If you have not done so already, read Chapter 2 in the finite element textbook.

1. Consider the following boundary value problem:

\[ -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right) = f(x) \quad 0 < x < 1, \]
\[ u = 0 \text{ at } x = 0, 1, \]

for the material modulus \( k \) given by

\[ k(x) = \frac{1}{\alpha} + \alpha(x - x_0)^2, \]

and the forcing function \( f \) given by

\[ f(x) = 2 \left\{ 1 + \alpha(x - x_0) \left[ \tan^{-1}(\alpha(x - x_0)) + \tan^{-1}(\alpha x_0) \right] \right\}. \]

The exact solution to this problem is

\[ u(x) = (1 - x) \left[ \tan^{-1}(\alpha(x - x_0)) + \tan^{-1}(\alpha x_0) \right], \]

and has an “interior layer” near \( x = x_0 \). The solution can be made to vary from smooth to near discontinuous by increasing the parameter \( \alpha \). We are interested in studying the behavior of the finite element method for this problem.

We wish to solve this problem using piecewise linear basis functions for the values \( x_0 = 0.5 \) and \( \alpha = 5 \) (i.e. a smooth problem). Use the FEniCS codes provided\(^1\) to solve the boundary value problem given above. We assume a mesh of the form

\[ 0 = x_1 < x_2 < ... < x_N < x_{N+1} = 1, \text{ so that } h = 1/N. \]

a. Plot \( u_h(x) \) for \( N = 4, 8, \) and 16 and compare these approximate solutions to the exact solution.

b. Compute the finite element approximation \( u_h \) for \( N = 2^k \), for \( k = 2, ..., 7 \) and compute the error norms \( \|e\|_E \) and \( \|e\|_{L^2} \) for each mesh\(^2\). Plot \( \log \|e\| \) versus \( \log h \). What are the rates of convergence for each choice of norm? Does your convergence rate agree with the theoretical estimates? Discuss your results.

c. Repeat (a) and (b) for \( \alpha = 50 \) (i.e. a “rough” problem), and compare your results with the “smooth” problem \( (\alpha = 5) \). Do the numerical solutions confirm the theory?

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\(^1\)FEniCS examples shown in the class as well as files required for this assignment can be downloaded from [http://users.ices.utexas.edu/~noemi/teaching_docs/hw4/](http://users.ices.utexas.edu/~noemi/teaching_docs/hw4/).

\(^2\)The formulas for the errors have been given in hw3.
2. We would like to study the behavior of the finite element method with higher order shape functions, and in the presence of singularities. Consider the following boundary value problem

\[-\frac{d^2u}{dx^2} = f(x) \quad 0 < x < 1,\]

\[u(0) = 0, \quad \frac{du}{dx}(1) = 1,\]

where \(u(x)\) is the unknown displacement and the forcing function \(f(x)\) given by

\[f(x) = -\alpha x^{\alpha - 1},\]

with \(\alpha\) a parameter. The exact solution to this problem is

\[u(x) = \frac{1}{1 + \alpha} (x^{1+\alpha}).\]

When \(\alpha < 0\), the stress \(\frac{du}{dx}\) approaches infinity at \(x = 0\). In this case, the stress is said to have a singularity of order \(\alpha\). Because of this singularity, the problem can be taken as a model for elasticity problems with re-entrant corners. Thus, the behavior of the finite element method on this model problem will provide insight into singular problems in two and three dimensions.

(a) Show that for \(-\frac{1}{2} < \alpha < 0\), the stress is singular at \(x = 0\), but the strain energy is bounded.

(b) Define \(s\) to be the highest derivative of \(u(x)\) that is square integrable, i.e.

\[\int_0^1 \left(\frac{d^s u}{dx^s}\right)^2 dx < \infty.\]

Show that for a given \(\alpha\), \(s\) is given by

\[s < \alpha + \frac{3}{2}.\]

(c) We would like to verify the error estimate

\[\| u - u_h \|_E = \leq C h^\mu,\]

where \(C\) is a constant independent of \(h\) and

\[\mu = \min(k, s - 1).\]

Here, \(k\) is the order of the shape functions. In other words, if the solution is sufficiently smooth, a convergence rate of \(k\) is observed; otherwise, the convergence rate depends on the smoothness of the solution, and increasing \(k\) will not improve the rate. Using FEniCS, solve this problem using both linear and quadratic shape functions, for the following three cases:

i. \(\alpha = -\frac{1}{4}\)

ii. \(\alpha = 1.25\)

iii. \(\alpha = 2\)

In all cases, estimate the rate of convergence by plotting log of the error in the energy norm vs. log of the mesh size, and uniformly refining the mesh until the asymptotic rate of convergence is apparent. Discuss the convergence of the finite element method in the context of solution smoothness.