If you have not done so already, read Chapter 4 in the finite element textbook.

1. Consider the boundary value problem

\[ L[u] = f \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma, \]

where \( \Omega \) is a two-dimensional domain, \( \Gamma \) is its boundary, and the differential operator \( L \) is

\[ L = -\nabla \cdot k \nabla = -\frac{\partial}{\partial x} \left( k \frac{\partial}{\partial x} \right) - \frac{\partial}{\partial y} \left( k \frac{\partial}{\partial y} \right). \]

Recall that in Assignment 2, you showed that if \( L \) is a linear, self-adjoint, and positive definite operator, then the boundary value problem (1) is equivalent to an energy minimization problem, the Ritz method minimizes the error in the energy norm, and the Ritz and Galerkin methods are identical.

Here we would like you to show that the Poisson operator \( L \) given in (2) above is indeed linear, self-adjoint, and positive definite. Use the definitions of linearity, self-adjointness, and positive definiteness from Assignment 2, extended appropriately from one to two dimensions.

2. Consider the problem of heat conduction in a solid slab with Dirichlet boundary conditions:

\[ -\nabla \cdot (k(x) \nabla u(x)) = f(x) \text{ in } \Omega \]
\[ u(x) = 0 \text{ on } \Gamma, \]

where \( x = (x, y) \in \Omega, u(x) \) is the temperature, \( f(x) \) is the distributed heat source, and \( k(x) \) is the thermal conductivity. For this problem, we assume a two-dimensional square domain \( \Omega = (-a, a) \times (-a, a) \), and for simplicity we take \( k(x) = f(x) = 1 \) and \( a = 1 \).

(a) Derive (or recall) the weak form of the boundary value problem given by (3) and (4) and its finite element discretization.

(b) Making use of three axes of symmetry, reduce the square domain to a triangle which is one-eighth the size of the square (as shown in Figure 1). Using the appropriate boundary conditions and four linear triangles, solve for the nodal values of \( u_h \) and for the heat flux vector \( k \nabla u_h \) in each element.

(c) Repeat part (b), but use two axes of symmetry and a quarter model of the square, and use two bilinear rectangular elements for the finite element approximation (see Figure 2). Since the finite element approximation of the heat flux varies within each element, find the maximum value of the flux in each element.

(d) Compare your finite element approximations from (b) and (c) to the exact solution,

\[ u(x, y) = \frac{16a^2}{\pi^3} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^3} (-1)^{n+1} \left[ 1 - \frac{\cosh(n\pi y/2a)}{\cosh(n\pi/2)} \right] \cos \frac{n\pi x}{2a}. \]
Figure 1: Illustration for Problem 2 (b)

Figure 2: Illustration for Problem 2 (c)