FINITE ELEMENT FORMULATION OF GENERAL ELECTROTHERMOELASTICITY PROBLEMS

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SUMMARY
This paper is concerned with the development of consistent discrete models, via the concept of finite elements, of linear and non-linear electrothermomechanical behaviour of continuous bodies. In the development, general energy balances are utilized to derive equations governing electromagnetic fields over an element and coupled equations of motion and heat conduction of a typical element of the continuum. These equations make it possible to study a general class of field problems involving arbitrary geometries and boundary conditions. Sample problems are included.

INTRODUCTION
Classical physics has long recognized the fact that mechanical, electrical and thermal field quantities appear to be coupled in most natural phenomena. When current is applied to a continuous body it becomes hotter, and when it becomes hotter it undergoes a certain amount of deformation. Conversely, heating a body or subjecting it to prescribed deformations is accompanied by the development of electromagnetic fields. Although such phenomena are discussed liberally in most textbooks on elementary physics, relatively few solutions of the coupled field equations appear to be available in the literature owing to their inherent complexity. Indeed, solutions of the coupled field equations, the general forms of which themselves have been the subject of current investigations, are virtually intractable by classical analytical methods in the case of irregular geometries and complicated boundary and initial conditions. For quantitative solutions to such problems it is generally necessary to resort to numerical methods which effectively represent the continuum by an appropriate discrete model.

In this paper, we investigate the application of the increasingly popular concept of finite elements to the formulation of rather general discrete models of dynamic electrothermoelastic phenomena. Briefly, the finite element concept involves the representation of a continuum as a collection of a finite number of component parts, each of relatively simple geometric shape, which are connected together at prescribed nodal points. Various field quantities are represented locally over each element and are assumed to be defined uniquely in terms of their values at the element’s nodes. An important aspect of this procedure is that each element can be considered to be disjoint for the purpose of describing its behaviour locally. Thus, once the behaviour of a typical element has been defined, a discrete model of virtually any shape with arbitrary boundary and initial conditions can be obtained by connecting the elements appropriately together. The method, introduced in 1956 by Turner and co-workers, has found wide application in solid and fluid mechanics. Recent applications to certain electrical field problems have also been presented.
In applications of the finite element method, it is customary to attempt to formulate an appropriate variational principle governing the problem at hand so as to obtain a consistent means for generating analogues of the field equations which hold in an average sense over an element. We do not follow this procedure in this paper; firstly, because no well-defined functional appears to be readily available which is appropriate for the general class of problems we wish to consider, and secondly, because the use of variational principles is not necessary if we develop appropriate forms of the law of conservation of energy for an element. We feel that the latter procedure is not only more general than the conventional one but that it is more primitive in the sense that it merely involves the application of a minimum number of fundamental physical laws to an isolated portion of a continuum over which various fields are prescribed. In the present investigation, however, this approach requires that we develop a theory of electrothermoelectricity which differs in form from those we have found in the literature. We outline the development of these field equations in the following section. Next, we represent the continuum as a collection of finite elements and approximate over a typical element the displacement, temperature, scalar potential and vector potential in terms of their nodal values. Upon introducing these approximations into global forms of the laws of conservation of energy, charge and magnetic flux density, we obtain a system of equations governing the electrical, thermal and mechanical behaviour of a typical element. These equations can be modified so as to apply to a number of interesting special cases, including formulations of discrete models for coupled thermoelasticity, electroelasticity, electrostatic problems, etc. In the final section of the paper we present a sample problem which involves time-dependent heating of an anisotropic electrothermoelectric half-space.

Notation

In this paper we adopt Einstein's index notation and the summation convention. Lower case Latin indices represent components of vectors and tensors and range from 1 to 3. Upper case Latin indices indicate local nodal points of an element and range from 1 to \( N_e \) where \( N_e \) is the number of nodes belonging to element \( e \). When a distinction between local and global nodes is required, upper case Greek indices are used to represent global nodes. Other letter symbols are defined where they first appear in the paper.

THEORY OF ELECTROTHERMOELASTICITY

General

The theory of piezoelectricity, which is the study of quasi-linear electrothermoelectricity in anisotropic crystalline solids, was formulated by Voigt in 1894 on the basis of the works of Becquerel, the Curies, Duhem, Lord Kelvin, Pockels and Maxwell. This formulation appeared in Voigt's monumental *Lehrbuch der Kristallphysik* in 1910. Cady's comprehensive work contains a list of other references in this field. Recent work by Truesdell and Toupin, Toupin, Jordan and Eringen, Eringen, Jordan and Eringen, and Dixon and Eringen on electromagnetic elastic and thermoelastic solids, in addition to the earlier works of Debye and Stratton, provide the basis for a general non-linear theory of electrothermoelectricity in dielectric solids. Piezoelectricity is contained as a special case of the general non-linear theory which includes electrostriction and other non-linearities as well as electroelastodynamic effects. Supplementing this basic theory are the works of Landau and Lifshitz and Penfield and Haus on the electrodynamics of a moving medium and the works of Fano and others, and Ramo and others on electromagnetic fields, and Walker on elastic conductors. In this section, we develop a theory of electrothermoelasticity which is based on the existence of a differentiable free energy functional which depends upon the current values of the formation, temperature, electric field and magnetic field.
Consider a continuous body which, at time $\tau = 0$, occupies a configuration $C_0$ in three-dimensional Euclidean space. A material point $P_0$ in $C_0$ is located by intrinsic (material) co-ordinates $X^i$ which are initially rectangular Cartesian. At $\tau = t$, the body occupies another configuration $C$ and the location of the particle $P_0$ relative to a fixed reference frame in $C_0$ is given by the Cartesian co-ordinates $x^i$. The relations $x^i = x^i(X, t)$ define the trajectory of $P_0$ and thereby describe the motion of the body. The components of displacement are given by

$$u_i = u_i(X, t) = x_i(X, t) - X^i$$

and the deformation of the body is characterized by the Green–Saint–Venant deformation tensor

$$G_{ij} = \delta_{ij} + 2\gamma_{ij}$$

where $\delta_{ij}$ is the Kronecker delta and $\gamma_{ij}$ is the strain tensor

$$\gamma_{ij} = \frac{1}{2}(\varepsilon_{ij} + \varepsilon_{ji} + \eta_{ij} + \eta_{ji})$$

Here commas denote partial differentiation with respect to $X^i$, (i.e. $u_i = \frac{\partial u_i}{\partial X^i}$).

Apart from the purely kinematical relations embodied in equations (1)–(3), there is associated with each particle a measure of mass and electric charge. which, for a perfect continuum, are accounted for by introducing the scalar-valued functions $\rho_0$ and $\rho$, the mass densities in $C_0$ and $C$, $\rho_e$ and $\rho_s$, the volume and surface charge densities in $C$. We postulate that the behaviour of the body must be such that the following physical laws are satisfied:

**Conservation of mass**

$$\int_{C_0} \rho_0 \, dv = \int_{C} \rho \, dv$$

**Balance of linear momentum**

$$\frac{d}{dt} \int_{C_0} \rho_0 \, u_i \, dr_0 = \int_{C_0} \rho_0 F_i \, dr_0 + \int_{A_0} T^j x_{j,i} \, dA_0$$

**Balance of angular momentum**

$$\frac{d}{dt} \int_{C_0} \rho_0 \varepsilon^{ijk} x^i u_j \, dr_0 = \int_{C_0} \rho_0 \varepsilon^{ijk} x^i F_j \, dr_0 + \int_{A_0} \varepsilon^{ijk} x^i t^{mn} n_m x_{j,n} \, dA_0$$

**Conservation of energy**

$$\frac{d}{dt} \int_{C_0} \frac{1}{2} \rho_0 u_i u_i \, dr_0 + \frac{d}{dt} \int_{C_0} \rho_0 \frac{\partial \xi}{\partial t} \, dr_0$$

$$= \int_{C_0} \rho_0 F_i u_i \, dr_0 + \int_{C_0} \rho_0 h \, dr_0 + \int_{A_0} (T^i x_{j,i} u_j + q^i n_i + S^i n_i) \, dA_0$$

where $\int_A S^i n_i \, dA$ is defined by

$$\int_{A_0} S^i n_i \, dA_0 = \int_{C_0} (\dot{D}^i E_i + \dot{H}^i + J^i E_i) \, dr_0$$

**The Clausius–Duhem inequality**

$$\int_{C_0} \left( \rho_0 \dot{n} - \theta J^i E_i - \rho_0 \frac{h}{\theta} \right) \, dr_0 - \int_{A_0} \frac{q^i}{\theta} n_i \, dA_0 \geq 0$$

**Conservation of charge**

$$\int_{C_0} D^i n_i \, dA_0 = \int_{C_0} \rho_e \, dr_0 + \int_{A_0} \rho_s \, dA_0$$
In these equations, \( F_i \) are the components of body force per unit mass in \( C_0 \) (and are generally functions of the electromagnetic field), \( n_i \) are components of a unit normal to \( A_0 \), \( t^{ij} \) are components of the stress tensor per unit undeformed area referred to the convected lines \( X^i \), \( \varepsilon^{ijk} \) is the permutation symbol. \( h \) is the heat supplied per unit mass from internal sources; \( T^i \), \( q^i \), \( S^i \), \( J^i \) and \( D^i \) are components of the surface traction, heat flux, Poynting vector, conduction current density and electric flux density, respectively; \( \xi \) and \( \eta \) are the internal energy and entropy densities with respect to \( C_0 \), respectively; \( T \) is the absolute temperature; \( G = \det G_{ij} \) and \( E_i \) and \( H_i \) are components of the electric and magnetic fields. Observe that \( S_i = \varepsilon^{ijk} H_j E_k \) for polarizable and magnetizable media. In these equations, we ignore the influence of relativistic effects. It is understood that the body force density \( F_i \) is a function of the electric and magnetic fields. As is seen by the form of equation (6), we develop here an apolar theory, i.e. couple stresses, body couples, etc. are assumed to be zero.

Assuming that these quantities satisfy the usual requirements of smoothness, the surface integrals can be transformed to volume integrals using the Green-Gauss theorem. This leads to the following local forms of the above equations:

\[
\rho_0 = \sqrt{\det(G_0)} \rho
\]  
(11)

\[
(t^{ij} n_j)_i + \rho_0 F_k = \rho_0 n_i k
\]  
(12)

\[
t^{ij} = t^{ij}
\]  
(13)

\[
\rho_0 \dot{\xi} = t^{ij} \dot{\gamma}_{ij} + q^i_i + \rho_0 h + S^i
\]  
(14)

\[
S^i_i = D^i E_i + \dot{B}^i H_i + J^i E_i
\]  
(15)

\[
\rho_0 \theta \dot{\eta} - q^i_i - \rho_0 h + \frac{1}{\theta} q^i_i \theta - J^i E_i \geq 0
\]  
(16)

\[
D^i_l = \rho_e \text{ in } r_0, \quad \rho_e^* = [D^j]^l_{ij} \text{ on } A_0
\]  
(17)

Here \( t^{ij} \) is the stress tensor (i.e. \( T^i = t^{ij} n_j \)) and \([D^j]^l_{ij}\) is the jump in \( D^j \) at the boundary.

We now introduce the free energy density function \( \Pi \) and the internal dissipation \( \sigma^* \) defined by

\[
\Pi = \xi - \eta \dot{T} - \frac{1}{\rho_0} D^j E_j
\]  
(18)

\[
\sigma^* = t^{ij} \dot{\gamma}_{ij} - \rho_0 (\Pi + \eta \dot{T}) - D^j \dot{E}_j^* + H^j \dot{B}_j
\]  
(19)

Introducing these quantities into equation (14) and using equation (15), we find that

\[
\rho_0 \dot{\Pi} = t^{ij} \dot{\gamma}_{ij} - \rho_0 \eta \dot{T} - D^j \dot{E}_j^* + H^j \dot{B}_j - \sigma^*
\]  
(20)

\[
\rho_0 \theta \dot{\eta} = q^i_i + \rho_0 h + J^i E_i - D_j d_{ij} (\varepsilon^{ijk} u_i B_k) + \sigma^*
\]  
(21)

In addition, from equation (16) we obtain the dissipation inequality

\[
\sigma^* - \frac{1}{\theta} q^i_i \theta \geq 0
\]  
(22)

Equations (20) and (21) provide the necessary basis for the development of a theory of electrothermoelasticity. In equations (19) and (20), \( E_j^* \) is the stationary-medium electric field.
Electrothermoelasticity

We now consider a medium which is non-dissipative in the sense that \( \sigma^* = 0 \) and which is characterized by a free energy \( \Pi \) which is a differentiable function of the current strain \( \gamma_{ij} \), absolute temperature \( \theta \), electric field \( E_j^* \) and effective magnetic flux density \( B_j \):

\[
\Pi = \Pi(\gamma_{ij}, \theta, E_j^*, B_j)
\]  

(23)

Then

\[
\Pi = \frac{\partial \Pi}{\partial \gamma_{ij}} \gamma_{ij} + \frac{\partial \Pi}{\partial \theta} \theta + \frac{\partial \Pi}{\partial E_j^*} E_j^* + \frac{\partial \Pi}{\partial B_j} B_j
\]  

(24)

Substituting equation (24) into equation (22) with \( \sigma^* = 0 \), and noting that the resulting equation must hold for arbitrary \( \gamma_{ij} \), \( \theta \), \( E_j^* \) and \( B_j \), we arrive at the following relations:

\[
t^{ij} = \rho_o \frac{\partial \Pi}{\partial \gamma_{ij}}, \quad \eta = -\frac{\partial \Pi}{\partial \theta}
\]

(25a, b)

\[
D^j = -\rho_o \frac{\partial \Pi}{\partial E_j^*}, \quad H^j = \rho_o \frac{\partial \Pi}{\partial B_j}
\]

(25c, d)

Thus, the forms of the constitutive equations for \( t^{ij} \), \( \eta \), \( D^j \) and \( H^j \) are determined from the free energy function \( \Pi \).

Additional constitutive equations for the heat flux \( q^i \) and the conduction current density \( J^i \) must be introduced to complete the formulation of the theory. Technically, according to the equipresence principle, these should involve all of the dependent variables \( \gamma_{ij} \), \( \theta \), \( E_j^* \) and \( B_j^* \). However, in applications we shall use as a first approximation the classical linear Fourier law of heat conduction

\[
q^i = K^{ij} \theta_j
\]

(26)

where \( K^{ij} \) is the thermal conductivity tensor, and the linear Ohm's law,

\[
J^i = Z^{ij} E_j
\]

(27)

where \( Z^{ij} \) is the electrical conductivity tensor. However, there is no need to introduce these linearized approximations at this point.

Returning to equation (25), we can now obtain the equations of motion, heat conduction, charge and magnetic flux density of an electrothermoelastic continuum by introducing equation (25) into equations (12), (21), (15) and (17):

\[
\left( \rho_o \frac{\partial \Pi}{\partial \gamma_{ij}} \gamma_{ij} \right)_i + \rho_o F_k = \rho_o \dot{u}_k
\]

(28)

\[
-\rho_o \theta \frac{d}{dt} \left( \frac{\partial \Pi}{\partial \theta} \right) - q^i = \rho_o \theta + J^i E_i
\]

(29)

\[
-(e^{ijk} E_k H_j)_k = -\frac{d}{dt} \left( \rho_o \frac{\partial \Pi}{\partial E_j^*} \right) E_i + \rho_o \frac{\partial \Pi}{\partial B_j} \dot{B}_i + J^i E_i
\]

(30)

\[
-\left( \rho_o \frac{\partial \Pi}{\partial B_j} \right)_i = \rho_e
\]

(31)

It is understood that by \( E^k \) and \( H^k \) in these equations we mean the effective fields, i.e.

\[
E^i = E_i^* + e^{ijk} \dot{u}_j B_k
\]

(32)

\[
H^i = H_i^* - e^{ijk} \dot{u}_j D_k
\]

(33)

where \( E_i^* \) and \( H_i^* \) are the stationary-medium fields.
FINITE ELEMENT FORMULATION

**General**

We now outline the procedure for constructing finite element models of continuous fields. Let \( F(X) \) denote a continuous function whose domain is a region \( \mathcal{R} \) of three-dimensional Euclidean space \( \mathbb{E}^3 \). Following the usual procedure, we replace \( \mathcal{R} \) by a discrete model consisting of a finite number \( E \) of finite elements, all connected together at \( G \) prescribed nodal points (shown in Figure 1 for \( E^2 \)). The values of \( F(X) \) (or its approximation) at a nodal point \( X_\Delta \) are denoted \( F_\Delta \).

\[
\Delta = 1, 2, \ldots, G \quad (i.e. \; F_\Delta = F(X_\Delta)) \text{. Now consider a typical finite element } e. \text{ isolated from the collection, which contains } N_e \text{ prescribed nodal points. The function } F(X) \text{ is approximated locally over element } e \text{ as a linear combination of its local values } f^N_{ieol} \text{ at the } N_e \text{ nodal points:}
\]

\[
F(X) \approx \sum_{N=1}^{N_e} \psi^N_{ieol}(X) f^N_{ieol} \quad N = 1, 2, \ldots, N_e \tag{34}
\]

where the interpolation functions \( \psi^N_{ieol}(X) \), generally polynomials, satisfy

\[
\psi^N_{ieol}(X_{\Delta}) = \delta^N_{\Delta} \quad \sum_{N=1}^{N_e} \psi^N(X) = 1 \tag{35}
\]

The global nodal values \( F_\Delta \) are mapped into the local values \( f^N_{ieol} \) by

\[
f^N_{ieol} = \Omega^N_{ieol} F_\Delta \tag{36}
\]

where \( \Omega^N_{ieol} \Delta \) is unity if node \( N \) of element \( e \) coincides with node \( \Delta \) of the connected model, and is zero if otherwise. In equation (34), the repeated index \( N \) is summed from 1 to \( N_e \); in equation (36) \( \Delta \) is summed from 1 to \( G \).

Except for a set of Lebesgue measure zero, the final discrete model of \( F(X) \) is given by

\[
F(X) \approx \sum_{\epsilon=1}^{E} \Omega^\epsilon_{ieol} F_\Delta \tag{37}
\]

For further details on the topological properties of the model consult Reference 3.

An important feature of this discrete model is that the process of connecting elements or of decomposing \( \mathcal{R} \) into elements [e.g. equation (36)] is purely a formal one dependent only on geometric and topological properties of the model. Indeed, the local approximation (34) can be
constructed for an isolated element, independent of other elements in the system or of its ultimate location in the connected model. For this reason, we omit the element identification label \( e \) in the following developments and confine our attention to local approximations over a typical element. It is understood that global representations can then be obtained by simple Boolean transformations of the type indicated in equation (36).

**Approximation of electrical, thermal and mechanical fields**

In the present investigation, we select as primary unknowns those field quantities observed most naturally in the phenomena (or the potentials from which they can be derived): the displacement field \( u_i \), the temperature \( \theta \), the scalar potential \( \psi \) (voltage) and, in the case of magnetic fields, the vector potential \( A_i \). In the case of temperature, it is convenient to represent the absolute temperature \( \theta \) as the sum of a uniform reference temperature \( T_0 \) and an increment \( T(X, t) \), and to approximate \( T \) instead of \( \theta \). Thus, according to equation (34), we have for a typical finite element

\[
\begin{align*}
\psi(X, t) &= \psi^N(X) \phi_N(t) \\
T(X, t) &= T_0 + \psi^N(X) T_N(t) \\
\varphi(X, t) &= \psi^N(X) \psi_N(t) \\
A_i(X, t) &= \psi^N(X) A_N(t)
\end{align*}
\]

Here \( u_{Ni}, T_N, \varphi_N \) and \( A_{N_i} \) are the components of displacement, the temperature, the voltage and the components of the magnetic potential at node \( N \) of element \( e \). It is understood that \( u_{Ni}, \ldots, A_{N_i} \) are functions of only time \( t \), and that the repeated nodal index \( N \) is to be summed from 1 to \( N_e \), where \( N_e \) is the total number of nodes belonging to element \( e \).

It follows that for a finite element, the strain tensor is

\[
\gamma_{ij} = \frac{1}{2}(\psi_{Ni}^N u_{Nj} + \psi_{Nj}^N u_{Ni} + \psi_{Ni}^N \psi_{Nj}^N u_{Nk} u_{Mk})
\]

the absolute temperature is

\[
\theta = T_0 + \psi^N T_N
\]

the electric field is

\[
E^i = -\psi^N \varphi_N - \psi^N A_{N_i}
\]

and the magnetic flux density is

\[
B^k = \varepsilon^i_{jk} \psi^N A_{N_i}
\]

Here \( \psi^N = \partial \psi^N(X)/\partial X^i \), \( A_{Ni} = dA_{Ni}(t)/dt \) and the dependence of \( \psi^N \) on \( X \) and \( u_{Ni}, \ldots, A_{Ni} \) on \( t \) is understood.

**ELECTROTHERMOMECHANICAL BEHAVIOUR OF A FINITE ELEMENT**

The behaviour of an element is governed by four systems of coupled equations: motion, heat conduction, electric field and magnetic field.

**Equations of motion of an element**

Introducing equations (16)-(20) into the energy equation (7), and incorporating equation (38) into the resulting expression, we obtain for the law of conservation of energy of a electrothermoelastic finite element

\[
m^{N,M} \ddot{u}_{Mi} \dot{u}_{Ni} + \int_{r_e} \rho_0 \psi_{Nj}^N (\delta_{jk} + \psi_{Mj}^N u_{Mj}) d\gamma_i \dot{u}_{Ni} = p^{N,i} \dot{u}_{Ni}
\]
where

\begin{align}
m^{\alpha\beta} & = \int_{V_e} \rho_0 \psi^\alpha \psi^\beta \, dV_e \tag{47} \\
p^{\alpha\beta} & = \int_{V_e} \rho_0 \psi^\alpha F_i \, dV_e + \int_{\Gamma_e} T^i \psi^\alpha \, dA_e \tag{48}
\end{align}

Here \( r_e \) is the initial volume of the element, \( m^{\alpha\beta} \) is the mass matrix for the element and \( p^{\alpha\beta} \) are the components of generalized force at node \( N \). For finite deformations, it is understood that \( T^i \) acts on a material area \( A_e \) in the deformed element and, consequently, will depend upon the deformations (see Reference 4).

Since equation (46) must hold for arbitrary nodal velocities, we have

\begin{align}
m^{\alpha\beta} \ddot{u}_{\alpha\beta} + \int_{V_e} \rho_0 \frac{\partial}{\partial t} \psi^\alpha (\delta_{\beta\gamma} + \psi^\gamma_{,\gamma} \mu_{\alpha\beta}) \, dV_e = p^{\alpha\beta} \tag{49}
\end{align}

Appropriate forms of \( \Pi() \) must be introduced for specific materials. Equation (49) represents the general equations of motion for an electrothermoplastic finite element.

**Equations of heat conduction for an element**

Multiplying equation (21) by the temperature increment \( T(X,t) = T \) and observing that

\begin{align}
\nabla \cdot T \mathbf{q} & = T \nabla \cdot \mathbf{q} + \mathbf{q} \cdot \nabla T \\
\int_{V_e} \nabla \cdot T \mathbf{q} \, dV_e & = \int_{A_e} T \mathbf{q} \cdot \mathbf{n} \, dA_e
\end{align}

where \( \mathbf{q} \) is the heat flux vector, we arrive at the alternate form of the conservation of energy:

\begin{align}
\int_{V_e} \left[ \rho_0 \theta \mathbf{q} - \rho_0 \mathbf{h} T + \mathbf{q} \cdot \nabla T - J^i E_i + D_k \frac{d}{dt} (\varepsilon^{ijk} \dot{u}_j B_k) \right] \, dV_e = \int_{A_e} T \mathbf{q} \cdot \mathbf{n} \, dA_e \tag{52}
\end{align}

wherein \( \sigma^* \) has been equated to zero. Now for a finite element we recall that \( \theta \) and \( T \) are given by equations (43) and (39) and, accordingly, \( E^i \) is given by equation (44). Moreover, \( \eta \) is defined by equation (25b) for an electrothermoplastic media. Thus, introducing the finite element approximations into equation (52) and noting that the result must hold for arbitrary nodal temperatures, we arrive at the coupled equation of heat conduction for an electrothermoplastic finite element:

\begin{align}
- \int_{V_e} \left[ \rho_0 (T_0 + \psi^M T_M) \psi^\alpha \frac{d}{dt} \left( \frac{\partial \Pi}{\partial T} \right) - q^i \psi^\alpha - J^i \psi^\alpha (\psi^M \mu_{\alpha\beta}) + \psi^M \dot{A}_M + \psi^\alpha \psi^M \frac{\partial}{\partial E^i} \frac{d}{dt} (\varepsilon^{ijk} \dot{u}_j B_k) \right] \, dV_e = q^\alpha \tag{53}
\end{align}

where \( q^\alpha \) is the generalized heat flux at node \( N \):

\begin{align}
q^\alpha = \int_{V_e} \rho h q^\alpha \, dV_e + \int_{A_e} \psi^\alpha q^i \mathbf{n} \cdot \mathbf{d}A_e \tag{54}
\end{align}

Again, forms of equation (53) for specific materials can be obtained after identifying the appropriate constitutive equations for \( \Pi \), \( q^i \) and \( J^i \). We observe that the contributions of the current and potentials \( \phi \) and \( A_i \) appear non-linearly in equation (53).
Electromagnetic field equations for a finite element

To develop the equations governing the electromagnetic field in an element, we utilize three potential functions: the free energy $\Pi$, the scalar potential $\varphi$ and the vector potential $A$. These potentials have the properties:

$$
B^k = \epsilon^{ijk} A_{j,i}, \quad H^j = \rho_0 \frac{\partial \Pi^0}{\partial B^j}, \quad E_i = -\varphi_{,i} - A_i \tag{55a, b, c}
$$

Additionally we must use the finite element representation for the conservation of charge, equation (17). These equations are formulated in the following development.

Multiplying equation (17) by $\varphi$ and using the identity

$$
\nabla \cdot (\varphi \mathbf{D}) = \varphi (\nabla \cdot \mathbf{D}) + (\nabla \varphi) \cdot \mathbf{D} \tag{56}
$$

we obtain the following global form of the conservation of charge:

$$
- \int_{r_e} D^j \varphi_{,j} \, dr_e = \int_{r_e} \rho_e \varphi \, dr_e - \int_{A_e} \varphi D^j n_i \, dA_i \tag{57}
$$

Introducing equations (25c) and (40) into equation (57) and observing that the resulting equation must hold for arbitrary nodal voltages, we obtain the equations of charge for an electrothermoelastic finite element

$$
\int_{r_e} \left( -\rho_0 \frac{\partial \Pi^0}{\partial E_i^*} \right) \psi_N^i \, dr_e = c^N \tag{58}
$$

where $c^N$ is the generalized electric charge at node $N$:

$$
c^N = -\int_{r_e} \psi_N^j \rho_e \, dr_e + \int_{A_e} \psi_N^j D^j n_i \, dA_i \tag{59}
$$

Using equations (8) and (55a, b, c) and expressing $S^j_i$ in terms of $E_i$ and $H_i$ we see that the electromagnetic conservation principle can be written in the form

$$
- \int_{r_e} \left( \rho_0 \epsilon^{ijk} \frac{\partial \Pi^0}{\partial B^j} (\varphi_{,k} + A_k) \right) \psi_N^i \, dr_e = \int_{r_e} \left[ \rho_0 \frac{d}{dt} \left( \frac{\partial \Pi^0}{\partial E_i^*} \right) (\varphi_{,k} + A_k) \right] \psi_N^i \, dr_e \\
- \int_{r_e} \left( \rho_0 \epsilon^{ijk} \frac{\partial \Pi^0}{\partial B^j} (A_{k,i}) \right) dr_e - \int_{A_e} J^k(\varphi_{,k} + A_k) \, dA_e \tag{60}
$$

Using equations (40) and (41), the equations for a typical finite element are given by:

$$
\int_{r_e} \epsilon^{ijk} \left( \rho_0 \frac{\partial \Pi^0}{\partial B^j} \psi_k^i \right) \varphi_{,k} + \left( \rho_0 \frac{\partial \Pi^0}{\partial B^j} \right) \psi_N^i A_{k,i} \, dr_e = \int_{r_e} \left[ \frac{d}{dt} \left( \rho_0 \frac{\partial \Pi^0}{\partial E_i^*} \right) - J^k \right] \left[ \psi_N^i \varphi_{,k} + \psi_N^i A_{k,i} \right] \, dr_e \tag{61}
$$

Since equation (61) must hold for arbitrary nodal values of $\varphi_{,k}$ and $A_{k,i}$, we conclude that

$$
\int_{r_e} \epsilon^{ijk} \left( \rho_0 \frac{\partial \Pi^0}{\partial B^j} \psi_k^i \right) \frac{d}{dt} \left( \rho_0 \frac{\partial \Pi^0}{\partial E_i^*} \right) \psi_N^i + J^k \psi_N^i = 0 \tag{62}
$$

and

$$
\int_{r_e} \epsilon^{ijk} \left( \rho_0 \frac{\partial \Pi^0}{\partial B^j} \psi_N^i \right) - \frac{d}{dt} \left( \rho_0 \frac{\partial \Pi^0}{\partial E_i^*} \right) \psi_N^i \, dr_e = J^{Nk} \tag{63}
$$

where $J^{Nk}$ is the generalized current density at node $N$:

$$
J^{Nk} = -\int_{A_e} \psi_N^j J^k n_i \, dA_e + \int_{r_e} \epsilon^{ijk} H_j \psi_N^i \, dr_e \tag{64}
$$
Equations (58), (62) and (63) govern the electromagnetic fields in a finite element. Additionally we observe that equation (55a) does not uniquely define the vector potential $A$, i.e. a vector is not uniquely defined by specifying its curl. However, we will defer a complete specification of $A$ (i.e. specify div $A$) until specific forms of the free energy function are considered, since we desire the simplest formulation in each specific case.

**Finite element analogues of Maxwell's equations**

Although a multitude of special cases can be derived from equations (49), (53), (58), (62) and (63), we shall cite here only one example of special interest: the finite element analogues of the classical Maxwell field equations for stationary media (i.e. $E^*_I = E^I$, $B^*_I = B^I$, etc.)

\[
\text{div} \, B = 0, \quad \text{curl} \, E + \partial B / \partial t = 0 \quad (65a, b)
\]

\[
\text{div} \, D = \rho_e, \quad \text{curl} \, H = J + \partial D / \partial t \quad (65c, d)
\]

Also, we have

\[
\text{div} \, J = - \frac{\partial \rho_e}{\partial t} \quad (65e)
\]

Equations (67a) and (67b) are identically satisfied over an element due to the definitions of $\varphi$ and $A$ and the fact that the interpolation functions $\phi^N(x)$ are continuously differentiable. Suppose that $\gamma_{ij} = T \equiv 0$, and that the free energy is given by

\[
\rho_0 \| \vec{B} - \frac{\varepsilon}{2} \vec{E} \| = -2 \varepsilon^T \frac{\partial \vec{P}}{\partial \vec{E}} - \varepsilon^T \vec{E} (66)
\]

where $\mu$ is the permeability and $\varepsilon$ is the permittivity of the medium. Then

\[
H_i^* = \rho_0 \frac{\partial \| \vec{B} \|}{\partial B_i} = \frac{1}{\mu} B^i \quad \text{and} \quad D^i = -\rho_0 \frac{\partial \| \vec{P} \|}{\partial E_i} = \varepsilon E^i \quad (67a, b)
\]

Thus, for a finite element,

\[
H_i^* = \frac{1}{\mu} \varepsilon^T \phi^N A^N \quad \text{and} \quad D^i = -\varepsilon (\psi^N \phi^N + \phi^N A^N) \quad (68a, b)
\]

Hence, equations (58), (62) and (63) can be rewritten, respectively, in the forms

\[
\int_{V_e} \psi^N D^i \, dr_e = c^N \quad (69)
\]

and

\[
\int_{V_e} [\varepsilon^{ijk} H^i_j \psi^N_k] \, dr_e = c^N + \int_{V_e} \psi^N_k J^k \, dr_e \quad (70)
\]

\[
\int_{V_e} [\varepsilon^{ijk} H^i_j \psi^N_k] \, dr_e = \frac{d}{dt} \int_{V_e} \psi^N D^k \, dr_e + \int_{V_e} \psi^N J^k \, dr_e \quad (71)
\]

Equation (69) is the finite element analogue of equation (65c) and equation (71) is the finite element analogue of equation (65d). Equation (70) is the finite element analogue of the continuity equation (65e). The term on the left side of equation (70) represents a finite element approximation of div (curl $H$) which, for the continuum, is zero, but in the discrete model is only approximately zero over a finite element. As the finite element network is refined, this term vanishes. Consequently, it is meaningful to rewrite equation (70) in the form

\[
c^N = - \int_{V_e} \psi^N J^k \, dr_e \quad (72)
\]

which is now analogous to equation (65e).
LINEAR ELECTROThERMOELASTICITY

Up to this point, we have placed no restrictions on the form of the constitutive equations for \( \Pi(\gamma_{ij}, \theta, E_i, B_j) \), \( q^i \) or \( J^i \) or on the magnitudes of \( \gamma_{ij}, T, E_i \) or \( B_j \). Henceforth, we restrict our attention to linear electrothermoelasticity, for which the heat flux and current density are given by Fourier's law [equation (26)] and Ohm's law [equation (27)], the strain tensor is linear in the displacement gradients \( 2\gamma_{ij} = u_{ij} + u_{ji} \). \( C \) and \( C_0 \) are approximately the same. \( E_i^e = E_i \), \( H_k^e = H_k \), etc. Further, we assume that the material is characterized by a free energy function of the following quadratic form:

\[
\rho_0 \Pi = \frac{1}{2} E^{ijkm} \gamma_{ij} \gamma_{km} + B^{ij} \gamma_{ij} T + \frac{e_r}{2T_0} T^2 + \frac{1}{2} X^{ij} E_i E_j
\]

\( + R^i B_i T + \frac{1}{2} Y^{ij} B_i B_j + G^i E_i T + W^{ijk} \gamma_{ij} E_k + L^{ijk} \gamma_{ij} B_k \)  

(73)

Here \( E^{ijkm}, B^{ij}, X^{ij}, Y^{ij}, W^{ijk} \) and \( L^{ijk} \) are arrays of material parameters which, for homogeneous materials, are regarded as constants, and \( e_r \), \( R^i \) and \( G^i \) are the specific heat, pyromagnetic constant and pyroelectric constant, respectively. These arrays exhibit the following symmetries:

\[
E^{ijkm} = E^{jikm} = E^{jmk} = E^{mki} 
\]

(74a)

\[
X^{ij} = X^{ji}, \quad Y^{ij} = Y^{ji} 
\]

(74b, c)

\[
B^{ij} = B^{ji}, \quad W^{ijm} = W^{jim}, \quad L^{ijm} = L^{jim} 
\]

(74d, e, f)

According to equations (25)-(27), the constitutive equations for stress, entropy, etc. for such materials are

\[
t^{ij} = \rho_0 \frac{\partial \Pi}{\partial \gamma_{ij}} = E^{ijkm} \gamma_{km} + B^{ij} T + W^{ijm} E_m + L^{ijm} B_m 
\]

(75)

\[
\rho_0 \eta = \rho_0 \frac{\partial \Pi}{\partial T} = - \left( B^{ij} \gamma_{ij} + \frac{e_r}{T_0} T + R^i B_i + G^i E_i \right) 
\]

(76)

\[
D^i = - \rho_0 \frac{\partial \Pi}{\partial E_i} = -(X^{ij} E_j + G^i T + W^{jim} \gamma_{jm}) 
\]

(77)

\[
H^i = \rho_0 \frac{\partial \Pi}{\partial B_i} = R^i T + Y^{ij} B_j + L^{jim} \gamma_{jm} 
\]

(78)

\[
q^i = K^{ij} T_j, \quad J^i = Z^{ij} E_j 
\]

(79)

where \( K^{ij} = K^{ji} \) and \( Z^{ij} = Z^{ji} \). We can now obtain explicit equations for typical finite elements by introducing these relations in those derived in the previous section.

**Equations of motion**

\[
m^{NM} \ddot{u}_{Mi} + k^{NMm} u_{Mm} + b^{NM}_{xM} T_M + W^N_i \phi_M + W^{XMm} A_{Mm} + c^{NMm}_x A_{Mm} = p^{Xi} 
\]

(80)

**Equations of heat conduction**

\[
a^{NM} \Phi_M - k^{NM} T_M + T_0 (h^{NM}_{xM} u_{Mi} + g^{NM} \phi_M + g^{NM} A_{Mx} + r^{NM}_{xM} A_{Mi}) = q^N 
\]

(81)

**Equations of electromagnetic field**

\[
d^{NM} \Phi_M + e^{NM}_{xM} A_{Mx} - W^N M u_{Mx} - G^{NM} T_M = e^N 
\]

(82a)

\[
h^{NM} \Phi_M + h^{NM}_{xM} T_M - S^{NM}_{xM} A_{Mx} + q^{XM} A_{Mx} + s^{NM}_{xM} u_{Mx} + f^{NM}_{xM} u_{Mm} = J^{Xn} 
\]

(82b)

\[
Z^{MN} A_{Mx} = Y^{MN}_{xM} \Phi_M 
\]

(82c)
In these equations, $m_{NM}^N$ is the mass matrix defined in equation (47), $k_{NM}^{N,m}$ is the stiffness matrix, $a_{NM}^N$ is the specific heat matrix and $k_{NM}^N$ is the thermal conductivity matrix:

$$k_{NM}^{N,m} = \int_{r_e} E_{ijkmn} \psi^N_{m} \psi^N_{j} \psi^N_{k} \, dv_e,$$

$$a_{NM}^N = \int_{r_e} c_e \psi^N \psi^N \, dv_e$$  \hspace{1cm} (83)

$$k_{NM}^N = \int_{r_e} K^{ijk} \psi^N_{i} \psi^N_{j} \, dv_e$$  \hspace{1cm} (84)

Also $p_{N}^N$, $q_{N}$, $c_{N}$ and $J_{N}^N$ are the generalized nodal forces, heat flux, charge and current density at node $N$. Other matrices for the discrete model are defined in Table 1.

Table 1. Element matrices

<table>
<thead>
<tr>
<th>Matrix</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{NM}^N$</td>
<td>$\int_{r_e} B_{ij} \psi^N \psi^N_{j} , dv_e$</td>
</tr>
<tr>
<td>$W_{i}^{N,m}$</td>
<td>$\int_{r_e} W_{i}^{mnk} \psi^N_{k} \psi^N_{j} , dv_e$</td>
</tr>
<tr>
<td>$g_{NM}^N$</td>
<td>$\int_{r_e} G_{i} \psi^N \psi^N_{j} , dv_e$</td>
</tr>
<tr>
<td>$r_{i}^{N,M}$</td>
<td>$\int_{r_e} R_{i} \psi^N \psi^N_{j} , dv_e$</td>
</tr>
<tr>
<td>$e_{i}^{N,M}$</td>
<td>$\int_{r_e} X_{i}^{j} \psi^N_{j} \psi^N_{k} , dv_e$</td>
</tr>
<tr>
<td>$n_{i}^{N,m}$</td>
<td>$\int_{r_e} \epsilon_{ijk} \psi_{i}^{N} \psi^N_{j} P_{nk} R_{i} , dv_e$</td>
</tr>
<tr>
<td>$q_{i}^{N,M}$</td>
<td>$\int_{r_e} \phi_{i}^{N} \psi^N_{j} \psi^N_{k} , dv_e$</td>
</tr>
<tr>
<td>$f_{i}^{N,mm}$</td>
<td>$\int_{r_e} \epsilon_{ijk} \psi^N_{i} \psi^N_{k} P_{nk} R^{i,m} , dv_e$</td>
</tr>
<tr>
<td>$y_{i}^{N,M}$</td>
<td>$\int_{r_e} \phi_{i}^{N} \psi^N_{j} P_{nk} X_{i}^{k} , dv_e$</td>
</tr>
</tbody>
</table>

In Table 1 we have used the formulae

$$P_{nk} Y_{jr} = \delta_{jn} \delta_{rk}, \quad J_n^N = P_{nk} J_{N}^k$$  \hspace{1cm} (85)

In the case of homogeneous bodies for which the material coefficients in equation (73) are constants, some simplification in the definitions of the element matrices is possible. In this case

$$a_{NM}^N = \frac{c_e m_{NM}^N}{\rho_o} = \frac{c_e}{\rho_o} m_{NM}^N$$

$$= \frac{c_e P_{nk} G_k}{P_{ij} G_{i} G_{m}} h_{NM}^N = \frac{c_e P_{nk} X_{km}}{P_{ir} X_{i}^{r} P_{ik} X_{is}} X_{N}^{km}$$  \hspace{1cm} (86)

$$w_{i}^{N,m} = (W^{mnk})^{-1} B_{i}^{r} h_{NM}^N = (W^{mnk})^{-1} X_{i}^{kr} e_{r}^{N} S_{N}^{M,m}$$  \hspace{1cm} (87)

etc. Further reduction is possible if isotropy is assumed.
In this section, we consider a sample problem which involves time-dependent heating of an anisotropic electrothermoelastic half-space. A simplified linear model utilizing one-dimensional spatial and material forms of the finite element equations is used. This formulation represents the finite element equivalent of Voigt's classical linear equations for piezoelectricity. Although this problem is, in some respects, of a simple nature due to the absence of magnetic effects, it illustrates the merits of the finite element representation as we have developed it in this paper.

Using a standard simplex model, the non-dimensionalized, coupled finite element equations of motion, heat conduction and electromagnetic field for a typical interior node \( N \) are given by

**Equations of motion**

\[
\ddot{U}_N = \frac{1}{\xi^2} \left( \ddot{U}_{N-1} + \ddot{U}_{N+1} - 2\ddot{U}_N \right) + \frac{1}{\xi} \left( \dot{\theta}_{N-1} + \dot{\theta}_{N+1} - 2\dot{\theta}_N \right)
\]  

(88)

**Equations of heat conduction**

\[
\ddot{\theta}_N = \frac{1}{\xi^2} \left( \ddot{\theta}_{N-1} + \ddot{\theta}_{N+1} - 2\ddot{\theta}_N \right) + \frac{\delta_1}{\xi} \left( 2\dot{U}_N - \dot{U}_{N-1} - \dot{U}_{N+1} \right) + \frac{\delta_2}{\xi} \left( 2\dot{U}_N - \dot{U}_{N-1} - \dot{U}_{N+1} \right)
\]  

(89)

**Equations of electric field**

\[
\ddot{\Phi}_N = \ddot{\eta}\ddot{U}_N - \frac{\xi}{\xi^2} \lambda \ddot{\theta}_N
\]  

(90)

where we have utilized the non-dimensional equation parameters listed in Table II.

<table>
<thead>
<tr>
<th>Table II. Non-dimensional equation parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi = \frac{a}{\kappa} \bar{x}_1 )</td>
</tr>
<tr>
<td>( \tau = \frac{a^2}{\kappa} t )</td>
</tr>
<tr>
<td>( \bar{U} = \frac{a E^<em>}{\kappa B^</em> T_0} U )</td>
</tr>
<tr>
<td>( \ddot{\theta} = \frac{T}{T_0} )</td>
</tr>
<tr>
<td>( \ddot{\Phi} = \frac{a B}{K^* \Phi_0} \Phi )</td>
</tr>
<tr>
<td>( \delta_1 = \frac{T_0 B^<em>}{p_c E^</em>} )</td>
</tr>
<tr>
<td>( \delta_2 = \frac{\delta_1 \ddot{\eta} \lambda}{BT_0 \alpha^2} )</td>
</tr>
</tbody>
</table>

In the above table, \( \bar{x}_1 \) is a characteristic length, \( t \) the real time, \( \Phi_0 \) a reference voltage, \( c_v \) the specific heat at constant volume, \( \rho \) the mass density, \( B, G, E, X, K \) material parameters, \( T_0 \) a reference temperature, \( \ddot{\eta} \) an electromechanical coupling constant, \( \lambda \) an electrothermal coupling constant and \( \delta_1, \delta_2 \) are electrothermomechanical coupling constants.

We consider a linear elastic half-space \( (x_1 > 0) \) subject to a ramp surface heating of the form

\[
T_1 = \frac{T}{t_0} \quad 0 \leq t \leq t_0
\]  

(91)

\[
T_1 = T_f \quad t_0 < t
\]  

(92)
where $T_1$ is the surface temperature, $T_f$ is the final surface temperature and $t_0$ is the boundary temperature rise time. Additionally we define the following non-dimensional parameters

$$\tau_0 = \frac{a^2}{k} t_0$$  \hspace{1cm} (93)

$$\theta_j = \frac{\tau}{\tau_0}, \hspace{0.5cm} 0 \leq \tau \leq \tau_0$$  \hspace{1cm} (94)

$$\theta_j = 1, \hspace{0.5cm} \tau > \tau_0$$  \hspace{1cm} (95)

Our numerical results for the half-space are presented in Figures 2–9. This system of over two hundred ordinary differential equations was solved using standard Runge-Kutta numerical integration. Figures 2–5 contain the dimensionless temperature $\bar{\theta}$, displacement $\bar{U}$ and voltage $\bar{\Phi}$ as a function of dimensionless time $\tau$. whereas Figures 6–9 contain the same dimensionless variables as a function of dimensionless distance $\bar{x}$. In all figures the values of $\bar{\xi}$, $\delta_1$ and $\tau_0$ are 1.0.

Figures 2 and 3 illustrate the effects of the electromechanical coupling parameters $\delta_2$ on temperature $\bar{\theta}$ and displacement $\bar{U}$. The curves for $\delta_2 = 0.0$ are identical to the thermoelastic curves of Oden and Krosz.27 Figure 4, which contains the dimensionless voltage $\bar{\Phi}$ as a function of dimensionless time $\tau$, illustrates that the thermoelastic curves ($\delta_2 = 0.0$) are also valid for an electrothermoeelastic half-space if the parameter definitions developed in this paper are utilized. Figure 5 illustrates the effect on the non-dimensional voltage $\bar{\Phi}$ of a non-zero $\delta_2$. It is apparent that these curves are significantly different in character from the ‘uncoupled’ curves of Figure 4 and additionally that the sgn $\delta_2$ is of major importance in determining the shape and amplitude of these curves.

![Figure 2. Temperature variation at unit distance in electrothermoelastic half-space subjected to ramp heating](image)
Figure 3. Displacement variation at unit distance in electrothermoelastic half-space subjected to ramp heating

Figure 4. Voltage variation (uncoupled at unit distance) in electrothermoelastic half-space subjected to ramp heating
Figure 5. Voltage variation (coupled at unit distance) in electrothermoelastic half-space subjected to ramp heating.

Figures 6 and 7 illustrate the temperature and displacement as a function of half-space non-dimensional distance $\xi$ for different values of $\tau$. Only the curves for $\delta_2 = 0.0$ are included as they are considered representative.

Figure 6. Temperature distribution for various time parameters ($\tau$) of an electrothermoelastic half-space subjected to ramp heating.
Figure 7. Displacement distribution for various time parameters ($\tau$) of an electrothermoelastic half-space subjected to ramp heating

Figure 8. Voltage distribution for various time parameters ($\tau$) of an electrothermoelastic half-space subjected to ramp heating
The non-dimensional voltage curves of Figures 8 and 9, which are plotted as a function of non-dimensional distance for various $\delta_2$, are significantly different for the different cases. The 'coupled' curves of Figures 8 and 9 are linear combinations of the temperature and displacement curves. In Figures 8 and 9 the $\text{sgn} \delta_2$ is a major contribution to the shape and amplitude of these curves.

![Figure 9. Voltage distribution for various time parameters (t) of an electrothermoelastic half-space subjected to ramp heating](image)

and since the $\text{sgn}$-amplitude combinations can be different for various materials, almost any amplitude or shape combination of the voltage curves is possible.

The merits of the formulation are clearly demonstrated by this example. While more complicated geometries and boundary conditions involve lengthier and more cumbersome calculations, the general approach is essentially the same as that used here.

ACKNOWLEDGEMENT

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