POLYNOMIAL EXTENSIONS AND PROJECTION-BASED INTERPOLATION IN THREE DIMENSIONS

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H^1-Conforming Projection-Based Interpolation

Master tetrahedron,

$$T = \{(x_1, x_2, x_3) \; : \; x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 \leq 1\} \, ,$$

with faces f and edges e.
Min rule assumed:

$$p_f \leq p, \forall f, \quad p_e \leq p_f \text{ for adjacent faces } f \, .$$

Finite Element Spaces:

$$\begin{aligned} \mathcal{P}^p_{p_f,p_e} &= \{u \in \mathcal{P}^p(T) \; : \; u|_f \in \mathcal{P}^{p_f}(f), u|_e \in \mathcal{P}^{p_e}(e), \quad \forall f, e\} \\ \mathcal{P}^{p_f}_{p_e} &= \{u \in \mathcal{P}^{p_f}(f) \; : \; u|_e \in \mathcal{P}^{p_e}(e), \quad \forall e\} \, . \end{aligned}$$

Definition:

$$\begin{cases} u_1(a) = u(a) \\ \|u - \Pi u\|_{\epsilon,e} \to \min \\ |u - \Pi u|_{\frac{1}{2}+\epsilon,f} \approx \|\boldsymbol{\nabla}_f(u - \Pi u)\|_{-\frac{1}{2}+\epsilon,f} \to \min \\ |u - \Pi u|_{1,K} = \|\boldsymbol{\nabla}(u - \Pi u)\|_{0,K} \to \min \end{cases}$$

Remark: Due to the imbedding of $H^r(T)$ into $C(\bar{T})$ only for $r > \frac{1}{2}$, the interpolation operator is continuous only on $H^{\frac{1}{2}+\epsilon}(T)$. The standard argument based on reproducability of polynomials and continuity of the interpolation operator,

$$\|u - \Pi u\|_{1,T} = \|(u - \phi) - \Pi(u - \phi)\|_{1,T} \leq (1 + \|\Pi\|)\|u - \phi\|_{\frac{1}{2}+\epsilon,T} \,,$$

leads to a suboptimal estimate,

$$\|u - \Pi u\|_{1,T} \leq (1 + \|\Pi\|) \inf_{\phi \in \mathcal{P}^p} \|u - \phi\|_{\frac{1}{2}+\epsilon,T} \leq C p^{-(r-\frac{1}{2}-\epsilon)} \|u\|_{r,T}, \quad r > \frac{1}{2}.$$

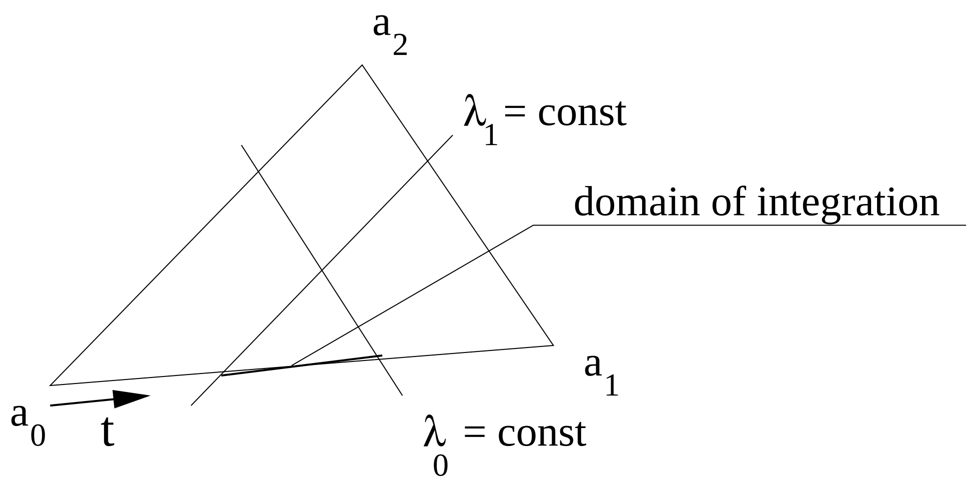
Polynomial preserving, continuous extension operator for a triangular face f

$$\begin{array}{c} H^{\frac{1}{2}+\epsilon}(f) \\ E^{grad}\Big\uparrow\Big\downarrow Tr^{grad} \\ H^{\epsilon}(\partial f) \end{array}$$

Here $\epsilon \in (0, \frac{1}{2}]$.

$$\begin{array}{c} \mathcal{P}^{p}_{p_e}(f) \\ E^{grad}\Big\uparrow\Big\downarrow Tr^{grad} \\ \mathcal{P}^{p_e}(\partial f) \end{array}$$

H^1 extension from one edge to triangle (Babuška):



$$\tilde{\phi}(\lambda_1, \lambda_2) = \int_0^1 \phi(\lambda_1 + \xi\lambda_2)\, d\xi = \frac{1}{\lambda_2}\int_{\lambda_2}^{\lambda_1+\lambda_2} \phi(t)\, dt\,.$$

Edge to edge operators:

$$\tilde{\phi}(\lambda_1, \lambda_2) = \int_0^1 \phi(\lambda_1 + \xi\lambda_2)\, d\xi = \frac{1}{\lambda_2}\int_{\lambda_2}^{\lambda_1+\lambda_2} \phi(t)\, dt\,.$$

$$\phi \to A^{right}\phi = \tilde{\phi}|_{\lambda_0=0}$$

$$\phi \to A^{left}\phi = \tilde{\phi}|_{\lambda_1=0}$$

The three edge problem:

Given f_i defined on edge $\lambda_i = 0, i = 0, 1, 2,$

$$\begin{cases} \text{Find } \phi_i \text{ such that} \\ \phi_i + A^{right}\phi_{i-1} + A^{left}\phi_{i+1} = f_i \ . \end{cases}$$

Multiply by $\omega^i, \omega^3 = 1$, to decouple,

$$\begin{cases} \text{Find } \phi_i\omega^i \text{ such that} \\ \phi_i\omega^i + \omega A^{right}\left(\phi_{i-1}\omega^{i-1}\right) + \omega^{-1}A^{left}\left(\phi_{i+1}\omega^{i+1}\right) = f_i\omega^i \ . \end{cases}$$

Define then,

$$E^{grad}(f_1, f_2, f_3) = \sum_i \tilde{\phi}_i \ .$$

Solution of the decoupled problems:

$$f = f^{even} + f^{odd}, \quad \phi = \phi^{even} + \phi^{odd}$$

(Easy) case $\omega = 1$:

$$\phi^{even}(x) = \frac{1}{2}\frac{d}{dx}\left\{(1-x^2)\int_{-1}^{x}\frac{f^{even}(s)}{1-s}\,ds - (1-x^2)\int_{x}^{1}\frac{f^{even}(s)}{1+s}\,ds\right\}$$

$$\phi^{odd}(x) = \frac{1}{2}\frac{d}{dx}\left\{\frac{1}{1+x}\int_{-1}^{x}(1-s^2)f^{odd}(s)\,ds - \frac{1}{1-x}\int_{x}^{1}(1-s^2)f^{odd}(s)\,ds\right\}$$

(Tricky) case $\omega \neq 1$:

much more complicated but **explicit !**

Projections

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & H^1 \\ \Big\downarrow id & & P^{grad} \Big\downarrow \\ \mathbb{R} & \longrightarrow & \mathcal{P}^{p+1}_{p_e+1,p_f+1} \end{array}$$

$$\begin{array}{ccc} \mathbb{R} & \longrightarrow & H^{\frac{1}{2}+\epsilon} \\ id \Big\downarrow & & P^{-\frac{1}{2}+\epsilon,grad} \Big\downarrow \\ \mathbb{R} & \longrightarrow & \mathcal{P}^{p_f+1}_{p_e+1} \end{array}$$

- Operator $P^{grad} \; : \; H^1 \ni u \rightarrow P^{grad} u \in \mathcal{P}^p_{p_f,p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla}(P^{grad}u - u)\| \rightarrow \min \\ (P^{grad}u - u, 1) = 0 \end{cases}$$

- Operator $P^{-\frac{1}{2}+\epsilon,grad} \; : \; H^{\frac{1}{2}+\epsilon} \ni u \rightarrow P^{-\frac{1}{2}+\epsilon,grad} u \in \mathcal{P}^p_{p_e}$,

$$\begin{cases} \|\boldsymbol{\nabla}(P^{-\frac{1}{2}+\epsilon,grad}u - u)\|_{-\frac{1}{2}+\epsilon} \rightarrow \min \\ (P^{-\frac{1}{2}+\epsilon,grad}u - u, 1) = 0 \end{cases}$$

Poincare inequalities

Poincare's inequalities:
There exist $C > 0$ such that,

$$\|u\|_{0,T} \leq C\|\boldsymbol{\nabla} u\|_{0,T} \ ,$$

for every function $u \in H^1(K)$ belonging to either of the two families:
Case 1: $(u, 1)_{0,T} = 0$,
Case 2: $u = 0$ on ∂K.

Poincare's inequalities for fractional spaces defined on a face f:
There exist $C > 0$ such that

$$\|u\|_{0,f} \leq C|u|_{\frac{1}{2}+\epsilon,f} \approx C\|\boldsymbol{\nabla} u\|_{-\frac{1}{2}+\epsilon,f} \ ,$$

for every function $u \in H^{\frac{1}{2}+\epsilon}(f)$ belonging to either of the two families:
Case 1: $(u, 1)_{0,f} = 0$,
Case 2: $u = 0$ on ∂f.

ϵ-(sub) optimal p estimate

Step 1: Comparison with the projection on the element level

The interpolation error is bounded by the projection error and the interpolation error on the element boundary,

$$\begin{aligned}
\|u-\Pi u\|_{1,T} &\leq \|u-P^{grad}u\|_{1,T}+\|P^{grad}u-\Pi u\|_{1,T}\\
&\preceq \|u-P^{grad}u\|_{1,T}+\|E\|_{L(H^{\frac{1}{2}}(\partial T),H^{1}(T))}\|P^{grad}u-\Pi u\|_{\frac{1}{2},\partial T}\\
&\preceq \|u-P^{grad}u\|_{1,T}+\|u-P^{grad}u\|_{\frac{1}{2},\partial T}+\|u-\Pi u\|_{\frac{1}{2},\partial T}\\
&\preceq \|u-P^{grad}u\|_{1,T}+\|u-\Pi u\|_{\frac{1}{2},\partial T}\,,
\end{aligned}$$

Step 2: Localization to faces

$$\|u\|_{\frac{1}{2},\partial T} \leq \|u\|_{\frac{1}{2}+\epsilon,\partial T}$$

Step 3: Comparison with the commuting projections on the face level

The face interpolation error is bounded by the face projection error and the interpolation error on the face boundary,

$$\|u - \Pi u\|_{\frac{1}{2}+\epsilon,f} \preceq \|u - P^{grad} u\|_{\frac{1}{2}+\epsilon,f} + \|u - \Pi u\|_{\epsilon,\partial f}$$

Step 4: Estimation of the interpolation error on edges

$$\begin{aligned}
\|u - \Pi u\|_{\epsilon,\partial f} &\preceq \sum_{e \in f} \|u - \Pi u\|_{\epsilon,e} \\
&= \sum_{e \in f} \|(u - \underbrace{u_1}_{\text{linear interpolant}}) - P^{\epsilon}(u - u_1)\|_{\epsilon,e} \\
&\preceq \sum_{e \in f} p_e^{r-\epsilon} \|u - u_1\|_{r,\epsilon} \\
&\preceq \sum_{e \in f} p_e^{r-\epsilon} \|u\|_{r,\epsilon}
\end{aligned}$$

Here $r > \frac{1}{2}$.

Crucial lemma:

$$\inf_{\phi \in \mathcal{P}_{-1}^{p}} \|u - \phi\|_{0,e} \leq C p^{-(1-\epsilon)} \|u\|_{1,e}, \quad u \in H^1(e)$$

Final estimate

$$\begin{aligned}\|u-\Pi u\|_{1,T} \preceq\ & \|u-P^{grad}u\|_{1,T} \\ & +\sum_f \|u-P^{\frac{1}{2}+\epsilon,grad}u\|_{\frac{1}{2}+\epsilon,f} \\ & +\sum_e \|(u-u_1)-P^{\epsilon}(u-u_1)\|_{\epsilon,e}\end{aligned}$$

Combining with the estimates for the projection errors, we get:

Theorem: *For every ϵ, there exists a constant $C = O(\epsilon^{-\frac{1}{2}})$ such that,*

$$\|u-\Pi u\|_{1,T} \leq Cp^{-(r-\epsilon)}\|u\|_{1+r,T}$$

for $r > \frac{1}{2}$.

Final remarks

- The $H^r, r > \frac{1}{2}$ regularity is necessary to define the linear interpolant but it *does not* result in the deterioration of the convergence rate.
- The ϵ and $C = O(\epsilon^{\frac{1}{2}})$ results from the Trace Theorem and the localization step and it seems to be unavoidable.
- Bramble-Hilbert argument leads to the corresponding ϵ (sub)-optimal *hp* interpolation error estimate.
- The local interpolation at vertices can be traded for a quasi-local (Clement-like) interpolation. The regularity assumption can then be lowered to $u \in H^r,\ r > 0$.

and ....

The presented results have been generalized to the whole de Rham diagram

under a conjecture on the existence of $H(curl)$-extension, polynomial preserving, operators.

Commuting Projection and (Projection Based) Interpolation Operators for a tetrahedron and a triangular face

$$
\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^{\frac{3}{2}+\epsilon}(T) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{\epsilon}(\mathrm{curl},T)\cap \boldsymbol{H}^{\frac{1}{2}+\epsilon}(T) & \xrightarrow{\boldsymbol{\nabla}\times} & \boldsymbol{H}^{\epsilon}(\mathrm{div},T) & \xrightarrow{\boldsymbol{\nabla}\circ} & L^2 \\
\Big\downarrow id & & P^1 \Big\downarrow \Pi & & P^{curl} \Big\downarrow \Pi^{\mathrm{curl}} & & P^{div} \Big\downarrow \Pi^{\mathrm{div}} & & \Big\downarrow P \\
\mathbb{R} & \longrightarrow & \mathcal{P}^{p+1}_{p_e+1,p_f+1} & \xrightarrow{\boldsymbol{\nabla}} & \mathbf{P}^{p}_{p_e,p_f} & \xrightarrow{\boldsymbol{\nabla}\times} & \mathbf{P}^{p-1}_{p_f-1,p_e} & \xrightarrow{\boldsymbol{\nabla}\circ} & \mathcal{P}^{p-2}
\end{array}
$$

$$
\begin{array}{ccccccccc}
\mathbb{R} & \longrightarrow & H^{\frac{1}{2}+\epsilon}(f) & \xrightarrow{\boldsymbol{\nabla}} & \boldsymbol{H}^{-\frac{1}{2}+\epsilon}(\mathrm{curl},f) & \xrightarrow{\boldsymbol{\nabla}\times} & H^{-\frac{1}{2}+\epsilon}(f) & \longrightarrow & \mathbf{0} \\
\Big\downarrow id & & P^{\frac{1}{2}+\epsilon} \Big\downarrow \Pi & & \Big\downarrow \Pi^{\mathrm{curl}} & & \Big\downarrow P & & \\
\mathbb{R} & \longrightarrow & \mathcal{P}^{p_f+1}_{p_e+1} & \xrightarrow{\boldsymbol{\nabla}} & \mathbf{P}^{p_f}_{p_e} & \xrightarrow{\boldsymbol{\nabla}\times} & \mathcal{P}^{p_f-2} & \longrightarrow & \mathbf{0}
\end{array}
$$