ON THE NUMERICAL SOLUTION OF A CLASS OF PROBLEMS IN A LINEAR FIRST STRAIN-GRADIENT THEORY OF ELASTICITY

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SUMMARY

This paper is concerned with the development of general discrete models for the analysis of boundary-value problems in the first strain-gradient theory of elasticity. Extensions of the finite element method are constructed for this purpose, and general equations of motion are derived for finite elements of a class of micro-polar materials which are characterized by strain energy functions involving strains and second gradients of strains or displacements. The notion of generalized nodal doublets is introduced. The problem of a composite consisting of a strain-gradient sensitive microlayer embedded between semi-infinite bodies is examined as an example problem. Some of the results are compared with available exact solutions.

INTRODUCTION

Although the idea of couple stresses was introduced by Voigt in his 1887 study of elastic crystals,1 the first attempts at developing general notions of micro-structure in elastic materials is usually attributed to E. and F. Cosserat.2,3 Until recent years, the work of these early investigators was virtually unnoticed; but a paper of Toupin4 and a series of papers by Mindlin and his collaborators5-7 have apparently revived interest in the subject. Of particular interest are the so-called ‘strain-gradient’ theories of elasticity in which the potential energy of deformation is assumed to be a function of not only the strain, as in the classical theories, but also first- and higher-order gradients of the strain. A general, non-linear strain-gradient theory of elasticity, including general constitutive equations and boundary conditions, was first presented by Toupin.4 Subsequently, Mindlin5 investigated linear versions of the theory in which three forms of the strain energy functions were considered; and more recently Mindlin and Eshel8 presented detailed comparisons of these forms. Mindlin6 regards the strain-gradient theory as a low frequency, long wavelength micro-deformation approximation of the more general micro-structure theories. Applications of linear strain-gradient theories to various boundary-value problems have been considered by a number of authors, including, for example, Sternberg and Muki,9 Day and Weitsman,10 Cook and Weitsman11 and Hazen and Weitsman.12 These authors have shown, in particular, that the consideration of the strain-gradient theories has a significant effect on the stress concentrations around discontinuities.

The basic equations of the linear first strain-gradient theories of elasticity are considerably more involved than those of the classical theory, and solutions to specific boundary-value problems may be correspondingly more difficult if attacked by classical methods. The present paper is concerned with the development of general discrete models for the analysis of boundary-value problems in the linear first strain-gradient theory of elasticity. Extensions of the finite element concept are constructed for this purpose, and general equations of motion are derived for

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finite elements of a class of micro-polar materials (i.e. materials which are strain-gradient sensitive) described by strain energy functions which involve quadratic forms in the strains and the second gradients of displacements. It is shown that this approach leads to the concept of generalized nodal double forces called doublets, corresponding to double stresses and the usual generalized nodal forces which, in the discrete model, correspond to stresses. An alternate formulation leads to the notion of generalized nodal couples. The development of consistent finite element models of various fields is necessarily different than in the classical case, for nodal values of the displacement gradients must now be specified in addition to the displacements themselves.

Following this introduction, basic equations of the first strain-gradient theory of elasticity are reviewed. Finite element models of the displacement field appropriate to the problem are then discussed and several specific forms of these approximations are presented. The approximate fields are then introduced into appropriate energy balances for a finite element, and general equations of motion are obtained. Numerical results of applications of the finite element model to problems with known solutions are then examined.

**FIRST STRAIN–GRADIENT THEORY OF ELASTICITY**

Following the work of Mindlin and Eshel, we review briefly in this section several basic ideas of the first strain-gradient theory of elasticity. For an arbitrary continuous media, the basic kinematic variables needed are defined by the following linear relations:

\[
\begin{align*}
\gamma_{ij} &= \frac{1}{2}(u_{ij} + u_{ji}) = \text{strain} \\
\omega_{ij} &= \frac{1}{2}(u_{ij} - u_{ji}) = \text{rotation} \\
\omega_i &= \frac{1}{2}e_{ijk} u_{k,i} = \text{rotation vector} \\
\kappa_{ijk} &= u_{k,ij} = \text{second gradient of displacement} \\
\kappa_{ij} &= \frac{1}{2}e_{ijk} u_{k,ij} = \text{gradient of strain} \\
\kappa_{ijk} &= \frac{1}{2}e_{ijk} u_{k,ij} = \text{symmetric part of } \kappa_{ijk}
\end{align*}
\]

In these equations, \(u_i = u_i(x_j, t)\) are the Cartesian components of the displacement field of a continuum, \(x_j\) being a system of rectangular Cartesian co-ordinates, and the commas denote partial differentiation with respect to \(x_j\) (i.e. \(u_{ij} = \partial u_i / \partial x_j; \ u_{k,ij} = \partial^2 u_k / \partial x_i \partial x_j\)).

We now consider an elastic body constructed of a material which is characterized by a strain energy function \(W\) which represents the strain energy per unit of undeformed volume. At this point, we depart from the classical theory and assume that \(W\) is a function of not only the strains but also gradients of the strain. Since we intend to confine our attention to linear theories, we take \(W\) to be a quadratic form in \(\gamma_{ij}\) and \(\kappa_{ijk}\) or \(\hat{\kappa}_{ij}\). Mindlin and Mindlin and Eshel considered three such forms for the strain energy density for isotropic bodies:

\[
\begin{align*}
W &= \hat{W}(\gamma_{ij}, \hat{\kappa}_{ijk}) = \hat{W}(\gamma_{ij}, \tilde{\kappa}_{ij}) = \hat{W}(\gamma_{ij}, \tilde{\kappa}_{ijk}, \tilde{\kappa}_{ij}) \\
\hat{W} &= \frac{1}{2}\lambda\gamma_{ij}\gamma_{ji} + \mu\gamma_{ij}\gamma_{ji} + \beta_1 \kappa_{kij} \kappa_{kij} + \beta_2 \kappa_{kij} \kappa_{kij} + \beta_3 \kappa_{kij} \kappa_{kij} + \beta_4 \kappa_{kij} \kappa_{kij} + \beta_5 \kappa_{kij} \kappa_{kij} + \beta_6 \kappa_{kij} \kappa_{kij} + \beta_7 \kappa_{kij} \kappa_{kij} \\
W &= \frac{1}{2}\lambda\gamma_{ij}\gamma_{ji} + \mu\gamma_{ij}\gamma_{ji} + 2\beta_d \kappa_{ij} \kappa_{ij} + 2\beta_d \kappa_{ij} \kappa_{ij} + \beta_0 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_1 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_2 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_3 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_4 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_5 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_6 \kappa_{kij} \kappa_{kij} \kappa_{kij} + \beta_7 \kappa_{kij} \kappa_{kij} \kappa_{kij}
\end{align*}
\]

Here \(\lambda\) and \(\mu\) are the usual Lamé constants and \(\beta_1, \beta_2, \ldots, \beta_7\) are micro-moduli associated with strain-gradient sensitive materials. These three forms are equivalent to one another and simple relations between the micro-moduli of each form can be obtained.
Components $\sigma_{ij}$ of the stress tensor are derived from the strain energy functions as follows:

$$\sigma_{ij} = \frac{\partial W}{\partial \gamma_{ij}} = \frac{\partial W}{\partial \gamma_{ij}} = \frac{\partial W}{\partial \gamma_{ij}} = \sigma_{ij}$$

(6)

Likewise

$$\tilde{\mu}_{ijk} = \frac{\partial W}{\partial \chi_{ijk}} = \tilde{\mu}_{ijk}$$

(7)

$$\bar{\mu}_{ijk} = \frac{\partial W}{\partial \chi_{ijk}} = \bar{\mu}_{ijk}$$

(8)

$$\bar{\tilde{\mu}}_{ijk} = \frac{\partial W}{\partial \chi_{ijk}} = \bar{\tilde{\mu}}_{ijk}$$

(9)

and

$$\bar{\tilde{m}}_{ij} = \frac{\partial W}{\partial \tilde{\kappa}_{ij}}, \bar{\tilde{m}}_{ij} = 0$$

(10)

where $\bar{\mu}_{ijk}$, $\bar{\mu}_{ijk}$ and $\bar{\tilde{\mu}}_{ijk}$ are components of double stress and $\bar{\tilde{m}}_{ij}$ is the deviator of the couple stress tensor. In Reference 8 it is shown that

$$\bar{\mu}_{ijk} = \bar{\mu}_{ijk} + \bar{\mu}_{kji} - \bar{\mu}_{jki}$$

(11)

$$\bar{\tilde{m}}_{ij} = \frac{1}{2} \varepsilon_{ijpq} \bar{\mu}_{pq}$$

(12)

$$\bar{\tilde{\mu}}_{ijk} = \frac{1}{4} (\bar{\mu}_{ijk} + \bar{\mu}_{kji} + \bar{\mu}_{jki})$$

(13)

Thus, the doubles $\bar{\mu}_{ijk}$ and $\bar{\tilde{\mu}}_{ijk}$ and couple stresses $\bar{\tilde{m}}_{ij}$ can be calculated once the doubles $\bar{\mu}_{ijk}$ are known.

If the strain-gradient equations are viewed as long wavelength limit-of-difference equations for a simple crystal lattice in the micro-structure of the material, velocity gradient terms do not appear in the kinetic energy density $T$, and $T$ is given by the usual formula

$$T = \frac{1}{2} \rho \dot{u} \dot{u}$$

(14)

where $\rho$ is the mass density and $\dot{u}_i$ are the components of velocity. Then the equations of motion for the continuum can be shown to be:

$$\sigma_{jk,i} - \dot{\bar{\mu}}_{ijk} + F_k = \rho \ddot{u}_k$$

(15)

where $F_k$ are the components of body force per unit volume. Other forms of equation (15) in terms of $\bar{\mu}_{ijk}$, $\bar{\tilde{m}}_{ij}$ and $\bar{\tilde{\mu}}_{ijk}$ can be obtained by using the relation given in equations (11)-(13). Various forms of the associated traction boundary conditions can be found in Reference 8.

Now consider a finite volume $V$ and surface area $A$ of such a material which is subjected to body forces $F_k$, body couples $C_k$ and body doubles $\Phi_{ij}$ throughout, as well as surface tractions $S_j$, surface couples $m_j$ and surface double stresses $\tilde{\mu}_{ij}$. Then, according to Reference 8, a principle of conservation of energy can be postulated of the form

$$\frac{d}{dt} \int_V (T + W) \, dr = \int_V (F_k \dot{u}_k + C_k \dot{\omega}_k + \Phi_{ijk} \dot{\gamma}_{jk}) \, dr + \int_A (S_k \dot{u}_k + m_k \dot{\omega}_k + m_i \tilde{\mu}_{ij} \dot{\gamma}_j) \, dA$$

(16)

in which $\Phi_{ijk}$ is the symmetric part of $\Phi_{ijk}$ and $n_i$ are the components of a unit vector normal to $A$. The right side of equation (16) represents $\Omega$, the total mechanical power developed by the external forces, couples and doubles. Other forms of equation (16) can, of course, be derived using equations (11), (12) and (13).

**FINITE ELEMENT APPROXIMATIONS**

The finite element concept involves the representation of a continuous body by a collection of a finite number of component parts called finite elements. Ordinarily these finite elements are taken
to be of very simple geometric shapes, and various fields are approximated locally over each element in terms of their values at prescribed nodal points in the element and on its boundaries. On connecting the elements appropriately together, a discrete model of the original field is obtained in which the field is described by a finite number of its values at discrete nodal points in the model and by piecewise approximations over each finite element. The appealing aspect of this procedure is that the elements can be considered to be completely disconnected for the purpose of defining the local field approximations. Thus, the 'behaviour' of a typical element can be described locally, in an approximate manner, without consideration of its ultimate location in the model, the behaviour of other elements or its mode of connection with adjacent elements. This makes the procedure particularly well suited for bodies with irregular geometry and boundary conditions. Several generalizations of these ideas have recently been presented. 14, 15

In the application of the finite element concept to classical elasticity problems, the most primitive type of finite element approximation is the simplex model in which functions are represented by piecewise linear approximations over each finite element. Tetrahedral, triangular or linear finite elements are employed in three-, two- and one-dimensional simplex approximations, respectively, and the local fields over each element are uniquely determined in terms of their values at the vertices of these elements. Moreover, since the process of assembling elements amounts to matching nodal values of adjacent finite elements, simplex elements lead to an approximation of the given function which is continuous throughout the model, thereby satisfying one of the basic criteria for the convergence of the model to the actual field.

For the present type of problem, however, the often used simplex approximations cannot be used—at least without a great deal of difficulty—because they lead to displacement gradients which are uniform over each finite element. Consequently, the second gradients of displacements vanish over each element and the basic ingredients of the strain-gradient theories are lost. Therefore, for the problem at hand it is necessary to develop higher-order finite element approximations which are capable of depicting second gradients of displacement over a finite element.

Consider, then, a discrete model of a continuous elastic solid that consists of a finite number \( E \) of finite elements connected together at various nodal points in such a way that no gaps or discontinuities occur in the model that are not present in the actual body it represents. Consider in particular a typical finite element \( e \) which, for the moment, is isolated from the rest, and let \( u_i(x, t) \) denote the local displacement over the element. The local field is approximated over the element by functions of the form

\[
u_i = \psi_N(x) u_{Ni} + \varphi_N(x) u_{Nij}\]

where \( i, j = 1, 2, 3 \), and the repeated nodal indices \( N \) are summed from 1 to \( N_r \), \( N_r \) being the total number of nodal points of element \( e \). In this equation, \( u_{Ni} \) are the components of displacement of node \( N \) of the element and \( u_{Nij} \) are the displacement gradients at node \( N \) of the element. That is, if \( x_i^j (i = 1, 2, 3; N = 1, 2, ..., N_r) \) denote the co-ordinates of node \( N \),

\[
u_i(x_i^j) \equiv u_{Ni} \quad \text{and} \quad \frac{\partial u_i(x_i^j)}{\partial x^j} \equiv u_{Nij}
\]

The interpolating functions \( \psi_N(x) \) and \( \varphi_N(x) \) are defined so as to have the properties

\[
\psi_N(x_M) = \delta_{NM}, \quad \frac{\partial \psi_N(x_M)}{\partial x^i} = 0 \quad \text{and} \quad \varphi_N(x_M) = 0, \quad \frac{\partial \varphi_N(x_M)}{\partial x^j} = \delta_{ij} \delta_{MN}
\]

where \( \delta_{ij} \) and \( \delta_{MN} \) are Kronecker deltas. Thus, the functions \( \psi_N(x) \) and \( \varphi_N(x) \) may be regarded as generalized Hermite interpolation functions, since the approximate field in equation (17)
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coincides with the displacement field and its first partial derivatives at each node point of the element. Examples of these functions are to be cited subsequently.

To obtain the finite element approximation of the entire displacement field over the assembled system of elements, element identification labels \( e \) are affixed to each local field (e.g. \( u_{N(e)} \)) and global values of displacement and displacement gradient at a node \( \Delta \) of the assembly of elements are denoted \( U_{\Delta i} \) and \( U_{\Delta i,j} \) respectively. Then, a mathematically formal statement of the relation between local and global values at incident nodal points is given by References 16–18:

\[
\begin{align*}
\begin{split}
    u_{N(e)} &= \Omega_{N\Delta}^{(e)} U_{\Delta i}, \\
    u_{N(e,j)} &= \Omega_{N\Delta}^{(e)} U_{\Delta i,j}
\end{split}
\end{align*}
\]  

(20)

where \( u_{N(e)} \) and \( u_{N(e,j)} \) denote local values of \( u_i \) and \( u_{ij} \) at node \( N \) of element \( e \). \( U_{\Delta i} \) and \( U_{\Delta i,j} \) denote corresponding global values at node \( \Delta \) of the assembly of elements. and \( \Omega_{N\Delta}^{(e)} \) is defined as being unity if node \( N \) of element \( e \) is identical to node \( \Delta \) of the assembled system and zero if otherwise. The repeated global node index \( \Delta \) in equation (20) is to be summed from 1 to \( G \), \( G \) being the total number of nodes in the assembled system of elements. The final finite element representation of the displacement field \( U_i(x) \) over the entire collection of \( E \) elements is given by

\[
U_i(x) = \sum_{e=1}^{E} \psi_{N}^{(e)}(x) U_{\Delta i} + \sum_{e=1}^{E} \phi_{N}^{(e)}(x) U_{\Delta i,j}
\]

(21)

where \( \psi_{N}^{(e)}(x) \) and \( \phi_{N}^{(e)}(x) \) are the interpolating functions of equation (17) corresponding to finite element \( e \).

In general, the functions \( \psi_{N}^{(e)} \) and \( \phi_{N}^{(e)} \) are required to be such that the field \( U_i(x) \) and its first partial derivatives are continuous throughout the model. Such conditions in turn impose restrictions on the shape of the elements, and are introduced to fulfill one of the basic requirements for monotonic convergence of the finite element model to that of the actual field as the network of elements is refined. Below, several examples of admissible local field approximations are considered.

Three-dimensional element

The tetrahedron with five nodes, one at each vertex and a fifth located at an arbitrary point in the element, is an example of a three-dimensional finite element on which a function and its first partial derivatives can be approximated (Figure 1). Each component of displacement \( u_i \) can be

![Figure 1. Higher-order finite elements](image)
approximated by the general three-dimensional cubic

$$u_i = A_i + B_{ij} \chi^j + C_{ijk} \chi^j \chi^k + D_{ijkm} \chi^j \chi^k \chi^m$$  \hspace{1cm} (22)$$

where

$$C_{ijk} = C_{ikj} \hspace{1cm} (23a)$$

$$D_{ijkm} = D_{ijmk} = D_{imkj} = D_{ikjm} \hspace{1cm} (23b)$$

with \( i, j, k, m = 1, 2, 3 \). For each \( u_i \), there are twenty independent coefficients \( A_i, B_{ij}, C_{ijk} \) and \( D_{ijkm} \). There are determined in terms of the nodal values \( u_{ij} \) and \( u_{ij,j} \) by the sixty independent conditions

$$u_i(x_i) = u_{ij}, \quad \frac{\partial u_i(x_i)}{\partial x_j} = u_{ij,j}$$  \hspace{1cm} (24)$$

where \( N = 1, 2, 3, 4, 5 \) and \( x_i \) are the co-ordinates of node \( N \). On solving equation (24), the results are introduced into equation (22) and the local displacement fields are put in the form of equation (17).

**Two-dimensional element**

For plane problems, a triangular element with four nodes, one at each vertex and one at its centroid, can be used. Actually, the location of the fourth node is arbitrary, but the centroid of the element is often a convenient choice. In this case the values of the local field and of its first partial derivatives are prescribed at nodes 1, 2 and 3 but only the function itself is prescribed at node 4. Similar approximations have been used for plane stress problems by Felippa. 18 A cubic approximation is again used:

$$u_\alpha = A_\alpha + B_{\alpha \beta} \chi^\beta + C_{\alpha \beta \gamma} \chi^\beta \chi^\gamma + D_{\alpha \beta \gamma 1} \chi^\beta \chi^\gamma + D_{\alpha \beta \gamma 2} \chi^\beta \chi^\gamma + D_{\alpha \beta \gamma 3} \chi^\beta \chi^\gamma$$  \hspace{1cm} (25)$$

where \( C_{\alpha \beta \gamma} = C_{\gamma \beta \alpha} \) and \( \alpha, \beta, \gamma = 1, 2 \). For each component of displacement \( u_1 \) and \( u_2 \) there are ten unspecified coefficients. These are determined from the conditions

$$u_\alpha(x_i) = u_{\alpha i}, \quad N = 1, 2, 3, 4$$

$$\frac{\partial u_\alpha(x_i)}{\partial x_\beta} = u_{\beta \alpha i}, \quad N = 1, 2, 3$$  \hspace{1cm} (26)$$

Then equation (17) acquires a slightly modified form, since \( \phi_{1a}(x) \equiv 0 \).

**One-dimensional element**

A one-dimensional element is considered as a third example. In this case, one-dimensional Hermite interpolation polynomials are used and the displacement \( u_1 = u(x) \) is given by the cubic

$$u = A + Bx + Cx^2 + Dx^3$$  \hspace{1cm} (27)$$

with \( A, B, C \) and \( D \) determined from the conditions

$$u_1 = u(x_1), \quad u_1' = \frac{\partial u(x_1)}{\partial x}, \quad u_2 = u(x_2), \quad u_2' = \frac{\partial u(x_2)}{\partial x}$$  \hspace{1cm} (28)$$

where \( x_1 \) and \( x_2 \) are the co-ordinates of nodes 1 and 2 of the element. Solving equation (28) and introducing the results into equation (27), it is found that equation (17) is given by

$$u(x) = \psi_1(x) u_1 + \psi_2(x) u_2 + \phi_1(x) u_1' + \phi_2(x) u_2'$$  \hspace{1cm} (29)$$
Here \( \varphi_A(x) = \varphi_{N\cdot N}(x) \) \((N = 1, 2)\)

\[
\begin{align*}
\psi_1 &= 1 - 3\xi^2 + 2\xi^3, \\
\psi_2 &= 3\xi^2 - 2\xi^3, \\
\varphi_1 &= L(\xi - 2\xi^2 + \xi^3), \\
\varphi_2 &= L(-\xi^2 + \xi^3)
\end{align*}
\]

where \( \xi = (x-x_1)/L \) and \( L = x_2-x_1 \).

**Other elements**

It is emphasized that the above elements are cited only as examples. Several other types of finite elements can be developed which are capable of being used in first strain-gradient problems. In two-dimensional space, for example, rectangular elements over which the local field is given by two-dimensional Hermite interpolation polynomials, each given by sixteen-term bicubic polynomials, can be used. Similar functions were used by Bogner and others to develop flat plate elements. The higher-order plane stress elements presented by Tocher and Hartz could also be used. As a final comment in this regard, we cite the one-dimensional element in which the second derivatives of displacement \( u_{rr} \) are specified at each node, as well as the derivatives \( u_r \). In this case, a fifth-degree polynomial uniquely determined the local field in terms of the values of \( u, u_r, \) and \( u_{rr} \) at each of the two nodes.

**KINEMATICS AND EQUATIONS OF MOTION OF A FINITE ELEMENT**

We now consider an elastic body of material characterized by strain energy functions of the form given in equation (2). The body is of arbitrary shape, with arbitrary boundary conditions, and is in motion due to the action of a general system of external forces, couples and doublets. Following the usual procedure, we set out to construct a discrete model of this system by representing the body by a collection of a finite number \( E \) of finite elements connected together at their nodes. For the moment, the particular type of finite elements is unimportant. We now temporarily confine our attention to a typical finite element \( e \). Assuming that the displacement field throughout the element is given approximately by equation (17), we find from equation (1) that for the finite element

\[
\gamma_{ij} = \frac{1}{2}(\psi_{N\cdot N} u_{Nj} + \psi_{N\cdot N} u_{Nj} + \varphi_{N\cdot N} u_{Nj} + \varphi_{N\cdot N} u_{Nj} + \varphi_{N\cdot N} u_{Nj})
\]

\[
\omega_i = \frac{1}{2} \varepsilon_{ijk}(\psi_{N\cdot N} u_{Nk} + \varphi_{N\cdot N} u_{Nk})
\]

\[
\kappa_{ijk} = \psi_{N\cdot N} u_{Nk} + \varphi_{N\cdot N} u_{Nk}
\]

Here \( N = 1, 2, \ldots, N_e; \ i, j, k = 1, 2, 3; \) and the element identification label \( e \) has been omitted for convenience. Other kinematical variables can be evaluated by simply introducing equation (17) into the appropriate equation in expression (1). However, the kinematics of the element is satisfactorily specified by equations (17) and (31) if an energy function of the form given by equation (3) is used.

Let \( W_{ei} \) denote the strain energy density for finite element \( e \). Then, according to equations (2), (3), (6), (7) and (31), the stresses and doublet stresses in the finite element are given in terms of the nodal values \( u_{Nj} \) and \( u_{Nj} \) by

\[
\sigma_{ij} = \frac{1}{2} \left( \frac{\varepsilon_{ij} \varphi_{N\cdot N}}{\gamma_{ij}} + \frac{\varepsilon_{ij} \varphi_{N\cdot N}}{\gamma_{ij}} \right) = G_{ij}(x; u_{Nj}; u_{Nj})
\]

\[
\tilde{\sigma}_{ijk} = \frac{1}{2} \left( \frac{\varepsilon_{ijk} \varphi_{N\cdot N}}{\kappa_{ijk}} + \frac{\varepsilon_{ijk} \varphi_{N\cdot N}}{\kappa_{ijk}} \right) = F_{ijk}(x; u_{Nj}; u_{Nj})
\]
where
\[
G_{ij}(x; u_{Nr}, u_{N_{r,k}}) = \lambda \delta_{ij}(\psi_{X,j}u_{N_{r}} + \psi_{N_{r,k}}u_{N_{r,k}}) + \mu(\psi_{X,j}u_{N_{r}} + \psi_{N_{r,k}}u_{N_{r,k}} + \psi_{N_{r,k}}u_{N_{r,k}})
\]
and
\[
F_{ijk}(x; u_{Nr}, u_{N_{r,k}}) = \frac{1}{2}(\delta_{jk} \delta_{lm} + \delta_{ik} \delta_{jm} + 4\delta_{id} \delta_{km})(\psi_{X,rr}u_{N_{m}} + \psi_{N_{r,rr}}u_{N_{r,m}}) + \frac{1}{3}\delta_{ij}(\psi_{X,rr}u_{N_{r}} + \psi_{N_{r,rr}}u_{N_{r,r}}) + \frac{1}{2}\delta_{ij}(\psi_{X,rr}u_{N_{r}} + \psi_{N_{r,rr}}u_{N_{r,r}})
\]

Let \( T_{e} \) denote the kinetic energy density for the element. Noting that
\[
T_{e} = \int_{V_e} \rho \bar{u}_{ij} \bar{u}_{ij} \, dv
\]
and introducing equation (17) into the time derivative of equation (14), we find that the total time-rate-of-change of energy (kinetic plus internal) for the finite element is
\[
\frac{d}{dt} \int_{V_e} (T_{e} + W_{e}) \, dv = m_{NM} \ddot{u}_{N_{j}} + k_{NM} \dddot{u}_{N_{j}} + k_{NM} \dddot{u}_{M_{j}} + i_{NM} \dddot{u}_{M_{j}} + k_{NM} \dddot{u}_{M_{j}} + i_{NM} \dddot{u}_{M_{j}}
\]
\[
+ \bar{u}_{N_{j}} \int_{V_e} (\psi_{X,ij} \psi_{ij} + \psi_{X,ik} \psi_{ik}) \, dv + \bar{u}_{M_{j}} \int_{V_e} \psi_{NM} \, dv + \bar{u}_{N_{j}} \int_{V_e} (\psi_{N_{r,ij}} \psi_{ij} + \psi_{N_{r,ik}} \psi_{ik}) \, dv
\]
\[
+ \bar{u}_{M_{j}} \int_{V_e} \psi_{NM} \, dv + \bar{u}_{N_{j}} \int_{V_e} (\psi_{N_{r,ij}} \psi_{ij} + \psi_{N_{r,ik}} \psi_{ik}) \, dv + \bar{u}_{N_{j}} \int_{V_e} (\psi_{N_{r,ij}} \psi_{ij} + \psi_{N_{r,ik}} \psi_{ik}) \, dv
\]
\[
+ \bar{u}_{M_{j}} \int_{V_e} \psi_{NM} \, dv
\]
In equation (37) do not appear in the traditional finite element formulation. They may be interpreted as being due to the 'micro-structure' endowed in the element by assigning it to the higher-order displacement field defined in equation (17).

It is also interesting to note that any finite element equation represents a global description of a relation that, in the limit, pertains to a point in a continuum. We see that even though the micro-effects of velocity gradients were ignored in defining the kinetic energy density in equation (14), terms equivalent to velocity-gradient effects (as represented by the terms involving \( \dddot{u}_{N_{j}} \) in equation (37)) appear in the finite element equations. This suggests that higher-order finite element formulations are related naturally to the ideas of micro-structure in a continuum. Indeed, the basic equations describing micro-effects in elasticity follow after higher-order kinematical terms are introduced in describing the ‘micro-deformation’ of a differential element. Thus, a continuum with micro-structures is the limiting case of a finite element model constructed of higher-order finite elements. These observations also suggest that in the case in which strain-gradient equations are viewed as a moderately long wavelength limit of difference equations of simple crystal lattices, the terms involving nodal velocity gradients in equation (37) can be omitted. In fact, these terms should vanish in the limit if \( T_{e} \) is to approach \( T \) of equation (14) as the finite element network is refined.
Returning to equation (37), we see that the mechanical power $\Omega_{(e)}$ developed by the external forces, couples and doublets acting on the element is given by

$$\Omega_{(e)} = p_{Nk} \dot{u}_{Nk} + d_{Njk} \dot{u}_{Nj,k}$$

(41)

where

$$p_{Nk} = \int_V (F_k \psi_N + \frac{1}{2} \epsilon_{nmk} C_n \psi_{N,m} + \Phi_{(jk)} \psi_{N,j}) \, dv$$

(42)

and

$$d_{Njk} = \int_V (F_k \varphi_{Nj} + \frac{1}{2} \epsilon_{nmk} C_n \varphi_{NJ,m} + \Phi_{(mk)} \varphi_{NJ,m}) \, dv$$

(43)

Here $p_{Nk}$ is the generalized force acting at node $N$ of element $e$ and $d_{Njk}$ is a generalized nodal double-force acting at node $N$ of element $e$. Both the nodal forces and the nodal doublets are 'consistent' in the sense that the mechanical power in equation (41) is the same as that developed by the distributed external forces $F_k$, $S_k$, couples $C_n$, $m_n$, and doublets $\Phi_{(jk)}$ and $\mu_{ijk}$ due to the assumed velocity field calculated using equation (17).

According to equation (16), energy is conserved in the finite element if

$$\int_V (T_{(e)} + W_{(e)}) \, dv = \Omega_{(e)}$$

(44)

Thus, substituting equations (37) and (41) into equation (44) and simplifying, we arrive at the energy balance for a finite element:

$$[m_{NM} \ddot{u}_{Mj} + k_{NM} \dot{u}_{Mj} + \int_V (\psi_{N,i} G_{ij} \psi_{N,j} + \psi_{N,ik} F_{ik}) \, dv - p_{Nj}] \ddot{u}_{Nj} + [k_{NMr} \ddot{u}_{Mjr} + i_{MNr} \dot{u}_{Mjr} \dot{u}_{Mjr} + \int_V (\varphi_{N,r,i} G_{ij} \varphi_{N,r,j} + \varphi_{N,r,ik} F_{ik}) \, dv - d_{Njr}] \ddot{u}_{Njr} = 0$$

(45)

Now the nodal values $u_{Nj}$ and $\dot{u}_{Nj}$ are regarded as linearly independent functions of time, and equation (45) must hold for arbitrary values of the nodal velocities $\dot{u}_{Nj}$ and nodal velocity gradients. It follows that the terms in brackets in equation (45) must vanish. Thus, we arrive at the pair of equations

$$m_{NM} \ddot{u}_{Mj} + k_{NM} \dot{u}_{Mj} + \int_V (\psi_{N,i} G_{ij} \psi_{N,j} + \psi_{N,ik} F_{ik}) \, dv = p_{Nj}(t)$$

(46)

$$k_{NMr} \ddot{u}_{Mjr} + i_{MNr} \dot{u}_{Mjr} \dot{u}_{Mjr} + \int_V (\varphi_{N,r,i} G_{ij} \varphi_{N,r,j} + \varphi_{N,r,ik} F_{ik}) \, dv = d_{Njr}(t)$$

(47)

Equations (46) and (47) are the general equations of motion of a finite element of an elastic body constructed of a material characterized by a strain–energy function of the form in equation (3). The functions $G_{ij}(\cdot)$ and $F_{ik}(\cdot)$ are defined in equations (34) and (35) and we are reminded once again that the repeated indices are to be summed throughout their admissible range. In this case $i, j, k, r, s = 1, 2, 3$; $M, N = 1, 2, \ldots, N_v$, where $N_v$ is the total number of nodes of element $e$. 
Equations (46) and (47) pertain to a typical finite element. It is now a simple matter to connect all of the finite elements together and to obtain the corresponding global equations of motion for the entire discrete model of the body. From the invariance of mechanical power, we find

\[ P_{\Delta j} = \sum_{\varepsilon} E \Omega_{\Delta}^{(\varepsilon)} r_{\Delta}^{(\varepsilon)} \quad D_{\Delta j} = \sum_{\varepsilon} \Omega_{\Delta}^{(\varepsilon)} d_{\Delta}^{(\varepsilon)} \]  

(48)

where \( P_{\Delta j} \) and \( D_{\Delta j} \) are the global generalized forces and double forces at node \( \Delta \) of the connected system. \( \Omega_{\Delta}^{(\varepsilon)} \) is the array given in equation (20) and \( E \) is the total number of finite elements. Introducing equations (46) and (47) into equation (48) and incorporating equation (20) into the result, we arrive at the global equations of motion

\[ M_{\Delta r} \ddot{U}_{r} + K_{\Delta r} U_{r} + \sum_{\varepsilon} E \Omega_{\Delta}^{(\varepsilon)} \int_{V_{\varepsilon}} [\psi_{N,i} G_{ij}(x; \Omega_{\varepsilon}^{(\varepsilon)} U_{\varepsilon}, \Omega_{\varepsilon}^{(\varepsilon)} U_{\varepsilon r})] \, dr = P_{\Delta r}(t) \]  

(49)

\[ K_{\Delta r} U_{r} + L_{\Delta r} U_{r} + \sum_{\varepsilon} \Omega_{\Delta}^{(\varepsilon)} \int_{V_{\varepsilon}} [\varphi_{N,i} G_{ij}(x; \Omega_{\varepsilon}^{(\varepsilon)} U_{\varepsilon}, \Omega_{\varepsilon}^{(\varepsilon)} U_{\varepsilon r})] \, dr = D_{\Delta r}(t) \]  

(50)

in which

\[ M_{\Delta r} = \sum_{\varepsilon} \Omega_{\Delta}^{(\varepsilon)} \eta_{\eta}^{(\eta)} \Omega_{\eta}^{(\eta)} \]  

(51)

\[ K_{\Delta r} = \sum_{\varepsilon} \Omega_{\Delta}^{(\varepsilon)} k_{\eta}^{(\eta)} \Omega_{\eta}^{(\eta)} \]  

(52)

\[ L_{\Delta r} = \sum_{\varepsilon} \Omega_{\Delta}^{(\varepsilon)} l_{\eta}^{(\eta)} \Omega_{\eta}^{(\eta)} \]  

(53)

Equations (49) and (50) represent the final system of linear differential equations describing the motion of the finite element model of the body. Upon applying appropriate boundary conditions, these are solved for the nodal displacements \( U_{\Delta r} \) and displacement gradients \( U_{\Delta r} \). Local values \( u_{N} \) and \( u_{N r} \) are calculated using equation (20) and the element strains, strain gradients, stresses and double stresses are evaluated using equations (31)-(35).

SAMPLE PROBLEMS

To demonstrate the application of the theory developed in this paper, we now examine some simple special cases of the general finite element equations. In particular, we consider the static problem of the elastic half-space, subjected to uniform stress at infinity, in which two strain-gradient-sensitive materials are welded together in such a way that a microlayer of width \( 2h \) results. This geometry might be used to represent a simplified model of a composite material formed by microplates or whiskers embedded in a soft matrix, as indicated in Figure 2. This problem was investigated by Day and Weitsman, who obtained exact solutions using the first strain-gradient theory of elasticity.

In the present case the displacement field of each element is of the form given in equation (27) with the nodal displacements \( u_{N} \) and displacement gradients \( u_{N r} \) \((N = 1, 2)\) being the unknowns. The strain energy function reduces to

\[ W_{(m)} = \frac{2}{\lambda_{(m)} + 2\mu_{(m)}} [\psi_{N,x} \dot{u}_{N} + \varphi_{N,r} \dot{u}_{N r}]^{2} + \mu_{(m)} [\psi_{N,x} \ddot{u}_{N} + \varphi_{N,r} \ddot{u}_{N r}]^{2} \]  

(54)
where $m = 1, 2$ and $\lambda_{(1)}, \mu_{(1)}, I_{(1)}$ are material constants for the exterior material and $\lambda_{(2)}, \mu_{(2)}, I_{(2)}$ are those for the microlayer. Here $\lambda_{(m)}, \mu_{(m)}$ are the Lamé constants and

$$I^2 = \frac{2}{\lambda + 2\mu} (\tilde{a}_1 + \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4 + \tilde{a}_5) = \frac{1}{\lambda + 2\mu} (3\tilde{a}_1 + 2\tilde{a}_2) \tag{55}$$

where $\tilde{a}_1, \ldots, \tilde{a}_5$ are the micromoduli for equation (3) and $\tilde{a}_1$ and $\tilde{a}_2$ are those of equation (5). We may also have use for the ratio

$$\beta = (\lambda_{(2)} + 2\mu_{(2)}) (\lambda_{(1)} + 2\mu_{(1)}) \tag{56}$$

which is a measure of discontinuity of the classical strain at the interface of the microlayer and the exterior material.

![Figure 2. Composite body](image)

According to equation (32), the stress $\sigma^{(1)}_{11}$ in the microlayer and the stress $\sigma^{(1)}_{11}$ outside the microlayer are given by

$$\sigma_{11}^{(1)} = (\lambda_{11} + 2\mu_{11}) u_{,x}, \quad \sigma_{11}^{(2)} = (\lambda_{12} + 2\mu_{12}) u_{,x} \tag{57}$$

and from equation (33).

$$\bar{\sigma}_{111} = (3\tilde{a}_1 + 2\tilde{a}_2) u_{,xx} \tag{58}$$

For purposes of comparing our results with those of Reference 10, we are also interested in the double stress $\bar{\mu}_{122}$ which can be shown to be given by

$$\bar{\mu}_{122} = \tilde{a}_1 u_{,xx} \tag{59}$$

Following Day and Weitsman,\textsuperscript{10} we also introduce a parameter $\alpha$ such that

$$\tilde{a}_1 = \frac{1 - 2\alpha}{3} \mu (\lambda + 2\mu), \quad \tilde{a}_2 = \alpha \mu (\lambda + 2\mu) \tag{60}$$

Then, if $\rho = h/l$, equation (59) can be written in the form

$$\frac{1}{h} \bar{\mu}_{122}^{(m)} = (1 - 2\alpha_{(m)}) \frac{l_{(m)}}{3 \rho_{(m)}} (\lambda_{(m)} + 2\mu_{(m)}) u_{,xx} \tag{61}$$
Also, in the present example the equations of motion (46) and (47) for an element reduce to the linear static relations

\[
\begin{align*}
\rho_1 &= \left(\frac{6R}{5L} + \frac{24s}{L^3}\right) (u_1 - u_2) + \left(\frac{R}{10} + \frac{12s}{L^2}\right) (u_{1,x} + u_{2,x}) \\
\rho_2 &= \left(\frac{6R}{5L} + \frac{24s}{L^3}\right) (u_2 - u_1) - \left(\frac{R}{10} + \frac{12s}{L^2}\right) (u_{1,x} + u_{2,x}) \\
d_1 &= \left(\frac{2RL}{15} + \frac{8s}{L}\right) u_{1,x} + - \left(\frac{LR}{30} + \frac{4s}{L}\right) u_{2,x} + \left(\frac{R}{10} + \frac{12s}{L^2}\right) (u_1 - u_2) \\
d_2 &= \left(\frac{2RL}{15} + \frac{8s}{L}\right) u_{2,x} + - \left(\frac{LR}{30} + \frac{4s}{L}\right) u_{1,x} + \left(\frac{R}{10} + \frac{12s}{L^2}\right) (u_1 - u_2)
\end{align*}
\]

where \( R = \lambda + 2\mu \) and \( s = R\beta/2 \).

**DISCUSSION**

In the examples considered, we examine two cases characterized by the following values of the material parameters \( \beta, \rho_1, \rho_2, \alpha_1 \) and \( \alpha_2 \)—Case I: \( \beta = \rho_1 = 10, \rho_2 = 20, \alpha_1 = \alpha_2 = 0.1 \); Case II: \( \beta = \rho_2 = 10, \rho_1 = 2, \alpha_1 = \alpha_2 = 0.1 \). For each case, finite element solutions were obtained for 3, 5, 10 and 15 elements inside the microlayer \( 0 \leq x/h \leq 1 \). For the results shown the length of the finite element model was \( 4h \). This rather short model gave surprisingly good results as compared with the exact solution. For all solutions cited here, the externally applied force per unit area is unity.

Figures 3 and 4 show the dimensionless displacement \( R_{(2)} u/h \) plotted as a function of dimensionless distance \( x/h \). Notice that the discontinuity in strain at the interface which is predicted by the classical (non-polar) theory of elasticity does not appear in the strain-gradient solution. This more realistic result is, of course, due to the fact that second gradients of displacement are now used to characterize the deformation so that it is possible to maintain continuity in the first derivatives of the displacement. It is also noted that for \( x/h > 1 \) the displacements obtained from the strain-gradient solution do not approach those of the classical solution. The finite element solution is seen to be in good agreement with the exact solution for all cases considered.

In Figures 5 and 6 the Cauchy stress \( \sigma_{xx} = \sigma_{yy} \) is shown as a function of the dimensionless distance \( x/h \). We observe that with only three finite elements in the microlayer, good results are obtained. Unlike the classical case, we see that the stress experiences a finite discontinuity at the interface \( x/h = 1 \). According to Reference 10, the value of the dimensionless stress always lies between the classical value of unity and a value equal to the parameter \( \beta \). With \( \rho_1 \) finite, the stress approaches \( \beta \) as \( \rho_2 \) becomes infinitely larger and approaches unity as \( \rho_1 \) becomes infinite with \( \rho_2 \) finite. If \( \beta > 1.0 \) (that is, if the material of the microlayer is stiffer than that outside the microlayer), then the stress at the interface is always greater than the classical stress. We note that these strain-gradient effects are of a local character and are significant only in a neighbourhood of the interface. As \( x/h \) becomes larger, the stress approaches values predicted by classical theory.

Figure 7 shows the dimensionless double stress \( (1/h) \tilde{\mu}_{,x} = (1/h) \tilde{\mu}_{,y} \) plotted versus \( x/h \) for fifteen finite elements. As expected, finite element solutions for the double stress are less accurate than the corresponding solutions for displacements and stresses since they are functions of the second derivatives of an approximate displacement field. Averaged values of the double stress over each element are in qualitative agreement with the exact solutions.
Figure 3. Displacements. Three elements in microlayer

Figure 4. Displacements. Fifteen elements in microlayer
Figure 5. Cauchy stress $\sigma_{xx}$. Three elements in microlayer

Figure 6. Cauchy stress $\sigma_{xx}$. Fifteen elements in microlayer
Figure 7. Double stress $\bar{\sigma}_{x=0}$. Fifteen elements in microlayer

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