Use of Variational Methods for the Analysis of Contact Problems in Solid Mechanics

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ABSTRACT

We describe several variational formulations of the problem of contact of an incompressible elastic body with a lubricated rigid foundation. These include variational inequalities, Lagrange multiplier methods, and penalty methods. Existence theorems are presented as well as results of finite element approximations.

1. INTRODUCTION

In this paper, we outline an analysis of a class of nonlinear boundary-value problems in the theory of incompressible elastic bodies which arise in the study of contact problems; i.e., the problem of the contact of an incompressible elastic body with a rigid foundation. We develop variational principles for this class of problems which embody the use of variational inequalities, Lagrange multipliers, and penalty methods. We study the existence and uniqueness of solutions to such problems, their approximation using finite elements, and we present some representative numerical solutions.

Our first approach combines features of saddle point theory and the theory of variational inequalities. In particular, much of our analysis is based on applications of the following theorems:

Theorem 1: Let $U$ and $V$ denote real reflexive Banach spaces, $K \subset U$ a nonempty closed convex subset of $U$ and $M \subset V$ a nonempty closed convex subset of $V$.

Let $L : K \times M \rightarrow \mathbb{R}$ be a real functional satisfying the following conditions:

(i) $\forall q \in M$, $v \rightarrow L(v, q)$ is convex and lower semicontinuous;

(ii) $\forall v \in K$, $q \rightarrow L(v, q)$ is concave and upper semicontinuous;

(iii) $\exists q_0 \in M$ such that $L(v, q_0) \rightarrow +\infty$ as $||v||_U \rightarrow +\infty$;

(iv) $\lim_{v \rightarrow K} (\inf L(v, q)) = -\infty$, $||q||_V \rightarrow +\infty$.

Then there exists a saddle point $(u, p) \in K \times M$ of $L$; i.e.,

$$L(u, p) \leq L(u, p) \leq L(v, p) \quad \forall v \in K, \forall q \in M.$$ (1.1)

Theorem 2: Let the conditions of Theorem 1 hold except in place of (i) we have

(i') $\forall q \in M$, $v \rightarrow L(v, q)$ is convex and Gâteaux differentiable, with Gâteaux derivative

$$\forall v \in K: \left< \frac{\partial L(u, p)}{\partial u}, v \right>_U = \lim_{t \rightarrow 0^+} \frac{L(u + tv, p) - L(u, p)}{t},$$

and, in addition to (ii), we have

(ii') $\forall v \in K$, $q \rightarrow L(v, q)$ is Gâteaux differentiable with derivative

$$\forall q \in M: \left< \frac{\partial L(u, p)}{\partial q}, q \right>_V = \lim_{t \rightarrow 0^+} \frac{L(u, p + tq) - L(u, p)}{t}.$$ (1.1)

Then there exists a saddle point $(u, p) \in K \times M$ satisfying (1.1) and, moreover, $(u, p)$ is characterized by
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\[ <\frac{\partial L(u,p)}{\partial u}, u - v>_U > 0 \quad \forall v \in U \]

\[ <\frac{\partial L(u,p)}{\partial p}, p - \rho>_P < 0 \quad \forall \rho \in P \]

In (1.2), \(<\,,\,>_U\) and \(<\,,\,>_P\) denote duality pairing on \(U' \times U\) and \(P' \times P\) respectively.

It is possible to extend Theorem 1 to cases in which \(K\) is only weakly sequentially closed; see, for example, Ekeland [1]. A proof of Theorems 1 and 2 in the forms given here can be found in Ekeland and Temam [2]. Note that (1.2) represents a pair of variational inequalities. Both reduce to variational equalities whenever \(u \in \text{int } K\) and \(p \in \text{int } P\).

2. A SIGNORINI PROBLEM FOR INCOMPRESSIBLE ELASTIC BODIES

We begin by considering the problem of the contact of an incompressible linearly elastic body with a rigid frictionless (lubricated) foundation. The body is characterized as the closure of an open bounded domain \(\Omega \subset \mathbb{R}^n, n = 1, 2, 3\), with a smooth boundary \(\Gamma\) which consists of three parts \(\Gamma = \Gamma_D \cup \Gamma_F \cup \Gamma_C\), where \(\Gamma_D, \Gamma_F\) denote the portions of the boundary on which the displacements and tractions are prescribed respectively and \(\Gamma_C\) is a portion of \(\Gamma\) containing the unknown contact surface. The "classical" problem is governed by the equations

\[ \sigma_{ij,j} + f_i = 0 \quad \varepsilon_{ij}^s(v) = (v_{ij} + v_{ij})/2, \quad \text{in } \Omega \]

and boundary conditions

\[ v_i = 0 \text{ on } \Gamma_D; \quad \sigma_{ij}n_j = s_i \text{ on } \Gamma_F \]

and the contact conditions

\[ \sigma_n = \sigma_{ij}n_in_j = \begin{cases} 0 & \text{if } u_n - s < 0 \\ \leq 0 & \text{if } u_n - s = 0 \end{cases} \text{ on } \Gamma_C. \]

Here the usual notations are employed; \(u_n = u_{i1}\) and \(s = s(x)\) is the initial "gap" representing the normalized distance between material particles \(x\) and the contact surface \(\Gamma_C\). In addition, we have the incompressibility constraint

\[ \varepsilon_{ii}^s(v) = \text{div } v = 0. \]

Constraints (2.3) and (2.4) and homogeneous (fixed) conditions on \(\Gamma_D\) characterize the constraint set

\[ C = \{ v \in U : \varepsilon_{ii}^s(v) = 0 \text{ a.e. in } \Omega, \quad v_i = 0 \text{ on } \Gamma_D; \quad v_n - s \leq 0 \text{ a.e. on } \Gamma_C \}, \]

where \(U\) is an appropriate real Banach space of measurable functions on \(\Omega\).

It is easily shown that solutions of the problem (2.1) - (2.4) can be characterized by the following variational equality:

Find \(u \in C\) such that

\[ \int_{\Omega} 2\varepsilon_{ij}(u) \varepsilon_{ij}^s(v-u) dx \geq \int_{\Omega} f_i(v_i - u_i) dx + \int_{\Gamma_F} s_i(v_i - u_i) ds \quad \forall v \in C. \]

Moreover, (2.6) is equivalent to the constrained minimization problem

\[ F(v) = \int_{\Omega} \mu \varepsilon_{ij}(v) \varepsilon_{ij}^s(v) dx - \int_{\Omega} f_i v_i dx + \int_{\Gamma_F} s_i v_i ds \]

Note that if \(u \in L^p(\Omega), f_i \in L^2(\Omega), s_i \in L^2(\Gamma_F)\), then \(F\) is a functional on the Sobolev space \((H^1(\Omega))^n\) and we may take \(\bar{U} = (H^1(\Omega))^n, n = 1, 2, 3\).
3. A RESOLUTION BY LAGRANGE MULTIPLIERS

We reformulate (2.7) by relaxing the incompressibility constraint (2.4). Then, instead of the set \( C \) of (2.5) we have

\[
K = \{ v \in U: v_n - s \leq 0 \text{ a.e. on } \Gamma_C, \ v_\lambda = 0 \text{ a.e. on } \Gamma_D \} .
\] (3.1)

Next we introduce the Lagrangian,

\[
L(v,q) = F(v) - \int_\Omega q e_{ij}(v) dx .
\] (3.2)

Since \( e_{ij}(v) \in L^2(\Omega) \), the Lagrange multiplier \( q \in L^2(\Omega) \); i.e. in this case the constraint set \( M \) in Theorem 1 is a subspace of a complete normed linear space.

Clearly, conditions (i) and (ii) of Theorem 1 (indeed i') and (ii') of Theorem 2) are satisfied. In fact, \( L(\cdot,q) \) is strictly convex for every \( q \in \text{mes } (\Gamma_D) > 0 \). Condition (iii) also holds by Korn's inequality. To verify (iv) we note that for \( U_0 = (H^1_0(\Omega))^n \), \( \inf_{v \in K} L(v,q) \leq \inf_{v \in U_0} L(v,q) \), since \( U_0 \subset K \) (because of \( s \geq 0 \)). Then it suffices to show that \( \inf_{v \in U_0} L(v,q) \to - \infty \) as \( ||q||_0 \to \infty \).

From the characterization of a minimizer \( v_q \) of \( L(\cdot,q) \) on \( U_0 \),

\[
\int_\Omega q e_{ij}(v) dx = \int_\Omega v_\lambda dx - \int_\Omega e_{ij}(v) \delta_{ij} \rho dx
\]

\[
\leq (2||v||_{L^1})^2 + ||e_{ij}(v)||_{L^2}^2 \leq \sum_{i,j=1}^n ||e_{ij}||_{L^2}^2 .
\]

Here \( ||v||_{L^1} = \text{ess. sup } x \in \Omega |v| \), and \( ||e_{ij}||_{L^2}^2 = \sum_{i,j=1}^n ||e_{ij}||_{L^2}^2 \). We recall the lemma proved by Tartar [3],

**Lemma 1:** For \( q \in L^2(\Omega) \), there exists a \( v \in U_0 \) such that \( \text{div } v = q \) and \( ||v||_1 \leq C ||q||_{L^1} \), \( C > 0 \) provided the compatibility condition \( \int_\Omega q dx = 0 \) is satisfied.

Let the set \( M \) be defined by

\[
M = \{ q \in L^2(\Omega): \int_\Omega q dx = 0 \} .
\] (3.3)

Then, for every \( q \in M \), it can be shown that \( ||e_{ij}(v_q)||_0 \to + \infty \text{ as } ||q||_0 \to + \infty \). On the other hand,

\[
\inf_{v \in U_0} L(v,q) = \int_\Omega e_{ij}(v) \delta_{ij} \rho dx \leq -m ||e_{ij}(v)||_0^2
\]

provided there exists a positive number \( m \) such that

\[
\rho(x) \geq m \text{ a.e. in } \Omega .
\] (3.4)

Therefore, if \( ||q||_0 \to + \infty \), \( \inf_{v \in U_0} L(v,q) \to - \infty \). Thus all conditions of Theorem 1 are satisfied.

In conclusion, we have

**Theorem 3:** Let \( \text{mes } \Gamma_D > 0 \) and (3.4) hold. Then there exists a unique saddle point \( (u,p) \) of the functional \( L \) of (3.2) and \( (u,p) \in K \times M \). Moreover, \( u \) is a solution of the variational inequality (2.6) and of the minimization problem (2.7).

4. A RESOLUTION BY PENALTY METHODS

An alternate formulation of the Signorini problem described in Section 2 can be constructed using penalty methods. That is, instead of satisfying the constraint \( e_{ij}(u) = 0 \text{ a.e. in } \Omega \), the penalty functional

\[
P(v) = \frac{1}{2} \int_\Omega (e_{ij}(v))^2 dx
\] (4.1)

is appended to the functional \( F \) of (2.7). It is important to note that \( P(\cdot) \) is weakly lower semicontinuous on \( U \), \( P(v) \geq 0 \) for every \( v \in U \), and \( P(v) = 0 \text{ iff } e_{ij}(v) = 0 \text{ a.e. in } \Omega \). The penalized problem is

\[
u_\lambda \in K: E(u_\lambda, \lambda) \leq E(v,\lambda) \text{ for } v \in K
\]

\[
E(v,\lambda) = F(v) + \lambda P(v) .
\] (4.2)
For every $\lambda > 0$, $E(\cdot, \lambda)$ is strictly convex, Gâteaux differentiable, and coercive on $K$ if $\text{mes } (\Gamma_D) > 0$. Then the penalized problem has a unique solution $u_\lambda$ for each $\lambda > 0$. Moreover, $u_\lambda$ is uniformly bounded for every $\lambda > 0$. Since the Sobolev space $(H^1(\Omega))^N$ is a reflexive Banach space there exists a subsequence $u_{\lambda}^{(n)}$ which converges weakly to some element $u$ in $(H^1(\Omega))^N$. Moreover, if the minimizer of $F$ on $C$ is $u_0$, then $u_\lambda$ converges weakly to $u$ for $\lambda \to +\infty$. Since $F(u_0)$ and $F(u_\lambda)$ are bounded, $F(u) = 0$ as $\lambda \to +\infty$, i.e. $\varepsilon_{\lambda}^{(u)}(u_\lambda)$ converges strongly to $\varepsilon_{\lambda}^{(u)}$ in $L^2(\Omega)$. This means that $\varepsilon_{\lambda}^{(u)}(u_\lambda) = 0$ a.e. in the domain $\Omega$. Since $F(u_\lambda) \leq F(u_\lambda) + \lambda P(u_\lambda) \leq F(u_0)$, we obtain $F(u) \leq F(u_0)$ by taking the limit $\lambda \to +\infty$. Since $u \in C$, the limit $u$ has to be a minimizer of $F$ on $C$. Because of the uniqueness of minimizers of $F$ on $C$, $u$ is exactly the same as $u_0$. Similar arguments can be given to show that, for every convergent subsequence of $u_\lambda$, the limit is unique. Thus, the original sequence $u_\lambda$ converges weakly to the minimizer $u_0$ of $F$ on $C$.

Moreover, for the minimizer $u_\lambda$, we have
\[
\int_{\Omega} \lambda \varepsilon_{\lambda}^{(u_\lambda)} v \, dx = \int_{\Omega} v \, dx - 2\int_{\Omega} \varepsilon_{\lambda}^{(u_\lambda)} v \, dx
\]
for every $v \in U_0$, since $U_0 \subset K$. Then
\[
\left\| \text{grad } (\lambda \varepsilon_{\lambda}^{(u_\lambda)}) \right\|_{-1} \leq \sup_{v \in U_0} \left( \int_{\Omega} \lambda \varepsilon_{\lambda}^{(u_\lambda)} v \, dx \right) \left\| v \right\|_1
\]
\[
\leq \left\| \text{grad } (\lambda \varepsilon_{\lambda}^{(u_\lambda)}) \right\|_{-1} + 2 \left\| \varepsilon_{\lambda}^{(u_\lambda)} \right\|_0 \left\| u_\lambda \right\|_1
\]
Thus, $\text{grad } (\lambda \varepsilon_{\lambda}^{(u_\lambda)})$ is uniformly bounded in $(H^{-1}(\Omega))^n$ for each $\lambda > 0$. Since the image of $\text{grad } L^2(\Omega)$, $(H^{-1}(\Omega))^N$ is isomorphic to $L^2(\Omega)/N = \{ u = \text{constant} \}$ (see Tarter [4], Lemma 8, p. 30), $\lambda \varepsilon_{\lambda}^{(u_\lambda)}$ is bounded in $L^2(\Omega)/N$. Then there exists a subsequence (still denoted $\varepsilon_{\lambda}^{(u_\lambda)}$), which converges weakly to some limit $-p$ in $L^2(\Omega)$. We note that the limit $-p$ is unique within constants. Since $\varepsilon_{\lambda}^{(u_\lambda)}$ converges strongly to $\varepsilon_{\lambda}^{(u)}$, and since $F(.)$ is weakly lower semicontinuous, we have

**Theorem 4**: Let $\text{mes } (\Gamma_D) > 0$, and (3.4) hold. Then the sequence $u_\lambda$ of solutions of the penalized problem (4.2) converges weakly to the solution of the constrained problem (2.7) as $\lambda \to +\infty$. Moreover, the sequence $\lambda \epsilon_{\lambda}^{(u_\lambda)}$ also converges to $-p$, which is identified with the pressure, in $L^2(\Omega)$ modulo constants.

### 5. FINITE ELEMENT APPROXIMATIONS

The use of penalty methods as a basis for finite element approximations of constrained problems has been described by several authors; see, for example, Hughes, Liu, and Brooks [5], Reddy [6], and the references therein. We briefly outline results of examples solved numerically by finite element methods.

We consider the plane-strain rigid punch problem indicated in Fig. 1 and the half-cylinder Hertz problem shown in Fig. 2. Eight-node isoparametric elements are used. Our formulation employs the Lagrange multiplier method plus a penalty term for the incompressibility constraint:
\[
L(v, q) = F(v) + \lambda P(v) - \int_{\Gamma_C} q(v_n - s) \, ds
\]

Numerical results are indicated in the figures. Some remarks on these results are listed as follows:

1. When $3 \times 3$ Gaussian numerical integration is taken for both the strain energy and the penalty functional, convergence of deformations as $\lambda \to +\infty$ (i.e. Poisson's ratio $\nu = 0.5$ for plane strain problems) is not obtained. Indeed, for the Hertz problem, adequate results are obtained only for Poisson's ratio $\nu \leq 0.455$. If $3 \times 3$ Gaussian quadrature is used for the strain energy, and reduced $2 \times 2$ Gaussian quadrature is used for the penalty functional, then rapid convergence of deformations as $\lambda \to +\infty$ is observed. The pressure $\lambda \epsilon_{\lambda}^{(u_\lambda)}$ seems to converge at the Gauss points for $2 \times 2$ Gaussian quadrature integrations.
Fig. 1. A rigid punch problem on a rigid foundation for incompressible body.

(2) If the resulting equations are solved by Uzawa's method, then the iteration factor can be chosen independently of the penalty parameter $\lambda$.

Space does not permit the inclusion of many important details. These will be discussed in a forthcoming companion paper.

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