NOTE ON AN APPROXIMATE METHOD FOR COMPUTING CONSISTENT CONJUGATE STRESSES IN ELASTIC FINITE ELEMENTS

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SUMMARY
A technique is described wherein improved stresses can be computed in finite element models based on displacement approximations. The method is based on the idea of consistent stress approximations and it approximates such stresses using the notion of a domain of influence of the stress intensity at a nodal point. Considerable improvement in accuracy of the stresses is obtained with little difficulty.

INTRODUCTION
The calculation of so-called ‘consistent’ or ‘conjugate’ approximations of stresses in displacement finite element formulations involves, in addition to the usual computations, the solution of an auxiliary system of linear equations, the order of which is comparable to that of the stiffness matrix itself. For this reason (and despite the fact that rigorous arguments can be made that such conjugate stresses represent ‘best approximations’ in a certain sense), the calculation of consistent stresses may be time consuming and expensive compared to conventional averaging methods. To overcome these shortcomings, an approximate method for computing consistent stresses in finite elements is presented in this paper. The method is also based on ideas drawn from the theory of conjugate approximations; it leads to smooth stress approximations in regions in which high stress gradients are experienced, and it involves only the solution of rather small systems of narrowly banded linear equations.

CONSISTENT STRESSES
We must first review some of the basic ideas of conjugate approximations. Consider a finite element model of an elastic body \( B \) which consists of a collection of \( E \) elements connected together at \( G \) nodes, and suppose that the finite element model of the displacement components \( u_i(x) \) is of the form

\[
u_i = u_i^\Delta \phi_\Delta(x)
\]

(1)

Here \( u_i^\Delta \) are the components of displacement at node \( \Delta, i = 1, 2, 3, \Delta = 1, 2, \ldots, G \) and \( \phi_\Delta(x) \) are interpolation functions which have the properties \( \phi_\Delta(x_i) = \delta_{\Delta i} \); \( \Delta, \Gamma = 1, 2, \ldots, G \). Moreover, the basis functions \( \phi_\Delta(x) \) have local support; i.e. they assume non-zero values only in some compact neighbourhood of node \( \Delta \). In addition, a fundamental feature of finite element formulations is that the global basis functions \( \phi_\Delta(x) \) are generated from families of local basis functions \( \psi^{(e)}_N(x) \) according to

\[
\phi_\Delta(x) = \sum_e^{E} \Omega^{(e)}_N \psi^{(e)}_N(x)
\]

(2)

where \( \Omega^{(e)}_N \) is the Boolean matrix which maps global modes \( x^\Delta \) into local nodes \( x^{N}_{\Delta(e)} \) of element \( e \). We remark that the approximation (1) may also include specified values of derivatives of \( u_i \) of various orders at the nodal points; then the interpretation of \( u_i^\Delta \) and the range of the index \( \Delta \) must be modified. Our conclusions and the basic method of attack, however, are unaltered.

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Once the generalized displacements $u^\alpha$ of (1) are determined using well-established direct stiffness schemes, stresses are calculated from the constitutive equations; i.e. if $\sigma^{ij}$ is the 'conventional' stress tensor obtained in finite element analyses, then

$$\sigma^{ij} = E^{ijmn} u^\alpha_{m} \phi_{\Delta,n}(x)$$

(3)

Here $E^{ijmn}$ is the elasticity tensor and $\phi_{\Delta,n} = \partial \phi_{\Delta} / \partial x_n$, $x_n$ being the material (Cartesian) co-ordinates of $x$.

While $\phi_{\Delta}(x)$ [and hence $u_{\alpha}(x)$] are, by construction, continuous, $\phi_{\Delta,n}(x)$ [and hence $\sigma^{ij}(x)$] are generally discontinuous at interelement boundaries. However, consistent stress approximations $S^{ij}(x)$ can be computed which are continuous throughout the model. As described in Reference 3, conjugate (consistent) stresses are of the form

$$S^{ij}(x) = S^{ij\Delta} \phi_{\Delta}(x) = S^{ij}_{\Delta} \phi_{\Delta}(x)$$

(4)

where

$$S^{ij\Delta} = \langle \sigma^{ij}, \phi_{\Delta} \rangle; \quad S^{ij}_{\Delta} = \langle \sigma^{ij}, \phi_{\Delta} \rangle$$

(5)

Here $\langle \cdot, \cdot \rangle$ denotes an inner product defined on the space spanned by $\phi_{\Delta}(x)$; for example,

$$\langle u, v \rangle = \int_{\Omega} uv \, d\Omega$$

(6)

The functions $\phi^{\Delta}(x)$ in (5) are the conjugate approximation functions and are given by

$$\phi^{\Delta}(x) = C^{\Delta \Gamma} \phi^{\Gamma}(x)$$

(7)

Here $C^{\Delta \Gamma}$ is the inverse of the fundamental matrix $C_{\Delta \Gamma}$, where

$$C_{\Delta \Gamma} = \langle \phi_{\Delta}, \phi_{\Gamma} \rangle$$

(8)

The major difficulty encountered in applying the theory of consistent stress approximations concerns the $G \times G$ matrix $C_{\Delta \Gamma}$ of equation (8): it is clear from equations (7) and (5) that its inverse is needed to determine the conjugate functions $\phi^{\Delta}(x)$. Moreover, if $S^{ij}_{\Delta}$ are known, the important quantities $S^{ij\Delta}$ can be obtained only by solving the system of linear equations

$$C_{\Delta \Gamma} S^{ij\Gamma} = S^{ij}_{\Delta}$$

(9)

**SOME PROPERTIES OF CONJUGATE STRESSES**

Equations (1)–(9) summarize the essential features of the theory of consistent stresses. We subsequently describe a technique for approximately implementing this theory which is based on the following observations:

1. The quantities $S^{ij\Delta}$ represent the values of the consistent stresses at node $\Delta$.
2. The quantities $S^{ij}_{\Delta}$ represent averaged stresses in a neighbourhood of node $\Delta$.
3. The quantities $S^{ij}_{\Delta}$ can be routinely generated from local data.
4. In the majority of existing direct stiffness programs which include capabilities for frequency calculations, the fundamental matrix $C_{\Delta \Gamma}$ is available or can be routinely generated. Moreover it is symmetric and banded.
5. Conventional finite element stresses $\sigma^{ij}$ or conventional stress averages are generally accurate enough for practical purposes at points at which stress gradients are low. It is only in regions containing high stress gradients that more accurate stress approximations are generally needed.

Proof or justification of each of these observations, respectively, is straightforward:

(a) This follows from the definition of $\phi_{\Delta}(x)$; indeed, from equation (4), the consistent stress at node $x^{\Gamma}$ is

$$S^{ij}(x^{\Gamma}) = S^{ij\Delta} \phi_{\Delta}(x^{\Gamma}) = S^{ij\Delta} S^{ij}_{\Delta} = S^{ij\Gamma}$$

which proves observation 1. On the basis of this result, we proceed on the assumption that $S^{ij\Delta}$ are the quantities sought in a stress analysis.
(b) The averaged consistent stress over a body $\mathcal{B}$ of volume $V = \int d\mathcal{B}$ is

$$S_{ij}^{(\Delta V G)} = \frac{1}{V} \int_{\mathcal{B}} S_{ij}^\Delta(x) d\mathcal{B} = \frac{1}{V} \sum_{\Delta=1}^N S_{ij}^\Delta$$  \hspace{1cm} (10)

Here we have used the fact that $\sum_{\Delta} \phi^\Delta(x) = 1$ and

$$\int_{\mathcal{B}} \phi^\Delta d\mathcal{B} = C^\Delta \int_{\mathcal{B}} \phi^\Gamma \sum_{\Delta} \phi^\Delta d\mathcal{B} = \sum_{\Delta} C^\Delta C^\Delta = \sum_{\Delta} \delta^\Delta = 1$$

Thus, $S_{ij}^{(\Delta V G)}$ is (to within a factor $V^{-1}$) the sum of the components $S_{ij}^\Delta$ over all nodes.

(c) To justify assertion 3, let $\sigma_{ij}^{(e)}$ denote the conventional stress in element $e$. Locally, we can compute

$$s_{ij}^{(e)N} = \langle \sigma_{ij}^{(e)}, \psi_{N}^{(e)} \rangle = \int_{\mathcal{B}} \sigma_{ij}^{(e)} \psi_{N}^{(e)} d\mathcal{B}$$  \hspace{1cm} (11)

Then

$$S_{ij}^\Delta = \sum_{e=1}^E \Omega_{\Delta}^{(e)} s_{ij}^{(e)N}$$  \hspace{1cm} (12)

Thus, the calculation of the 'consistent nodal averages' $S_{ij}^\Delta$ involves only a local integration (11), which can be easily accomplished by Gaussian quadrature, and the sum of appropriate nodal values (12). The latter operation involves only the summation of $r$ numbers for each stress component at each node joining $r$ elements.

(d) Observation 4 comes from the fact that the fundamental matrix $C_{\Delta \Gamma}$ is merely the consistent mass matrix for a homogeneous structure possessing a mass density of unity. Thus, for a homogeneous structure of mass density $\rho$ and mass matrix $M_{\Delta}$, $C_{\Delta \Gamma} = (1/\rho) M_{\Delta \Gamma}$. The symmetry and bandedness of $C_{\Delta \Gamma}$ follows, of course, from the symmetry of the inner product (6) and the local support of $\phi^\Delta(x)$.

(e) Observation 5 is based merely on experience and its premise can be relaxed to whatever degree is desired in the approximate method to be described.

**DOOMAINS OF INFLUENCE**

In view of the bandedness of $C_{\Delta \Gamma}$ and the 'dampened' character of the conjugate functions $\phi^\Delta(x)$, it appears likely that good approximations of the consistent stress values $S_{ij}^\Delta$ might be obtained by using only a small submatrix of $C_{\Gamma \Delta}$, which corresponds to a collection of elements over which stress gradients and concentrations are particularly high. Outside of such collections of elements, we assume that the conventional or averaged stresses are satisfactory; but, within such collections, we wish to employ the theory of consistent stresses, at least approximately, to obtain improved stress values.

The question arises as to whether or not a quantitative measure can be defined which will give the size of the collection of elements (or, equivalently, the size of the submatrix of $C_{\Gamma \Delta}$) needed to obtain good approximations of the consistent stresses. Our answer to this question involves the notion of a domain of stress influence: Let $\sigma_{ij}(x)$ denote the stress tensor at a particle $x$ of a continuous body $\mathcal{B}$. Suppose that for some $i$ and $j$ and a specific particle $\bar{x}$, there is a set of particles $\mathcal{D}_{\bar{x}} \subseteq \mathcal{B}$ containing $\bar{x}$ such that for every $x \in \mathcal{D}_{\bar{x}}$

$$| \sigma_{ij}(x) | \leq | \sigma_{ij}(\bar{x}) |$$  \hspace{1cm} (13)

Then we shall refer to a set $\mathcal{D}_{\bar{x}} \subseteq \mathcal{D}_{\bar{x}} = \{ x : \| x - \bar{x} \| \leq \rho_{\bar{x}}, \rho_{\bar{x}} \text{ being a real number } > 0 \}$ as a domain of influence of the stress at particle $\bar{x}$ if, for a given real number $\varepsilon$, $0 < \varepsilon \leq 1$,

$$| \sigma_{ij}(x) | \geq \varepsilon | \sigma_{ij}(\bar{x}) |$$  \hspace{1cm} (14)

whenever $x \in \mathcal{D}_{\bar{x}}$ and

$$| x_i - \bar{x}_i | \leq \rho_{\bar{x}}$$  \hspace{1cm} (15)

for all $i, j = 1, 2, 3$. The number $\rho_{\bar{x}}$ shall be referred to as the $\varepsilon$-radius of influence of the stress at $\bar{x}$. 
Suppose, for example, that the stress in a one-dimensional case is distributed as indicated in Figure 1(a). Then local maxima or minima exist at points A, B, C and D, as shown. If, for example, we set $\varepsilon = 0.3$, then the domains of influence $\mathcal{D}_A$, $\mathcal{D}_B$, $\mathcal{D}_C$ and $\mathcal{D}_D$ indicated by the hashed internals are obtained using definitions (14) and (15).

Figure 1. (a) Stress distribution and (b) the corresponding variation in stress gradient

We remark that in some cases, a restricted domain of influence can also be determined on the basis of stress gradients rather than stress values, provided the gradient exists and is non-zero at the local maxima and minima, $\mathfrak{R}$. If the stress gradient does not exist at $\mathfrak{R}$, then it can be evaluated at neighbouring points, $x \pm \varepsilon$. Then a restricted radius of influence $\rho_\varepsilon^*$ can be computed by determining, for a given $\varepsilon$, $0 < \varepsilon \leq 1$, the particles for which

$$|\sigma^{ij}(x)_k| \geq \varepsilon |\sigma^{ij}_{\mathfrak{K}}|$$

(16)

For the stress distribution in Figure 1(a), restricted domains of influence corresponding to $\varepsilon = 0.3$ are indicated in Figure 1(b). Note that at C, the gradient at $x_c \pm \varepsilon$ is used in (16) to determine $\mathcal{D}_c^*$. 

**APPROXIMATE METHODS**

The ingredients for approximate implementations of the consistent stress concept, particularly the solution of equation (9), are now clear. Assume that an existing finite element code has been used to obtain the conventional finite element stresses $\sigma^{ij}(x)$ and the fundamental matrix $C_{\Delta i}$. We wish to obtain, at least approximately, the nodal stresses $\bar{S}^{ij}_N$. We may proceed as follows:

1. Compute the local consistent stress components $S^{ij}_{\Delta i N}$ using equation (11).
2. Generate the global quantities $S^{ij}_N$ using equation (12).
3. On the basis of the stresses $S^{ij}_N$, identify those nodal points $\mathfrak{K}^\Delta$ in the body at which high stress concentrations appear to exist.
4. Specify an $\varepsilon$, $0 < \varepsilon \leq 1$, and determine an approximate domain of influence using the stresses $S^{ij}_N$; i.e. if $\mathfrak{K}^\Delta$ is a node of high stress concentration as identified in step 3, then the approximate 'discrete' domain of influence of C is calculated as those nodal points $x^i$ in the neighbourhood of $\mathfrak{K}^\Delta$ for which

$$|S^{ij}_N| \geq \varepsilon |S^{ij}_N|$$

(17)

for a given choice of $i$ and $j$.
5. Use matrix condensation to eliminate those stresses at nodes not belonging to domains of influence. That is, if the coefficient matrix $C = [C_{ij}]$ in equation (9) is partitioned so that, in matrix form it appears as

$$\begin{bmatrix}
C_{11} & C_{12} \\
C_{21} & C_{22}
\end{bmatrix} \begin{bmatrix}
\bar{S}_1 \\
\bar{S}_2
\end{bmatrix} = \begin{bmatrix}
S_1 \\
S_2
\end{bmatrix}$$

(18)

where $N + M$ is the total number of nodes in the model, $N$ is the number of nodes in the domain of influence $D_e$, $\bar{S}_1$ are the nodal stresses at nodes in $D_e$, etc., then the desired nodal stresses are determined by solving the system

$$[C_{11} - C_{12}C_{22}^{-1}C_{21}]\bar{S}_1 = S_1 - C_{12}C_{22}^{-1}S_2$$

(19)

At this point, the procedure amounts to little more than standard matrix condensation wherein the idea of domains of stress influence is used as a rationale for partitioning $C$. However, we can now take advantage of property 2 of the stresses $S_{ij}$ mentioned earlier, and obtain an alternate approximation.

(a) On the basis of conventional nodal stress averages $\sigma_{ij}$ identify nodal points $\bar{x}_i$ at which high stress concentrations appear to exist.

(b) Repeat steps 4 and 5 above, replacing $S_{ij}$ by $\sigma_{ij}$. Then the right-hand side of (19) is replaced by

$$\sigma_1 - C_{12}C_{22}^{-1}\sigma_2$$

(20)

A third method, generally easier to implement but less accurate than the two described above, can be obtained by noting that $C$ is strongly banded, $C_{ij}$ is very sparsely populated, and, by construction, $|S_1| > |S_2|$. This suggests that instead of (19) or (20) we simply use

$$C_{11}\bar{S}_1 = \sigma_1$$

(21)

Note that the accuracy of any of these approximate methods can be estimated by repeating the calculation in equation (17) for a smaller $\varepsilon$ and comparing the peak stresses so obtained.

AN EXAMPLE

As a simple example, we consider the plane stress problem indicated in Figure 2 wherein an accurate description of the stress $S_{11}$ at point $A$ is desired. The finite element model of constant strain triangles shown is used. We take $(2a/L) = 8$, $E = 30 \times 10^6$ lb/in$^2$, $v = 0.25$ and $L = 10.0$ in, with an applied stress of $\sigma_{11} = S$ at $x = \pm L/2$. 

![Figure 2. Plate with circular hole under uniaxial stress and finite element model of one quadrant](image)
Figure 3 contains a comparison of the exact stress, the piecewise constant conventional stress and the consistent stress, $\sigma_{11}$ along $x_1 = 0$. As can be seen, the full consistent stress approximation yields a peak stress only 6.7 per cent lower than the exact stress, while the conventional (unaveraged) peak stress is some 11.3 per cent in error. Unfortunately, the increase in accuracy of 4.6 per cent is obtained at the expense of inverting (solving) a system of order 1620. However, by using the scheme described previously, we establish a domain of influence consisting of only nineteen elements with an $\varepsilon = 0.5$. This involved the solution of only twenty equations and less than 4 CPU computing time on the UNIVAC 1108. This resulted in a peak stress which differed from the consistent stress only in the fifth significant figure. The result was insensitive to the choice of $S_{ij}^H$ or $\sigma_{ij}^H$ in (17) or (20) as a measure of stress gradients. By increasing $\varepsilon$ still further to $\varepsilon = 0.8$, we arrive at a matrix $C$ of order only 7. Remarkably, we again obtain a very accurate approximation differing only 0.5 per cent from the peak consistent stress and 7 per cent from the exact peak stress (see Figure 4). This improved stress calculations required 4 sec of CPU time.

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