| v_h^p - v_h^p \| = O(h^p)
$$

so that it makes sense asymptotically (as $h \to 0$) to approximate $\sup \langle r_h^1, v \rangle$ by $\sup \langle r_h^1, v_h^p \rangle$.

As an example of how this procedure can be implemented, consider the model problem,

Find $u$ in $V = \{ v \in H^1(\Omega) \mid v = 0$ on $\Gamma_1 \}$ such that

$$
\int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} fv dx + \int_{\Gamma_2} gv ds \quad \text{for all } v \text{ in } V
$$

This is the variational form of the model Poisson problem,

$$
-\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2
$$

$$
u = 0 \text{ on } \Gamma_1 \subset \partial \Omega
$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma_2 \subset \partial \Omega
$$
with $\Delta = \nabla^2, H^1(\Omega)$ the usual Sobolev space of functions with derivatives in $L^2(\Omega)$, and $\partial \Omega = \Gamma_1 \cup \Gamma_2$.

We define

$$\mathcal{V}_h = \{ v_h \in \mathcal{V} : v_h|_K \in Q_1(K), \partial \Omega = \Gamma \},$$

$$\mathcal{V}_h^p(K) = \{ v_h^p \in \mathcal{P}_p(K) \}$$

$$\mathcal{V}_h^{p,0}(K) = \{ v_h^p \in \mathcal{V}_h^p(K) ; p > 1, v_h^1 \text{ interpolant } v_h^0 = 0 \}$$

where $Q_1(K)$ is the usual set of bilinear functions defined on a quadrilateral element $K$ and $\mathcal{P}_p(K)$ is the space of polynomials of degree $p$ defined on $K$.

The residual $r_h^1$ satisfies

$$<r_h^1, v_h^p> = \sum_K \left( \int_K (-\Delta v_h^1 - f) v_h^p \, dx + \int_{\partial K \setminus \partial \Omega} \frac{\partial u_h^1}{\partial n} - \frac{\partial u^*_h}{\partial n} \right) v_h^p \, ds$$

$$+ \int_{\partial K \cap \Gamma_2} (\frac{\partial u_h^1}{\partial n} - g) \, v_h^p \, ds = \sum_K <r_h^1, v_h^p>$$

where $u_h^1$ is the (coarse-grid-initial) finite element approximation of $u$ determined using the space $\mathcal{V}_h$ and $r_h^1$ is the functional on $\mathcal{V}_h^p$ defined as indicated, with $\partial u_h^1/\partial n$ an approximation of $\partial u/\partial n$ obtained from an adjacent element to $K$.

It is not enough to simply calculate $r_h^1$ as an indicator of the error in element $K$. In general, we wish to have an indicator $\phi_K$ of the error which will bound the local error above and below and which will converge to zero at the same rate as the actual error; e.g.

$$C \| \phi_K \|_{1,K} \leq \| \text{error} \|_{1,K} \leq \| \phi_K \|_{1,K}$$

$$\| \phi_K \|_{1,K}^2 = \int_K \nabla \phi_K \cdot \nabla \phi_K \, dx$$

such an error indicator is obtained as a solution of the auxiliary problem,

$$\int_K \nabla \phi_K \cdot \nabla v_h^p \, dx = <r_h^1, v_h^p> \text{ for all } v_h^p \in \mathcal{V}_h^{p,0}(K)$$

We generally compute the solution of (11) using the concept of hierarchic elements in which the stiffness matrices are only modified by the addition of a row and column with the addition of each degree of freedom (see, e.g., Carey and Oden, 1981 for details). Using (11), (9), and (5), we have (to within terms of $O(h^p)$)

$$\int_K \nabla \phi_K \cdot \nabla v_h^p \, dx = <r_h^1, v_h^p>$$
where $C$ is a (hopefully) known constant. Though this estimate is global, we use $\| \phi \|_{1,K}$ as an estimate of the local error over each element $K$. In general, reducing $\| \phi \|_{1,K}$ implies a reduction in $\| r_h^1 \|_h$ which (particularly for linear self-adjoint problems on Hilbert spaces) implies a reduction in $\| u-u_h^1 \|$. 

**Interpolation Error Estimates**

It is well known (see, e.g., Oden and Carey, 1983) that for linear elliptic problems the approximation error $\| e_h \|_V = \| u-\tilde{u}_h^1 \|_V$ can be bounded above by the so-called interpolation error,

$$ \| e_h \|_V \leq C \| u-\tilde{u}_h^1 \|_V \quad \text{for all } v^1_h \text{ in } V^1_h $$

(13)

For the model problem (6), for example,

$$ \int_{\Omega} \nabla(u-u_h^1) \cdot \nabla(u-u_h^1) \, dx $$

$$ |u-u_h^1|_{1,\Omega} \leq C \inf_{v_h \in V_h} |u-\tilde{u}_h^1|_{1,\Omega} $$

(14)

If $u$ is smooth enough, a local interpolation error estimate can be derived of the type (for $Q_1$-elements)

$$ |u-\tilde{u}_h^1|_{1,K} \leq C h_K |u|_{2,K} $$

(15)

where

$$ |u|_{2,K}^2 = \int_K W \, dx $$

$$ W \, dx \equiv \left( \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} \right) \, dx \, dy $$

(16)

The basic problem we face when attempting to make use of any of these estimates is that we must calculate the higher order derivatives of the unknown solution using only available information, i.e., through use of the currently available finite element solution $u_h^1$. There are numerous a priori techniques for estimating the second derivatives $u_{xx}, u_{xy}$ or $u_{yy}$, but many are somewhat intuitive and not all are based on rigorous estimates. Exceptions are the techniques based on so-called "extraction formulas" introduced by Babuska and Miller (1984a, 1984b). Following their idea one can prove that, if $u$ is regular enough, then the second derivatives at an arbitrary point $(x_0,$
If we have $y_0$ that satisfy
\[
\frac{\partial^2 u}{\partial x^2}(x_0, y_0) = \frac{\partial^2 u}{\partial y^2}(x_0, y_0) = \int_\Omega \Delta u \, dxdy
\]
\[
\int_\Omega (\phi + \overline{\phi}) f \, dxdy - \int_{\partial\Omega} u \frac{\partial}{\partial n} (\phi + \overline{\phi}) ds + \int_{\partial\Omega} (\phi + \overline{\phi}) \frac{\partial u}{\partial n} ds
\]  
\tag{17}
\]

Here $\phi = \frac{1}{\pi} \cos^2 \frac{r}{2}$ where $(r, \theta)$ are the polar coordinates centered at the point $(x_0, y_0)$ under consideration and $\overline{\phi}$ is an arbitrary, regular function. By the proper choice of $\overline{\phi}$, one can eliminate the boundary terms in (17). Of course, $u$ on the right-hand side of (17) remains still unknown, but when replaced by its element approximations $u_h^n$ results in a formula for approximation of second-derivatives at $(x_0, y_0)$ of the same order of accuracy as the $L^2$-error in the approximation of $u$ by $u_h^n$. For example, for the first order approximation we can "extract" the difference of second order derivatives with $O(h^2)$ order of convergence! Formula (17), when combined with equation (7), allows us to calculate each of the derivatives separately. Also, by choosing $\phi = \frac{1}{\pi} \sin^2 \theta$ in the same formula, we can "extract" the mixed derivative $\frac{\partial^2 u}{\partial x \partial y}(x_0, y_0)$.

One method we have used successfully in applying the estimate (15) is to construct the function $\phi$ using a bivariate blending function of Gordon and Hall (1971, 1973) type.

Note that we still have a global estimate although we "apply it" to $K$
\[
|u - u_h^1|_1, \Omega \leq C \sum_{K} h^2_K |u|_2, K
\]  
\tag{18}
\]

MESH REFINEMENT STRATEGIES BASED ON THE A POSTERIORI ERROR ESTIMATES

While many issues remain open in the area of reliable a-posteriori error estimation, still further complications exist in designing efficient adaptive algorithms based on these estimates. The basic problem can assume the form of an optimal control problem in which one has to attain a discrete approximation which is optimal in some sense determined by the error measures and the strategy used to reduce error. The entire problem is further complicated by the fact that our a-posteriori estimates are global in nature (particularly the residual-type estimates discussed earlier) even though they are used locally as a basis for local enrichments of the solution.

In this section we describe three methods developed by Oden and colleagues, (1985), Demkowicz and Oden (1985), Demkowicz, Oden, and Strouboulis (1984), and Demkowicz, Oden, and Devloo (1985).

An $h$-Method

Consider a quadratic mesh and the associated error estimate (18). If we
define a function \( h(x, y) \) specifying a "density" of the mesh by:

\[
h(x, y) = h_K \quad \text{if} \quad (x, y) \in K
\]  \hspace{1cm} (19)

the error estimate (18) may be written in the form of a functional

\[
J(h) = \int_{\Omega} h^2 W \, dx \, dy
\]  \hspace{1cm} (20)

where \( W \) is given by (16).

This leads to a natural minimization problem:

Find \( h = h_{\text{opt}} \) minimizing the functional (20) and subjected to the constraint:

\[
\int_{\Omega} \frac{1}{h} \, dx \, dy = N
\]  \hspace{1cm} (21)

where \( N \) is simply the number of elements. The solution of this problem is

\[
h^2 \int_K W \, dx \, dy = \text{const. for every element } K
\]  \hspace{1cm} (22)

which results in a very simple iterative - mesh refinement scheme, provided we can estimate the function \( W \). For additional details we refer to Demkowicz, Oden, and Devloo (1985).

A p-Method (see Oden and others, 1985; Demkowicz, Oden, Strouboulis, 1985)

It is difficult to predict the rate of convergence of local interpolation error in the case of the p-version since it depends only on the unknown regularity of the solution (Babuska and all, 1981), and it is almost impossible to say anything about a local order of convergence of the error or the residual-type estimate. One way that we have used estimates of the type (12) effectively is to employ the following steps:

1. Solve local problem (11) and determine local contribution \( |\phi_K|_{1,K} \) to the global error estimate.
2. Normalize the local indicators \( |\phi_K|_{1,K} \) by subdividing by the largest one.
3. If
   \[
   \begin{align*}
   &0 < |\phi_K|_{1,K} < \delta_1 \quad \text{the first order approximation is retained.} \\
   &\delta < |\phi_K|_{1,K} < \delta_2 \quad \text{a second-order approximation is used.} \\
   &\delta_2 < |\phi_K|_{1,K} < 1 \quad \text{a third-order approximation is used.}
   \end{align*}
   \]

Here the numbers \( \delta_1 \) and \( \delta_2 \) are chosen rather arbitrarily and \( u_h^1 \) corresponds to first-order approximation on a uniform mesh.
Moving Mesh Strategy (see Demkowicz and Oden, 1985a)

A popular moving finite element method, due to Miller, is based on an $L^2$-residual estimate in its basic form which finds justification in only the one-dimensional case. In Demkowicz and Oden (1985) we present a moving mesh strategy based on the interpolation error estimate that can be considered as a generalization of the Brackbill and Saltzman strategy (1982).

We consider a fixed mesh of elements $K$ that is mapped onto a distorted mesh in such a way as to minimize the interpolation error. We may penalize the interpolation error functional further by appending to it additional terms to smooth the mesh and prevent the Jacobian of the map from vanishing.

In the method discussed in Demkowicz and Oden (1985a) the optimal mesh results from minimizing the functional

$$J = I_0 + \alpha I + \beta J$$  \hspace{1cm} (23)

where

$$I_0 = \sum_{K} \left( \int_{\hat{K}} \left( x^2 + y^2 + x_x^2 + y_y^2 \right) d\xi d\eta \right)$$  \hspace{1cm} (24)

$$I = \sum_{K} \left( \int_{\hat{K}} \frac{1}{J} \left[ \left( x_x^2 + y_y^2 \right) u_{\xi\xi} + \left( x_x^2 + y_y^2 \right) u_{\eta\eta} \right] d\xi d\eta \right)$$  \hspace{1cm} (25)

$$J = \sum_{K} \left( \int_{\hat{K}} \frac{1}{J} \left( x_x y_{\xi} + x_{\eta} y_{\eta} \right)^2 \right) d\xi d\eta$$  \hspace{1cm} (26)

Here $\alpha$ and $\beta$ are constant parameters, $I_0$ is the Dirichlet integral which treats as a constraint the invertibility of the transformation defining the mesh. In the absence of other terms, its minimization yields a conformal mesh and then the method is merely a generalization of the well-known conformal-map mesh generators. The functional $K$ forces the mesh lines to be orthogonal and this orthogonality allows us to simplify the form of the error functional $I$.

NUMERICAL EXPERIMENTS

In this section we present some representative examples.

A Heat Conduction Problem With a Moving Domain

We first consider a linear transient heat conduction problem defined on a moving domain $\Omega = (0, 4 + 0.1t) \times (0, 3)$ with purely homogeneous Dirichlet boundary and initial data. We choose the data on the right-hand side of the equation to correspond with a prescribed exact solution:

$$u = 10 e^{-5(x - 1 - 0.2t)^2} x (4 + 0.1t - x) y (3 - y) \quad C.$$  \hspace{1cm} Where:

$$C = \begin{cases} t \text{ for } 0 < t < 0.5 \\ 1 \text{ for } t > 0.5 \end{cases}$$

This problem has been solved for a mesh with 24 elements (see Fig. 1). The time step has been chosen $\Delta t = 0.1$ and the solution was computed for 20
time steps. The constants $\delta_1$ and $\delta_2$ described in the previous chapter have been chosen as follows:

$$\delta_1 = 1/20, \sigma_2 = 1/2.$$  

Figure 1 shows the mesh enrichment for terms $t = 0.5, 1.0, 1.5$ and $2.0$ while Figs. 2 and 3 present the computed first-order and enriched solutions on the section AA (see Fig. 1) compared to the exact solution. For additional details we refer to Demkowicz, Oden, and Strouboulis (1984).

A Stokes Problem

The second problem we consider deals with the application of the moving mesh strategy described earlier rather than the penalty-formulation. In this example $\Omega$ is the rectangle $(0, 2) \times (0, 1)$ and we define on this domain an initial uniform $16 \times 8$ mesh of rectangular elements. Dirichlet boundary conditions are used with the velocity $u$ prescribed as $u_0 = (1, 0)$ along the top edge and $u = (0, 0)$ along the remaining sides (driven cavity problem).

The optimal mesh results from a minimization of the functional

$$J = I_0 + \sum_{i=1}^{N} \alpha_i I_i + \beta J + \gamma K,$$

where $\alpha_i$, $i = 1, 2, \beta$ and $\gamma$ stand for positive (given) real numbers, $I_i$ is defined by (25) with $u$ replaced by the $i$-th component of the velocity field, $J_0$ is a similar term for the hydrostatic pressure given previously. Figure 4 presents a computed mesh for $\alpha_1 = \alpha_2 = 4, \beta = 4, \gamma = 0.1$.

A Model Elliptic Problem

As an example of the application of the h-method combined with the interpolation error estimate, we have solved the model elliptic problem (7) with homogeneous Dirichlet boundary conditions. The right-hand side of the equations correspond to the following (exact) solution defined in $\Omega = (0, 1)^2$:

$$u(x, y) = \phi(x) \phi(y),$$

where

$$\phi(x) = e^{-\phi} \frac{1}{e(x-x_0)^2 + \epsilon} + Ax + B$$

with $A$ and $B$ chosen in that way that $\phi(0) = \phi(1) = 0$. With $x_0 = 0.55$, $y_0 = 0.5$, $\epsilon_x = 0.02$, and $\epsilon_y = 0.05$; after 19 mesh refinements we have obtained a mesh presented in Fig. 5. The error together with its exact and computational error estimates are shown in Fig. 6.

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Figure 1. Mesh enrichment for the heat conduction problem.

Figure 2. Heat conduction problem. Computed solutions on section AA for \( p = 1 \) and adaptive correction for \( t = 0.5 \).

Figure 3. Heat conduction problem. Computed solutions on section AA for \( p = 1 \) and adaptive correction for \( t = 2.0 \).
Figure 4. Driven cavity problem. Optimal mesh after 8 FE recalculations; $\alpha = 4., \beta = 4., \gamma = 0.1.$

Figure 5. Refined mesh after 19 FE recalculations.

Figure 6. Convergence of the interpolation-error estimate based on the extraction formula.