1. Consider the problem of heat conduction in a solid slab with Dirichlet boundary conditions:

\[-\nabla \cdot (k(x)\nabla u(x)) = f(x) \quad \text{in } \Omega \tag{1a}\]
\[u(x) = 0 \quad \text{on } \Gamma, \tag{1b}\]

where \(x = (x, y) \in \Omega\), \(u(x)\) is the temperature, \(f(x)\) is the distributed heat source, and \(k(x)\) is the thermal conductivity. For this problem, we assume a two-dimensional square domain \(\Omega = (-a, a) \times (-a, a)\) with boundary \(\Gamma\), and for simplicity we take \(k(x) = f(x) = 1\) and \(a = 1\). We recall (from Assignment 5) that the exact solution of this problem is:

\[u(x, y) = \frac{16a^2}{\pi^3} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^3(-1)^{n+1}} \left[ 1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi/2)} \right] \cos \frac{n\pi x}{2a}. \tag{2}\]

(a) Recall the weak form of the boundary value problem given by (1).

(b) Use/adapt the FEniCS codes provided\(^1\) to solve the boundary value problem given above. We choose to use the same number of mesh points in all directions, hence assume a mesh of the form

\[\Omega_h = \{(x_i, y_j)^T : x_i = (i - 1)h, y_j = (j - 1)h, i, j = 1, ..., N + 1, h = 1/N\}.\]

i. Plot \(u_h(x, y)\) for \(N = 4, 8, 16\) and compare these approximate solutions to the exact solution given in Equation (2).

ii. Compute the finite element approximation \(u_h\) for \(N = 2^k\), for \(k = 2, ..., 8\) with linear and quadratic elements, and compute the error norms \(\|e\|_E\) and \(\|e\|_L^2\) for each mesh\(^2\). Plot \(\log \|e\|\) versus \(\log h\). What are the rates of convergence for each choice of norm? Does your convergence rate agree with the theoretical estimates? Discuss your results.

2. Consider the following boundary value problem:

\[-\nabla \cdot (k \nabla u) = f \quad \text{in } \Omega \tag{3a}\]
\[u = u_l \quad \text{on } \Gamma_{\text{left}}, \tag{3b}\]
\[u = u_r \quad \text{on } \Gamma_{\text{right}}, \tag{3c}\]
\[k \frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_{\text{top}}, \tag{3d}\]
\[k \frac{\partial u}{\partial n} + \alpha u = h \quad \text{on } \Gamma_{\text{bottom}}, \tag{3e}\]
where $k$, $\alpha$, $f$, $u_l$, $u_r$, $h$, and $g$ are given functions, and $n$ is the unit normal vector to the boundary.

a. Consider the above problem with $k = \alpha = 1$ on $\Omega = (0,1) \times (0,1)$. Let the exact solution be given by

$$u(x,y) = \sin(2\pi x) \sin(2\pi y).$$

(4)

Determine the source terms $f$, $u_l$, $u_r$, $h$, and $g$ so that their use in the boundary value problem (3) produces the exact solution (4). (Note that this technique of creating a problems with a known exact solution is often referred to as the method of manufactured solutions).

b. Derive the weak form of the boundary problem given by Equations (3).

c. Repeat (b) from Problem 1.