ANALYSIS OF NONLINEAR, DYNAMIC COUPLED THERMOVISCOELASTICITY PROBLEMS BY THE FINITE ELEMENT METHOD

J. T. Oden† and W. H. Armstrong‡

The University of Alabama, Huntsville, Alabama, U.S.A.

Abstract—This investigation deals with the numerical solution of a class of nonlinear problem in transient, coupled, thermoviscoelasticity. Equations of motion and heat conduction are derived for finite elements of thermomechanically simple materials and these are adapted to special classes of thermorheologically simple materials. The analysis involves the solution of large systems of nonlinear integrodifferential equations in the nodal displacements and temperatures and their histories. As a representative example, the general equations are applied to the problem of transient response of a thick-walled hollow cylinder subjected to time-varying internal and external pressures, temperatures, and heat fluxes. The integration scheme used to solve the nonlinear equations employs a linear acceleration assumption, representation of nonlinear integral terms by Simpson's rule, and the iterative solution of large systems of nonlinear algebraic equations at each reduced time step by the Newton-Raphson method. Various numerical results are given and are compared with the linearized, isothermal, and quasi-static solutions.

1. INTRODUCTION

The interconvertibility of mechanical work and heat was recognized long ago by Thompson (Count Rumford, 1797) and their equivalence was experimentally verified by Joule and Kelvin in the nineteenth century. Indeed, it is common experience that mechanical working of a solid body generates heat and that, conversely, heating a body produces mechanical work. It is also common experience that the thermomechanical properties of many materials may vary with time, temperature, and the rate at which they are deformed. Plastics, polymers, biological tissues, and most metals, for example, may continue to deform under constant applied load, and the rate of continued deformation may be highly sensitive to changes in temperature. The response of such materials is dependent upon the history of the local motion as well as the history of the temperature: i.e., the material possesses a 'memory' of its past response.

While such thermomechanical phenomena have been widely observed for centuries, general continuum theories aimed at describing them have emerged only recently in the works of Coleman [1] and others. Owing to the complexity of the general theories, they have provided largely qualitative information on the thermomechanical response of materials with memory. In fact, the use of such theories in the experimental characterization of actual materials constitutes, in itself, a difficult problem requiring much additional study: and applications of these theories to the quantitative solution of nonlinear boundary—and initial—value problems of practical importance do not appear to be available.

This paper is concerned with the formulation and numerical solution of a class of nonlinear problems in dynamic, coupled, thermoviscoelasticity. General finite-element...
models are derived for the equations of motion and heat condition of typical elements of dissipative materials. These include thermomechanical coupling and, for the classes of materials considered, assume the form of systems of nonlinear integrodifferential equations in nodal displacements and temperatures and their histories. In the case of the coupled heat conduction equation for an element, it is shown that forms for the discrete model differ significantly from that of the classical theory and cannot be obtained by simply adding a dissipation integral as is often done.

Applications of the finite-element method to problems in linear, isothermal, viscoelasticity have been presented [2-4] and solutions to linear heat conduction problems were given [5, 6]. References [7-10] considered applications of the method to problems in coupled thermoelasticity, and the formulation of general discrete models of thermomechanical behavior was discussed [11]. More recently, Taylor et al. [12] described a finite-element analysis of a class of thermorheologically simple materials in which a nonlinear temperature term was included. In their analysis, solutions of the uncoupled, linear heat conduction equation were used as input to the equations describing quasi-static motion of a finite element. Cost [13] considered thermomechanical coupling phenomena in thermorheologically simple materials and presented numerical solutions to representative linear non-isothermal problems.

In the present study, the general equations governing the finite element model are used to obtain special forms of the equations for a class of thermorheologically simple materials. These are applied to the problem of transient response of a thick-walled cylinder subjected to time-varying internal pressure and temperature. A description of the numerical procedures used in the analysis is also given. Briefly, the analysis employs a 'semi-explicit' integration scheme based on a quadratic difference approximation of displacements and temperatures (i.e., a linear acceleration scheme) and a technique in which nonlinear integral terms are expanded according to Simpson's rule and the displacement and temperature histories at only one time interval in the past need be retained. In this latter respect, the procedure is similar to that used in finite-element studies of linear thermoviscoelasticity problems [12]. However, in the present nonlinear analysis, a system of nonlinear algebraic equations in unknown nodal displacements and temperatures are encountered at each time increment, making the method not a true explicit scheme. At each step the system of nonlinear equations are solved iteratively by the Newton-Raphson method. Various numerical results are given and compared with the linearized, isothermal, and quasi-static solutions. It is found that including the effects of inertia, mechanically-dependent internal dissipation, and material nonlinearities can lead to significant departures from the results of the linear, quasi-static analysis.

2. THERMOMECHANICAL PRELIMINARIES

Consider a continuous body at a uniform temperature $T_0$ occupying a fixed homogeneous reference configuration $C_0$. Particles $X$ in $C_0$ are located by an intrinsic system of curvilinear coordinates $\xi=(\xi^1, \xi^2, \xi^3)$ being a fixed, cartesian, spatial frame of reference. We trace the behavior of the body from $C_0$ to other configurations $C$ by identifying the displacement vector $u(\xi, t)$ and the temperature change $T=T(\xi, t)$ at the particle $\xi$ at time $t$. The deformation and absolute temperature $\theta$ at a particle $\xi$ at time $t$ are determined by

$$\theta(\xi, t) = T_0 + T(\xi, t)$$

$$\gamma_{ij}(\xi, t) = u_{ij} + u_i \dot{u}_j + u_j \dot{u}_i + \mu_{ikj}$$

(2.1)
where \( \gamma_{ij} \) are the components of the Green–Saint–Venant strain tensor with respect to the basis \( g_i = \partial x/\partial \xi^i \) in the undeformed body, \( u_i \) and \( u^i \) are the covariant and contravariant components of \( u \), and the semi-colon denotes covariant differentiation with respect to the material coordinates \( \xi^i \). In (2.1) it is understood that \( T \) and \( \dot{u}_{ij} \) are functions of \( \xi \) and \( t \). The functions

\[
\gamma_{ij}(s) = \gamma_{ij}(\xi, t-s) \quad \dot{\theta}(s) = \dot{\theta}(\xi, t-s) \quad (2.2)
\]

where \( 0 \leq s < \infty \) are referred to as the total histories of the strain and temperature at \( \xi \) at time \( t \).

We are concerned with the behavior of a class of thermomechanically simple materials; that is, materials whose response at a particle \( \xi \) at time \( t \) depends upon the histories of the deformation and temperature at \( \xi \) [1]. Such materials may be characterized by a collection of constitutive equations for the free energy \( \varphi \), the stress tensor \( \sigma^{ij} \), the specific entropy \( \eta \), and the heat flux vector \( q^i \) of the form

\[
\begin{align*}
\varphi &= \Phi \left[ \gamma', \theta'; \gamma, \theta, g \right] \\
\sigma^{ij} &= \Phi^{ij} \left[ \gamma', \theta'; \gamma, \theta, g \right] \\
\eta &= \Phi^{i} \left[ \gamma', \theta'; \gamma, \theta, g \right] \\
q^i &= Q^i \left[ \gamma', \theta'; \gamma, \theta, g \right] \quad (2.3)
\end{align*}
\]

Here

\[
\Phi \left[ \gamma', \theta' \right], \ldots, \Phi^{ij} \left[ \gamma', \theta' \right]
\]

are functionals of the past histories \( \gamma', \theta' \), where \( \gamma', \theta' \) denote the restrictions of \( \gamma'(s), \theta'(s) \) to the open interval \( s \in (0, \infty) \), and they are functions of the current values \( \gamma = \gamma(0), \theta = \theta(0) \), and \( g = \text{grad} \theta \); \( \sigma^{ij} \) and \( q^i \) denote contravariant components of the stress and heat flux measured per unit deformed surface area and referred to the convected coordinate lines \( \xi^i \) in \( C \).

The constitutive functionals must be form invariant under transformations of the observer reference frame and under transformations of the material frame of references which belong to the isotropy group of the material (e.g., [14, 15]). In addition, the functionals (2.3) must be consistent with the basic physical laws of conservation of mass, balance of linear and angular momentum, conservation of energy, and the Clausuis-Duhem inequality. In particular, we require that (2.3) satisfies the local energy equation

\[
\rho \dot{\varphi} = \sigma^{ij} \dot{\gamma}_{ij} - \rho \eta \dot{\theta} - \sigma \quad (2.4)
\]

and the dissipation inequality

\[
\sigma + \frac{1}{\rho} q^i g_i \geq 0 \quad (2.5)
\]
where $\sigma$ is the internal dissipation,

$$
\sigma = \rho \theta - \rho h - q_{i;ij}
$$

(2.6)

Alternately, (2.4) can be used as a definition of $\sigma$. In these equations, $\rho$ is the mass density in $C$ and $h$ is the heat supplied from internal sources per unit mass.

By introducing (2.3) and (2.4) into (2.5) and observing the arbitrariness of the rates $\dot{\gamma}_{ij}$, $\theta$, and $\mathcal{Q}$ for fixed total histories of $\dot{\gamma}_{ij}$ and $\theta$, Coleman [1] arrived at the following fundamental properties of thermomechanically simple materials:

1. The free energy functional $\Phi [-]$ is independent of the temperature gradient $g$:

$$
\partial_g \Phi [-] = 0
$$

(2.7)

2. A thermomechanically simple material is characterized by only two constitutive functionals, one for the free energy and one for the heat flux. The constitutive functionals for stress and entropy are determined from the free energy functional by

$$
\sigma^{ij} = \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right] \equiv D_{\gamma^i_r} \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right]
$$

(2.8)

$$
\eta = D_{\gamma} \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right] \equiv -D_{\theta} \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right]
$$

(2.9)

where $D_{\gamma^i_r}$ and $D_{\theta}$ are partial differential operators and, in accordance with (2.7), we have eliminated the dependence on $g$ in the arguments of the constitutive functionals for $\sigma$, $\sigma^{ij}$, and $\eta$.

3. The internal dissipation (2.6) in thermomechanically simple materials is determined from the free energy functional by

$$
\sigma = \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right] \equiv -\rho \left\{ \delta_{\gamma^i_r} \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right] + \delta_{\theta} \Phi \left[ \gamma^i_r, \theta^i_r, \gamma, \theta \right] \right\}
$$

(2.10)

where $\delta_{\gamma^i_r}$, $\delta_{\theta}$ are Frechét differential operators and the vertical stroke indicates that the functionals are linear in arguments that follow: i.e.,

$$
\Phi \left[ \gamma^i_r, \theta^i_r \right]
$$

is linear in $\gamma^i_r$ and $\theta^i_r$, where

$$
\dot{\gamma}^i_r = -\frac{d}{d s} \gamma^i(s) \quad \dot{\theta}^i_r = -\frac{d}{d s} \theta^i(s)
$$

(2.11)

and $0 < s < \infty$. 


The operators $D_{\eta \eta}$, $D_{\theta}$, $\delta_{\eta \eta}$, and $\delta_{\theta}$ in (2.8), (2.9), and (2.10) are defined by

$$
D_{\eta \eta} \Phi [\gamma', \theta'; \gamma, \theta] = \lim_{s \to 0} \frac{\partial}{\partial x_s} \Phi [\gamma', \theta'; \gamma + s \Gamma', \theta]
$$

$$
D_{\theta} \Phi [\gamma', \theta'; \gamma, \theta] = \lim_{s \to 0} \frac{\partial}{\partial x_s} \Phi [\gamma', \theta'; \gamma, \theta + s f]
$$

$$
\delta_{\eta \eta} \Phi [\gamma', \theta'; \gamma, \theta] = \lim_{s \to 0} \frac{\partial}{\partial x_s} \Phi [\gamma', \theta'; \gamma + s \Gamma', \theta]
$$

$$
\delta_{\theta} \Phi [\gamma', \theta'; \gamma, \theta] = \lim_{s \to 0} \frac{\partial}{\partial x_s} \Phi [\gamma', \theta + s f; \gamma, \theta]
$$

(2.12)

where $\alpha$ is a scalar and $\Gamma'(s), f'(s)$ are total histories of arbitrary tensor-valued and scalar-valued functions $\Gamma$ and $f$. Due to the decomposition of total histories into past histories and current values, $D_{\eta \eta}$ and $D_{\theta}$ amount to ordinary partial differentiations with respect to $\gamma'$ and $\theta$.

In summary, we require for a specific thermomechanically simple material, functionals

$$
\Phi [\gamma', \theta'; \gamma, \theta]
$$

and

$$
Q^t [\gamma', \theta'; \gamma, \theta, s]
$$

for the free energy and the heat flux; the stress, entropy, and internal dissipation functionals are then determined from (2.8), (2.9), and (2.10). In the following section we examine specific forms of these functionals for an important subclass of simple materials.

### 3. THERMORHEOLOGICALLY SIMPLE MATERIALS

Experimental evidence obtained from tests on a large class of viscoelastic materials has led to the identification of an important subclass of materials with memory, commonly referred to as thermorheologically simple materials. This classification arose from the observation that, among certain amorphous high polymers which approximately obey established linear and nonlinear viscoelastic laws at uniform temperature, are a group which exhibit a simple property with a change of temperature; namely, a translational shift of various material properties when plotted against the logarithm of time at different uniform temperatures. The shift phenomenon is the basic characteristic of all thermorheologically simple materials and makes it possible to establish an equivalent relation between temperature and $ln t$.

To fix ideas, consider the case of isothermal deformations of an isotropic, linearly viscoelastic solid where stress is defined by a constitutive law of the form:

$$
\sigma_{ij} = \int_0^t J(t-s) \frac{\partial \gamma'}{\partial s} ds + \int_0^t K(t-s) \frac{\partial \gamma'}{\partial s} ds
$$

(3.1)
where $\gamma_{ij}$ and $\gamma$ are the deviatoric and dilatational components of $\gamma_{ij}$ and $J(t-s)$ and $K(t-s)$ are relaxation moduli. Now in the case of a thermorheologically simple material, it is possible to write the relaxation moduli at uniform temperature $\theta$, e.g. $J(t-s)$ as a function of $\ln t$, denoted $E_\theta(ln t)$. Then the shift property is apparent if

$$E_\theta(ln t) = E_{\theta_0}(ln t + f(\theta)) \tag{3.2}$$

where $f(\theta)$ is a shift function relative to $\theta_0$ such that $df/dt > 0$ and $f(\theta_0) = 0$. With this property, the relaxation modulus curve will shift toward shorter times with an increase in $\theta$. By introducing a shift factor,

$$b(\theta) = \exp f(\theta) \tag{3.3}$$

then

$$J_\theta(t) = E_\theta(ln t) = E_{\theta_0}[(ln(tb(\theta))) = J_{\theta_0}(\xi) \tag{3.4}$$

where $\xi$ is a reduced time given by

$$\xi = tb(\theta) \tag{3.5}$$

If it is possible to invert (3.5) so as to obtain $t = g(\xi)$, then a constitutive equation for stress can be obtained of the form

$$\xi = \int_0^t \frac{\partial \phi}{\partial \xi'} d\xi' + \frac{1}{2} \int_0^t K(\xi - \xi') \frac{\partial \phi}{\partial \xi'} d\xi' \tag{3.6}$$

wherein we have used the transformations

$$\delta_{ij}(\xi, \xi') = \sigma_{ij}(\xi, g(\xi)) \quad \gamma_{ij}(\xi, \xi') = \gamma_{ij}(\xi, g(\xi)) \tag{3.7}$$

That is, when transformed to a reduced time, the constitutive equations for a thermorheologically simple material assume the same form as the constitutive law for the isothermal case.

While the above development applies to the special case of linear response of a body at various uniform temperatures, the basic ideas can be extended to finite deformations of bodies subjected to transient, non-homogeneous temperatures [15]. For example, if it is assumed that the relaxation properties at a particle $\xi$ depend only on the current temperature $\theta(\xi, t)$ at $\xi$, it may be postulated that an increment $\Delta \xi$ in reduced time is related to an increment $\Delta t$ in real time according to $\Delta \xi = b[\theta(\xi, t)] \Delta t$. Then, instead of (3.5), we have

$$\xi = \int_0^t b[\theta(\xi, \xi')] d\xi' \tag{3.8}$$

Clearly, (3.8) reduces to (3.5) in the case of uniform, constant temperatures.
As a representative example of a constitutive functional for the free energy of a thermorheologically simple material, consider the functional proposed by Cost [13]:

\[ \rho \phi = \varphi_0 + \int_0^t D_{ij}(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} d\xi' - \int_0^t f(\xi - \xi') \frac{\partial \tilde{\theta}}{\partial \xi'} d\xi' \]

\[ + \int_0^t \int_0^1 [3K(\xi - \xi') - 2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'}] \frac{\partial \tilde{T}_{ij}}{\partial \xi'} d\xi' d\xi'' \]

\[ + \frac{1}{4} (\delta_{ij} \delta_{mn} + \delta_{in} \delta_{jm}) \int_0^t [2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' \]

\[ - \delta_{ij} \int_0^t [3\alpha K(\xi - \xi') - 2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' \]

\[ - \frac{1}{2} \int_0^t [m(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' . \]

(3.9)

Here all quantities have been transformed to reduced time: i.e., \( \dot{y}_{ij}(\xi, \xi') = y_{ij}(\xi, g(\xi), \ldots \) etc. \( t = g(\xi) \). Also \( \xi = f(t) \), as given by (3.8), \( \xi' = f(t') \) and \( \xi'' = f(t'') \) are dummy variables in reduced time. In (3.9), \( \varphi_0 \) is a constant. \( D_{ij}(\cdot), f(\cdot), K(\cdot), G(\cdot), \) and \( m(\cdot) \) are material kernels, and \( \alpha \) is the coefficient of thermal expansion. Integrating (3.9) by parts and introducing the resulting equation into (2.8), (2.9), and (2.10) yields the following constitutive equations for stress, entropy, and internal dissipation:

\[ \sigma_{ij} = \int_0^1 [2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} + \frac{1}{4} (\delta_{ij} \delta_{mn} + \delta_{in} \delta_{jm}) [3K(\xi - \xi') - 2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'}] \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' \]

\[ - \delta_{ij} \int_0^1 [3\alpha K(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' \]

\[ - \frac{1}{2} \int_0^1 [m(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' . \]

(3.10)

\[ \rho \dot{\theta} = f(0) + 3\alpha K(0) + m(0) \dot{\theta}(0) - \int_0^1 \frac{\partial}{\partial \xi'} \left[ 3\alpha K(\xi') \right] \dot{\theta}(\xi - \xi') d\xi'
\]

\[ - \int_0^1 \frac{\partial}{\partial \xi'} \left[ m(\xi') \right] \dot{\theta}(\xi - \xi') d\xi' \]

\[ = \int_0^1 \frac{\partial}{\partial \xi'} \left[ 3\alpha K(\xi - \xi') \right] \dot{\theta}(\xi - \xi') d\xi'
\]

\[ - \frac{1}{2} \int_0^1 \frac{\partial}{\partial \xi'} \left[ 2G(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' \]

\[ + \frac{1}{4} \int_0^1 \frac{\partial}{\partial \xi'} \left[ m(\xi - \xi') \frac{\partial \tilde{T}_{ij}}{\partial \xi'} \frac{\partial \tilde{T}_{mn}}{\partial \xi'} d\xi' d\xi'' . \]

(3.11)
In these equations $3K(\xi') = 3K(0, \xi')$, $m(\xi') = m(0, \xi') = 3\zeta^2 \cdot 3K(0, \xi')$. The material kernels $3K( \cdot )$ and $2G( \cdot )$ are relaxation bulk and shear moduli.

To conclude this section, an assumption is made concerning the heat constitutive equation for the thermorheologically simple material. Among a variety of forms that might be selected for the functional is the linear law proposed by Christensen and Naghdi [16].

$$q^i = \delta^{ij} \int_0^\xi \kappa(\xi - \xi') \frac{\partial \theta_j}{\partial \xi'} d\xi'$$

(3.13)

where $\kappa(\xi - \xi')$ is a thermal conductivity kernel and $\theta_j = \frac{\partial \theta}{\partial \xi^j}$. The fact that (3.13) is not completely reconcilable with (2.5) is the subject of a future note by the authors.

4. FINITE ELEMENTS OF DISSIPATIVE MEDIA

The construction of general finite-element models of nonlinear thermomechanical behavior has been discussed elsewhere [11, 17, 18]; consequently, only the main results need be stated here. Following the usual finite-element procedure, we decompose a continuum into a collection of elements connected smoothly together at nodal points. We isolate a typical element $e$ containing $N_e$ nodal points and represent the local displacement field and temperature field by approximations of the form

$$u_i(\xi, t) = \psi_N(\xi)u_i^N(t)$$

$$T(\xi, t) = \varphi_N(\xi)T^N(t)$$

(4.1)

where the repeated nodal index is summed from 1 to $N_e$. Here $u_i^N$ and $T^N$ are the values of the covariant displacement components and the temperature increment $T(\xi, t) \equiv \theta(\xi, t) - T_0$ at node $N$ and $\psi_N(\xi)$ and $\varphi_N(\xi)$ are local interpolation functions defined so that $\psi_N(\xi^M) = \delta^M_N$, at node $\xi^M$, and $\psi_N^N(\xi) = \varphi_N^N(\xi) = 0$ if $\xi$ does not belong to element $e$. By forming energy balances for the element (or, equivalently, by applying Galerkin’s method to local forms of the laws of conservation of linear momentum and energy), it can be shown [11, 17, 18] that the general equations of motion and heat conduction for a finite element are

$$m_{NM} \ddot{\xi}_M + \int_{v_e} \sigma^{ij} \varphi_j \bar{N} \bar{d}V = P_{Nk}$$

(4.2)

$$\int_{v_e} [P(T_0 + \varphi M T^M)\varphi_N \eta + \varphi_N \bar{q}] dV = q_N + \sigma_N$$

(4.3)

Here $v_e$ is the undeformed volume of element $e$, $m_{NM}$ is the mass matrix, $P_{Nk}$ are the components of generalized force at node $N$, $q_N$ is the generalized normal heat flux at node $N$, and $\sigma_N$ is the generalized dissipation at node $N$:

$$m_{NM} = \int_{v_e} \rho \psi_N \psi_N dm$$

$$P_{Nk} = \int_{v_e} \rho \psi_N F_k dm + \int_{A_e} \psi_N S_k dA$$

$$q_N = \int_{A_e} q^i n_i \varphi_N dA + \int_{v_e} \rho h \varphi_N dV$$

$$\sigma_N = \int_{v_e} \sigma \varphi_N dV$$

(4.4)
where $F_k$ and $S_k$ are components of body forces and surface tractions respectively and $n_i$ is a unit vector normal to $A_e$. Note also that for the finite element

$$
\sigma^{ij}\frac{\partial^2 y_{ij}}{\partial t^2} = \sigma^{ij}(\psi_N, \psi_M) + g^{mr}(\psi_M u^M_\tau)$$(4.5)

wherein $\Gamma^k_{jm}$ are the Christoffel symbols of the second kind, $g^{mr}$ is the contravariant metric tensor in $C_0$, and

$$
(\psi_M u^M_\tau)_{;\tau} = \psi_M u^M_\tau - \psi_M u^M_\tau \Gamma^m_{\tau i}$$(4.6)

To specialize (4.2) and (4.3) so that they apply to materials of the type defined by (2.8), (2.9), and (2.10), note that for the finite element

$$
2\eta_j(s) = g_l \cdot u_i(s) + g_{ij} \cdot u_i(s) + u_i(s) \cdot u_i(s)
$$

$$
\eta_i(s) = g^l \psi_N, \psi_M \Gamma^m_{\tau i}$$(4.7)

where $g_l$ and $g^l$ are the biorthogonal base vectors in the initial configuration and $u_i^{\psi M}(s)$ and $T^{N M}(s)$ are the total histories of the nodal displacement components and temperatures. Thus, the arguments in the functionals of (2.8), (2.9), (2.10), and (2.11) can be recast in terms of the histories of $u^N$ and $T^N$:

$$
D_{uj} \Phi \big[ - \big] = \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} [u_r^{\psi M}, T_r^{N M}; u^N, T^N]
$$

$$
D_{uj} \Phi \big[ - \big] = \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} [u_r^{\psi M}, T_r^{N M}; u^N, T^N]
$$

$$
\mathcal{Q} \big[ - \big] = \sum_{s=0}^{\infty} \sum_{s=0}^{\infty} [u_r^{\psi M}, T_r^{N M}; u^N, T^N]
$$

It follows that the equations of motion and heat conduction for finite elements of materials of this class are

$$
m_{NM} u_r^M + \int_{v_r}^{\infty} \mathcal{Q} \big[ - \big] \frac{\partial^2 y_{ij}}{\partial u_k^M} = P_{NK}$$(4.9)

$$
\int_{v_r} \{\rho(T_0 + \psi_M T^M) g - \phi_{N,1} \mathcal{Q} \big[ - \big] \} dv = q_0 + \int_{v_r} \phi_N \mathcal{Q} \big[ - \big] u_r^{\psi M} \ O_r^{N M} dv
$$

where, for simplicity, we denote by $\big[ - \big]$ the arguments listed in (4.8).
Thermorheologically simple materials

Further specialization of (4.9) and (4.10) can be obtained by restricting our attention to thermorheologically simple materials of the type described in the previous section. We shall now consider a class of such materials for which it is possible to transform all field quantities to corresponding quantities in reduced time: i.e. if \( f(\xi, t) \) is a function of real time \( t \), a function \( g(\xi) = t \) exists such that we can write

\[
\dot{f}(\xi, \xi) = f(\xi, g(\xi))
\]  

(4.11)

Then it is easily shown that

\[
f(\xi, t) = \dot{f}(\xi, \xi) \delta[\dot{\theta}(\xi, \xi)]
\]  

(4.12)

where \( \dot{f} = df/d\xi \) and \( \delta[ \] \) is the shift factor at \( \xi \) at reduced time \( \xi \) corresponding to \( t \):

\[
\delta(\dot{\theta}(\xi, \xi)) = \delta(\dot{\theta}(\xi, t))
\]  

(4.13)

Note that \( d\xi/dt = 0 \), since \( \xi \) denotes a material point (i.e., \( \xi^i \) are intrinsic coordinates). It follows that

\[
\dot{u}_i \equiv \dot{b}(\dot{\theta}) u_i^M
\]

\[
\dot{u}_i^M = b(\dot{\theta}) \dot{u}_i^M + b(\dot{\theta}) \dot{b}(\dot{\theta}) u_i^M
\]

\[
\dot{\theta} = b(\dot{\theta}) \dot{\theta}
\]  

(4.14)

e.g., wherein dependence of the quantities \( u_i^M, \dot{\theta}, \) etc. on \( \xi \) is understood. Similar relations hold for \( \eta, q_i, \) and \( \sigma \).

To cite a specific example, consider the material defined by (3.10)-(3.13): suppose \( 2\gamma_{ij} = u_{ij} + u_{ji} \), and \( \rho \theta \dot{\eta} \approx \rho T \dot{\eta} \). Then, introducing (3.10)-(3.13) into (4.9) and (4.10) after making the transformations (4.8), and transforming the resulting equations into reduced time with the aid of (4.11), (4.12), and (4.13), we obtain the following local equations of motion and heat conduction of a thermorheologically simple element in reduced time:

\[
m^{(1)}_{NM} \dot{u}_k + m^{(2)}_{NM} \dot{u}_k + a^{(N)}_{NM} \int_0^\epsilon \left[ 2G(\xi - \xi') \frac{\partial N}{\partial \xi'} d\xi' + b^{(M)}_{NM} \int_0^\epsilon \left[ 3K(\xi - \xi') - 2G(\xi - \xi') \right] \frac{\partial N}{\partial \xi'} d\xi' \right]
\]

\[
- c_{NM} \int_0^\epsilon 3K(\xi - \xi') \frac{\partial N}{\partial \xi'} d\xi' = \dot{p}_{NM}
\]  

(4.15)

\[
l^{(1)}_{MN} T^N + l^{(2)}_{MN} \dot{T}^N + l^{(3)}_{NM} \int_0^\eta \left[ m(\xi - \xi') \frac{\partial T}{\partial \xi'} d\xi' + h^{(1)}_{MN} \int_0^\eta \left[ 3K(\xi - \xi') \right] \frac{\partial T}{\partial \xi'} + d^{(2)}_{MN} \int_0^\eta \left[ 2G(\xi - \xi', \xi - \xi) \right] \frac{\partial T^N}{\partial \xi'} d\xi' + d^{(3)}_{MN} \int_0^\eta \left[ 2G(\xi - \xi', \xi - \xi') \right] \frac{\partial T^N}{\partial \xi'} d\xi' + d^{(4)}_{MN} \int_0^\eta \left[ 3K(\xi - \xi', \xi - \xi') \right] \frac{\partial T^N}{\partial \xi'} + d^{(5)}_{MN} \int_0^\eta \left[ 3K(\xi - \xi', \xi - \xi') \right] \frac{\partial T^N}{\partial \xi'} d\xi' \right]
\]

\[
- d^{(1)}_{MN} \int_0^\eta \left[ 3K(\xi - \xi', \xi - \xi') \right] \frac{\partial T^N}{\partial \xi'} d\xi' = \dot{q}_M
\]  

(4.16)
The multidimensional arrays appearing in these equations are defined in Table 1. The quantities $\Gamma_{jk}$ appearing in Table 1 are the Christoffel symbols of the second kind for the local element coordinates $\xi^i$.

We observe that the behavior of a collection of such finite elements is described by a large system of nonlinear integrodifferential equations. We discuss numerical solutions of special cases in the following sections.

**Table 1. Multidimensional arrays**

$$m_{NM}^{(1)} = \int_{v_e} \rho b^2 \psi_M \psi_N dv$$

$$m_{MN}^{(2)} = \int_{v_e} \rho b \phi_M \psi_N dv$$

$$a_{NM}^i = \int_{v_e} \frac{1}{2} [\psi_N, \phi(M, \beta \varphi_i + \psi_M, \delta \gamma_{jr} - 2 \psi_M \delta g_{kr} \Gamma_{jk}^i) - \psi_N \Gamma_{nj}^i f_{Mjn}] dv$$

$$b_{MN}^i = \int_{v_e} \frac{1}{2} [f_{Mnrr} \psi_N, i - \psi_N (\psi_M, \delta g_{nr} \Gamma_{jj}^i - \psi_M \delta g_{kr} \Gamma_{mn}^i \Gamma_{jj}^i)] dv$$

$$c_{MN} = \int_{v_e} b(\psi_M, \delta g_{nr} - \psi_M \delta g_{kr} \Gamma_{mn}^i) \psi_N dv$$

$$h_{MN}^{(1)} = T_0 m(0) \int_{v_e} \phi_N \phi_M dv$$

$$h_{MN}^{(2)} = 3\alpha T_0 K(0) \int_{v_e} \phi_M (\psi_N, i - \psi_N \Gamma_{jj}^i) dv$$

$$h_{MN}^{(3)} = T_0 \int_{v_e} \phi_N \phi_M dv$$

$$h_{MN}^{(4)} = 2 T_0 \int_{v_e} \phi_M, \phi(\psi_N, i - \psi_N \Gamma_{jj}^i) dv$$

$$k_{MN} = \int_{v_e} \phi_M, \phi_N, \phi_i dv$$

$$d_{MN}^{(1)} = \frac{1}{4} \int_{v_e} \phi_M (\delta \gamma_{ij} \psi_N, \psi_L, i - \psi_N \Gamma_{in}^i - 2 \psi_N \psi_L, a \Gamma_{jj}^i + 2 \psi_N \psi_L, \Gamma_{nm}^i \Gamma_{jj}^i + \psi_L, \psi_N, j) dv$$

$$d_{MN}^{(2)} = \frac{1}{4} \int_{v_e} \phi_M (\delta \gamma_{ij} \psi_N, \psi_L, i - \psi_N \Gamma_{ij}^i) dv$$

$$d_{MN}^{(3)} = \frac{1}{4} \int_{v_e} \phi_M (\delta \gamma_{ij} \psi_N, \psi_L, i - \psi_N \Gamma_{ij}^i) dv$$

$$J_{Mjn} = \psi_M, \beta \varphi_i - \psi_M \delta g_{kr} \Gamma_{nj}^k, M, N, L = 1, 2, \ldots, N_e$$

**5. NUMERICAL ANALYSIS**

To obtain quantitative solutions of (4.15) and (4.16), we shall assume that the kernels appearing in the integrals in these equations pertain to materials with fading memory; i.e. the current response is not appreciably effected by strains and temperatures at times distant in the past. This property is often assured by considering each kernel to
be representable in the form of a decaying Prony series; for example, a typical material kernel \( K(\xi) \) is defined by a series of the form

\[
K(\xi) = \sum_{i=1}^{n} K_i \exp(-\xi/v_i)
\]

(5.1)

where \( K_i \) and \( v_i \) are experimentally-determined constants, \( \xi \geq 0 \), and \( v_i > 0 \). Kernels of the type \( G(\xi, \xi') \) are assumed to have the property

\[
G(\xi - \xi', \xi - \xi') = G(2\xi - \xi' - \xi''),
\]

(5.2)

so that, in accordance with (5.1), we can write

\[
G(\xi - \xi', \xi - \xi'') = \sum_{i=1}^{n} G_i \exp\left[\frac{(-2\xi + \xi' + \xi'')}{\lambda_i}\right]
\]

(5.4)

etc., \( \lambda_i \) being a constant > 0.

Now consider a typical integral appearing in one of the constitutive equations cited earlier; e.g. define

\[
IG_{NJ} = \int_{0}^{\varepsilon} 2G(\xi - \xi') \frac{\partial \alpha_{ij}^N}{\partial \xi'} d\xi'
\]

(5.5)

We obtain an alternate representation of (5.5) which is easily adapted to numerical approximation by dividing the reduced time interval \([0, \varepsilon]\) into \( r \) subintervals \([\xi_m, \xi_{m+1}]\), \( m = 1, 2, \ldots, r+1 \). Then, using the representation (5.1) for the kernel in (5.5), we have

\[
IG_{NJ} = \sum_{i=1}^{n} G_i e^{-\xi/\lambda_i} \sum_{m=1}^{r} \int_{\xi_m}^{\xi_{m+1}} e^{\xi/\lambda_i} \frac{\partial \alpha_{ij}^N}{\partial \xi'} d\xi'
\]

(5.6)

If \( h_m = \xi_{m+1} - \xi_m \) is the time step, we may use Simpson’s rule to approximate the integrals in (5.6):

\[
\int_{\xi_m}^{\xi_{m+1}} \frac{\partial \alpha_{ij}^N}{\partial \xi'} \exp(\xi'/\lambda_i) d\xi' \approx \frac{h_m}{6} \left\{ \frac{\partial \alpha_{ij}^N(\xi_m)}{\partial \xi} \exp(\xi_m/\lambda_i) + 4 \frac{\partial \alpha_{ij}^N(\xi_m + h_m/2)}{\partial \xi} \exp((\xi_m + h_m/2)/\lambda_i) \right. \\
+ \left. \frac{\partial \alpha_{ij}^N(\xi_m + h_m)}{\partial \xi} \times \exp((\xi_m + h_m)/\lambda_i) \right\}
\]

(5.7)

To proceed further, the rates \( \partial \alpha_{ij}^N/\partial \xi \) must be replaced by difference approximations. In the present study, we assume a quadratic variation of \( \alpha_{ij}^N \) between nodes \( m - 1, m, m + 1 \).
which leads to approximations of $0(h^2)$ of the rates. Introducing these into (5.7) and rearranging terms, we arrive at the following form of (5.5):

$$I G_{Nj} = \sum_i G_i J_i^N(u, \lambda)$$

wherein

$$J_i^N(u, \lambda) = \bar{H} G(\hat{\alpha}_j^N, \lambda_i) \exp[-(\xi, + h)/\lambda_i] + \bar{C} G(\hat{\alpha}_j^N, \lambda_i)$$

and

$$\bar{H} G(\hat{\alpha}_j^N, \lambda_i) = \sum_{m=1}^{r-1} \left\{ \frac{1}{2}(\hat{\alpha}_{j(m+1)}^N - \hat{\alpha}_{j(m-1)}^N) \exp(\xi, m/\lambda_i) + \frac{1}{3} (\hat{\alpha}_{j(m+1)}^N - \hat{\alpha}_{j(m)}^N) \exp\left(\left(\xi, m + h/2\right)/\lambda_i\right) \right\}$$

(5.9a)

$$\bar{C} G(\hat{\alpha}_j^N, \lambda_i) = \frac{1}{3}(\hat{\alpha}_{j(r+1)}^N - \hat{\alpha}_{j(r-1)}^N) \exp(-\xi, m/\lambda_i) + \frac{1}{3} (\hat{\alpha}_{j(r+1)}^N - \hat{\alpha}_{j(r)}^N) \exp\left(-\left(\xi, + h/2\right)/\lambda_i\right)$$

(5.9b)

(5.9c)

In (5.9a) we have separated the histories $\bar{H} G(\hat{\alpha}_j^N, \lambda_i)$ from the current values $\bar{C} G(\hat{\alpha}_j^N, \lambda_i)$; that is, the value of the integral at the current time $\xi, + h$ is obtained by adding to the accumulated sum at $\xi, + h$, the contribution between $\xi, + h$. By the notation $\hat{\alpha}_{j(m),} \hat{\alpha}_{j(m-1),}$ etc., we mean $\hat{\alpha}_j^N(\xi, m)$, $\hat{\alpha}_j^N(\xi, m - h)$, etc.

Using the notation of (5.9), the equations of motion and heat conduction for a thermorheologically simple finite element assume the forms

$$\rho_{Nj} = m_{MN}^1 \frac{\hat{\alpha}_j^M + m_{MN}^2 \hat{\alpha}_j^M + (a_{MN}^1 - b_{MN}^1) \sum_k K_{k,j}^M(u, \lambda) + b_{MN}^1 \sum_k K_{k,j}^M(u, \lambda)}{\sum_k K_{k,j}^M(T, \mu)}$$

(5.10a)

$$\begin{align*}
\dot{\alpha}_N &= h_{MN}^1 \dot{T}^M + h_{MN}^2 \alpha_j^M - 3\alpha_j^M \sum_k K_{k,j}^M(T, \mu) - h_{MN}^1 \sum_k \frac{K_{k,j}^M(u, \mu)}{\mu_j^M} \\
&- 2d_{NML}^1 \sum_k \frac{K_{k,j}^M(u, \mu)}{\mu_j^M} J_{j}^L(u, \mu) + 2d_{NML}^2 J_{j}^L(u, \lambda) J_{j}^L(u, \lambda) \\
&+ 2d_{NML}^3 \sum_k \frac{K_{k,j}^M(u, \mu)}{\mu_j^M} J_{j}^L(T, \mu) + 6\alpha_j^M \sum_k \frac{K_{k,j}^M(T, \mu)}{\mu_j^M} J_{j}^L(T, \mu) \\
&+ k_{LM} \sum_k E_{k,j}^M(T, \mu)
\end{align*}$$

(5.10b)

Thus, upon connecting elements together, we obtain a system of highly nonlinear differential equations in the nodal values $\hat{\alpha}_j^N$, $\dot{T}^N$ and their histories.
6. INFINITE CYLINDER

For illustration purposes, we now consider applications of (5.10) to the problem of an infinite, thick-walled, circular cylindrical tube subjected to prescribed mechanical and thermal conditions at its inner and outer boundaries. For the local finite-element approximations of the displacement components and the temperature we use the simplex approximations $\psi_N(r)=a_N+b_Nr$, $N=1,2$, $a_N$ and $b_N$ being constants dependent only on the length of a radial element ($b_1=-b_2=-1/(r_2-r_1)$; $a_1=r_2/(r_2-r_1)$; $a_2=-r_1/(r_2-r_1)$ for an element between nodes at radii $r_1$ and $r_2$). By introducing this form of the interpolation functions into the arrays in Table I we obtain all properties of the equations of the discrete model that are independent of specific material properties.

To proceed further, we must identify the form of the shift function $b(\theta)$. Williams et al. [19] proposed the following empirical expression for $b(\theta)$ for a wide variety of polymers, polymer solutions, organic and inorganic glasses:

$$\log_{10}b(\theta) = \alpha(\theta - T_0)/{(\beta + \theta - T_0)}$$  \hspace{1cm} (6.1)

Here $\alpha$ and $\beta$ are constants. Experimental data [15] show that for $|\theta - T_0| \leq 50^\circ C$, $\alpha \approx 9$ and $\beta \approx 100$, so that if $T_0=0^\circ C$,

$$b(\theta) = 10^{9T_0/100+\theta}$$  \hspace{1cm} (6.2)

where, for the finite element, $T=\psi_N(r)T^N=(a_N+b_Nr)T^N$. Transforming to reduced time, we find that

$$\hat{b}(\theta) = 10^\xi, \hspace{0.5cm} \hat{b}=(\hat{b}ln(10))\xi$$  \hspace{1cm} (6.3a, b)

wherein

$$\xi = 9\Psi_N(r)\hat{T}^N/[100 + \Psi_L(r)\hat{T}^L]$$  \hspace{1cm} (6.3c)

and $\hat{T}^N$ is represented by the quadratic difference scheme described earlier.

In the numerical integration scheme, we treat the current value of $b(\theta)$ at time $\xi$ as a constant and allow $\hat{b}$ and $\hat{\xi}$ to lag the current values by the amount for the current step size $h_r$. The values of $\hat{b}$ and $\hat{\xi}$ are recomputed at the end of each time step and are carried forward into the next time interval to determine nodal displacements and temperatures. This procedure makes it possible to express the mass matrices $m^{(1)}_{MN}$ and $m^{(2)}_{MN}$ in the particularly simple forms

$$m^{(1)}_{MN} = 2\pi M_{NM}, \hspace{1cm} m^{(2)}_{MN} = l_4(10)\hat{\xi}m^{(1)}_{MN}$$  \hspace{1cm} (6.4a, b)

where

$$M_{NM} = \rho_0[b_n b_m (r_2^4 - r_1^4)/4 + (a_n b_m + a_m b_n) (r_2^3 - r_1^3)/3 + a_n a_m (r_2^2 - r_1^2)/2]$$  \hspace{1cm} (6.4c)

and $\Psi_N(r) \approx a_N + b_N(r_1 + r_2)/2$. Similar procedures are used to evaluate the remaining integrals in the equations of motion and heat conduction.
Neglecting body forces and internal heat sources, the generalized forces and heat fluxes of (4.4) become

\[ \hat{p}_N = r_N(a_N + b_N r_N)S_N(\xi) \quad \hat{q}_N = r_N(a_N + b_N r_N)Q_N(\xi) \]  
(6.5a, b)

(no sum on \( N \)), where \( S_N(\xi) \) and \( Q_N(\xi) \) are prescribed functions of reduced time. Alternatively, we can also prescribe arbitrary time-dependent nodal displacements and temperatures. For convective heat transfer problems, we set

\[ Q_N = f_N(\hat{T}_N - T_{0*}) \]  
(6.6)

where \( f_N \) is the film coefficient at node \( N \) and \( T_{0*} \) is the temperature of a medium in contact with the cylinder at a boundary.

To complete the statement of the problem, we consider a hypothetical material for which \( \rho_0 = 10^{-3} \text{ lb-sec}^2/\text{in.}^4, \alpha = 10^{-5} \text{ in./in.}^2/\text{C} \), the dilational relaxation kernel \( K(\xi) = 10 \times 10^5 \exp(-\xi/10^{-5}) + 5 \times 10^5 \exp(-\xi/\infty) \text{ lb/in.}^2 \), the shear relaxation kernel \( G(\xi) = 1 \times 10^5 \exp(-\xi/10^{-5}) + 0.1 \times 10^5 \exp(-\xi/\infty) \text{ lb/in.}^2 \), and the thermal conductivity kernel of the heredity-type heat flux law (3.13) of \( \kappa(\xi) = 0.10 \exp(-\xi/10^{-5}) + 0.01 \exp(-\xi/\infty) \) \text{ in./lb/in.}^2/\text{C sec.}

**Numerical procedure**

We now apply the above equations to the analysis of a cylinder with an inner radius of 10.0 in. and an outer radius of 11.0 in. for which the 1.0 in. wall is divided into twenty 0.05 in. elements. The system of nonlinear differential equations governing this model are integrated using the explicit quadratic scheme mentioned earlier; however, owing to the nonlinear heredity terms, we encounter at each time step a system of nonlinear algebraic equations in the current values of the nodal displacements and temperatures. In the examples to be cited below, these equations are solved by Newton-Raphson iteration. Starting values for the Newton-Raphson process are obtained from the initial conditions (e.g., if \( \hat{u}_N(\xi_0) = \hat{u}_N(\xi_0) = \hat{T}_N(\xi_0) = \hat{T}_N(\xi_0) = 0 \), then central difference approximations of \( u, \hat{u}_N, \hat{T}_N \) lead directly to expressions for \( \hat{u}_N, \hat{u}_N, \hat{T}_N \) and \( \hat{T}_N \) at time \( \xi_0 + h \) in terms of \( \hat{u}_N \) and \( \hat{T}_N \) at \( \xi_0 + h \). Initial values so computed are introduced directly into (5.10) to obtain displacements and temperatures at \( \xi_0 + h \) and the value of \( \hat{T} \) [see (6.3c)] to be used in the second time step. For the second time step, we again use the quadratic approximation \( \hat{u}_{m+1} \approx (3u_{m+1} - 4u_m + u_{m-1})/2h \) for various rates. This results in adequate data to initiate the Newton-Raphson iteration process. Once convergence of a specified degree is obtained, another reduced time increment is added and the process is continued.

Element stresses are computed at the end of each time step by introducing nodal displacements and temperatures into the stress constitutive equation (3.10). All quantities (nodal displacements, temperatures, and element stresses) are computed as functions of reduced time and are then transformed to real time by using the relation

\[ \Delta t = \Delta \xi/\hat{\eta}(\theta(\xi, \xi)) \]  
(6.7)

Here \( \Delta \xi = h \) is the reduced time increment and \( \hat{\eta}(\theta) \) is given by (6.2). The real time at step \( r + 1 \) is then \( t_r + h/\hat{\eta}(\theta(\xi, \xi)) \). Note, however, that different values of real time should be considered for \( \hat{u}_N \) and \( \hat{T}_N \) than for element stresses: more specifically, for the nodal values \( \hat{u}_N \). \( \hat{T}_N \) the shift factor is associated with a temperature change at a single node whereas
the stress tensor is associated with a shift factor that involves the entire element-temperature field and, consequently, depends upon the temperature changes at every node of the element. In the former case we simply set the real time increment \( \Delta t_r \) associated with nodal quantities \( \tilde{u}_r \), \( T_r \) equal to \( h/\tilde{b}_r \), where \( \tilde{b}_r \) is the (global) value of \( b(0) \) at node \( r \). In the latter case (transformation of stresses), we set the real time increment in element \( e \), \( \Delta t_{(e)} \), equal to \( h/\bar{b}_{(e)} \), wherein \( \bar{b}_{(e)} \) is an averaged shift factor for element \( e \).

**Numerical results**

We shall consider as an example the dynamic response of the cylinder due to purely mechanical pressure, step loading \( p = 10h(t) \) psi (\( h(t) \) being the unit step function) applied at the inner boundary, while the thermal boundary conditions correspond to convective heat transfer. Film coefficients of 0.1 and 1.0 in.lbf/in.\(^2\) °C sec. respectively are assumed for the inner and outer boundaries. The heat supply from internal sources is assumed to be zero and the initial temperature of the cylinder and the surrounding media are taken to be 300°C. Heat is thus generated through thermomechanical coupling. Since energy is dissipated in the system, the internal temperature increases during the transient part of the motion while heat is expelled at different rates at each boundary. For the material parameters and the loading considered, a steady-state condition is reached at approximately \( 10^{-3} \) sec. We can also obtain, for purposes of comparison, the solution of the isothermal case by ignoring all thermal effects, setting \( \tilde{b} = 1 \), \( \bar{b} = 0 \), and by treating the integration variable as real time.

Nondimensional plots of the computed radial and circumferential stress waves at various real times are shown for the isothermal and non-isothermal cases in Figs. 1 and 2: the displacement profiles are shown in Fig. 3. While stress and displacement profiles are obtained at different times for the isothermal and non-isothermal cases (owing to the necessity of transforming all computed quantities to real time in the non-isothermal case), the results do indicate a lag in the non-isothermal stress waves and a noticeable decrease in amplitude. Circumferential stresses are initially compressive but become tensile almost everywhere in the cylinder at \( t = 10^{-3} \) sec. The circumferential stress wave requires approximately \( 2.5 \times 10^{-5} \) sec to traverse the 1.0 in. thickness of the cylinder in the isothermal

![Fig. 1. Stress propagation in a thermorheologically simple cylinder—nondimensional radial stress versus radial distance for various real times.](image)
Fig. 2. Stress propagation in a thermorheologically simple cylinder—nondimensional circumferential stress versus the radial distance for various real times.

Fig. 3. Radial displacement profiles for various real times.
case and approximately $4.0 \times 10^{-5}$ sec in the non-isothermal case. Displacement profiles are qualitatively the same for the isothermal and non-isothermal cases. The non-isothermal displacements are higher than those predicted by the isothermal analysis at the outer boundary, but they are lower on the inner boundary. This is due chiefly to the relative magnitudes of the film coefficients at these boundaries. We remark that the results indicated were computed using a reduced time increment of $10^{-6}$ sec; rather rapid convergence of the Newton-Raphson scheme was observed at each time step.

Figure 4 indicates the computed temperature change over the thickness of the cylinder at various values of real time. Note that the mechanically loaded inner boundary is the first to experience an increase in temperature. The maximum temperature is reached at approximately $t = 4 \times 10^{-5}$ sec, and the temperature appears to decrease thereafter.

![Temperature Change](image)

**FIG. 4.** Nondimensionalized temperature distributions at various real times.

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**REFERENCES**


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