A NEW CLASS OF ADAPTIVE DISCONTINUOUS PETROV–GALERKIN
FINITE ELEMENT METHODS
WITH APPLICATION TO
SINGULARLY PERTURBED PROBLEMS

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Short course on DPG Method
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Lectures

Petrov-Galerkin Method with Optimal Test Functions.

Ultraweak variational formulation and the DPG method for convection-dominated diffusion.

1D analysis. Adaptivity.

Wave propagation as an example of a complex-valued problem.

Systematic choice of test norms. Robustness.

Convergence proofs.
Petrov-Galerkin Method with Optimal Test Functions.
Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
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PETROV GALERKIN METHOD WITH OPTIMAL TEST FUNCTIONS

≜

LEAST SQUARES (WITH A TWIST)
Least squares and optimal test functions

\[
\begin{cases}
  u \in U \\
  b(u, v) = l(v) & v \in V
\end{cases} \iff \quad Bu = l \quad B : U \rightarrow V'
\]

\[
< Bu, v > = b(u, v)
\]
Least squares and optimal test functions

\[
\begin{aligned}
\left\{ \begin{array}{l}
u \in U \\
b(u, v) = l(v)
\end{array} \right. \quad v \in V & \iff Bu = l \\
B : U \rightarrow V' \\
\langle Bu, v \rangle = b(u, v)
\end{aligned}
\]

**Least squares**: \( U_h \subset U , \)

\[
\frac{1}{2} \| Bu_h - l \|^2_{V'} \rightarrow \min_{u_h \in U_h}
\]
Least squares and optimal test functions

\[
\begin{align*}
& \left\{ \begin{array}{l}
    u \in U \\
    b(u, v) = l(v) \quad v \in V
\end{array} \right. \\
\iff & Bu = l \quad B : U \to V' \\
& \langle Bu, v \rangle = b(u, v)
\end{align*}
\]

- **Least squares:** \( U_h \subset U, \)
  \[
  \frac{1}{2} \| Bu_h - l \|_{V'}^2 \to \min_{u_h \in U_h}
  \]

- **Riesz operator:**
  \[
  R_V : V \to V', \quad \langle R_V v, \delta v \rangle = (v, \delta v)_V
  \]
  is an *isometry*, \( \| R_V v \|_{V'} = \| v \|_V. \)
\[
\begin{align*}
\begin{cases}
  u \in U \\
  b(u, v) = l(v)
\end{cases} & \iff Bu = l \quad B : U \rightarrow V' \\
\langle Bu, v \rangle & = b(u, v)
\end{align*}
\]

- **Least squares:** \( U_h \subset U, \)

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- **Riesz operator:**

\[
R_V : V \rightarrow V', \quad \langle R_V v, \delta v \rangle_V = (v, \delta v)_V
\]

is an *isometry*, \( \| R_V v \|_{V'} = \| v \|_V. \)

- **Least squares reformulated:**

\[
\frac{1}{2} \| Bu_h - l \|_{V'}^2 = \frac{1}{2} \| R_V^{-1} (Bu_h - l) \|_V^2 \rightarrow \min_{u_h \in U_h}
\]
Taking Gâteaux derivative,

\[ (R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h \]
Taking Gâteaux derivative,

\[
(R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u_h)_V = 0 \quad \delta u_h \in U_h
\]

or

\[
< Bu_h - l, R_V^{-1} B\delta u_h > = 0 \quad \delta u_h \in U_h
\]
Taking Gâteaux derivative,

\[(R^{-1}_V (B u_h - l), R^{-1}_V B \delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[< B u_h - l, R^{-1}_V B \delta u_h >_v = 0 \quad \delta u_h \in U_h\]
Taking Gâteaux derivative,

\[(R_V^{-1}(Bu_h - l), R_V^{-1} B \delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[< Bu_h - l, v_h >= 0 \quad v_h = R_V^{-1} B \delta u_h\]
Taking Gâteaux derivative,

\[(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[< Bu_h, v_h > = < l, v_h > \quad v_h = R_V^{-1}B\delta u_h \]
Least squares and optimal test functions

Taking Gâteaux derivative,

\[(R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u_h)_V = 0 \quad \delta u_h \in U_h\]

or

\[b(u_h, v_h) = l(v_h)\]

where

\[
\begin{align*}
  & v_h \in V \\
  & (v_h, \delta v)_V = b(\delta u_h, \delta v) \quad \delta v \in V
\end{align*}
\]
Least squares and optimal test functions

- Stiffness matrix is always hermitian and positive-definite (it is a least squares method...).
Least squares and optimal test functions

- Stiffness matrix is always hermitian and positive-definite (it is a least squares method...).
- The method delivers the best approximation error (BAE) in the “energy norm”:
  \[ \|u\|_E := \|Bu\|_V' = \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \]

  where the error representation function \( \psi \) comes from
  \[ \{ \psi \in V \mid (\psi, \delta v) = <l - Bu_h, \delta v > = l(\delta v) - b(u_h, \delta v), \delta v \in V \} \]
  (no need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques...).
Stiffness matrix is always hermitian and positive-definite (it is a least squares method...).

The method delivers the best approximation error (BAE) in the “energy norm”:

\[ \|u\|_E := \|Bu\|_V = \sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \]

The energy norm of the FE error \( u - u_h \) equals the residual and can be computed,

\[ \|u - u_h\|_E = \|Bu - Bu_h\|_V = \|l - Bu_h\|_V = \|R^{-1}_V(l - Bu_h)\|_V = \|\psi\|_V \]

where the error representation function \( \psi \) comes from

\[
\begin{cases}
\psi \in V \\
(\psi, \delta v)_V = <l - Bu_h, \delta v> = l(\delta v) - b(u_h, \delta v), \quad \delta v \in V
\end{cases}
\]

(no need for a-posteriori error estimation, note the connection with implicit a-posteriori error estimation techniques...)
A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.
Least squares and optimal test functions

- A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.

- Banach Closed Range Theorem $\implies$
  If $B' : V \rightarrow V'$ is injective, and we choose
  \[
  \|v\|_V = \sup_{u \in U} \frac{|b(u, v)|}{\|u\|_U}
  \]
  then
A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.

Banach Closed Range Theorem $\implies$
If $B' : V \to V'$ is injective, and we choose

$$
\| v \|_V = \sup_{u \in U} \frac{|b(u, v)|}{\| u \|_U}
$$

then

the energy norm coincides with the original norm in $U$.

$$
\| u \|_E = \| u \|_U
$$
Least squares and optimal test functions

- A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.
- Banach Closed Range Theorem $\implies$
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  \]
  then
- the energy norm coincides with the original norm in $U$.
  \[
  \| u \|_E = \| u \|_U
  \]

Indeed,
\[
\sup_u \sup_v \frac{|b(u, v)|}{\| u \| \| v \|_V} = \sup_v \sup_u \frac{|b(u, v)|}{\| u \| \| v \|_V} = \sup_v \frac{\| v \|_V}{\| v \|_V} = 1
\]
implies
\[
\sup_u \frac{\| u \|_E}{\| u \|} = 1 \implies \| u \|_E \leq \| u \|
\]
A lot depends upon the choice of the test norm $\| \cdot \|_V$; for different test norms, we get different methods.

Banach Closed Range Theorem $\implies$

If $B' : V \rightarrow V'$ is injective, and we choose

$$\|v\|_V = \sup_{u \in U} \frac{|b(u, v)|}{\|u\|_U}$$

then

the energy norm coincides with the original norm in $U$.

$$\|u\|_E = \|u\|_U$$

Also,

$$\inf_u \sup_v \frac{|b(u, v)|}{\|u\| \|v\|_V} = \inf_v \sup_u \frac{|b(u, v)|}{\|u\| \|v\|_V} = \inf_v \frac{\|v\|_V}{\|v\|_V} = 1$$

implies

$$\inf_u \frac{\|u\|_E}{\|u\|} = 1 \implies \|u\| \leq \|u\|_E$$
Petrov–Galerkin Method with Optimal Test Functions
Abstract $B^3$ Framework
(Repetitio Mater Studiorum Est)
Abstract Variational Problem

\[
\begin{cases}
  u \in U \\
  b(u, v) = l(v) \quad \forall v \in V
\end{cases}
\iff
Bu = l \
B : U \rightarrow V' \\
\langle Bu, v \rangle = b(u, v) \quad \forall v \in V
\]

where

- \( U, V \) are Hilbert spaces,
- \( b(u, v) \) is a continuous bilinear form on \( U \times V \),

\[
|b(u, v)| \leq M \|u\|_U \|v\|_V
\]

that satisfies the inf-sup condition (\( \iff \) \( B \) is bounded below),

\[
\inf_{\|u\|_U = 1} \sup_{\|v\|_V = 1} |b(u, v)| =: \gamma > 0
\]

- \( l \in V' \) represents the load and satisfies the compatibility condition \( l(v) = 0, \forall v \in V_0 \) where

\[
V_0 := \{ v \in V : b(u, v) = 0 \quad \forall u \in U \} \]
Let $b(u, v), u \in U, v \in V$ be a continuous bilinear form, $|b(u, v)| \leq M \|u\|_U \|v\|_V$, $l \in V'$. Consider the variational problem,

$$\begin{cases}
    u \in \tilde{u}_D + U \\
    b(u, v) = l(v), \quad \forall v \in V
\end{cases}$$

The inf-sup condition

$$\sup_{v \in V} \frac{|b(u, v)|}{\|v\|_V} \geq \gamma \|u\|_U$$

implies stability

$$\|u\|_U \leq \gamma^{-1} \|l\|_{V'}$$
Let \( b(u, v), u \in U, v \in V \) be a continuous bilinear form, \(|b(u, v)| \leq M \|u\|_U \|v\|_V\), \( l \in V' \). Consider the variational problem,

\[
\begin{aligned}
u_{hp} &\in \tilde{u}_D + U_{hp} \\
b(u_{hp}, v) &= l(v), \quad \forall v \in V_{hp}
\end{aligned}
\]

The discrete inf-sup condition

\[
\sup_{v \in V_{hp}} \frac{|b(u_{hp}, v)|}{\|v\|_V} \geq \gamma_{hp} \|u_{hp}\|_U
\]

implies discrete stability

\[
\|u_{hp}\|_U \leq \gamma_{hp}^{-1} \|l\|_{V'_{hp}}
\]
Let \( b(u, v), u \in U, v \in V \) be a continuous bilinear form, \(|b(u, v)| \leq M \|u\|_U \|v\|_V\), \( l \in V'\). Consider the variational problem,

\[
\begin{cases}
  u_{hp} \in \tilde{u}_D + U_{hp} \\
  b(u_{hp}, v) = l(v), \quad \forall v \in V_{hp}
\end{cases}
\]

The discrete inf-sup condition

\[
\sup_{v \in V_{hp}} \frac{|b(u_{hp}, v)|}{\|v\|_V} \geq \gamma_{hp} \|u_{hp}\|_U
\]

implies discrete stability

\[
\|u_{hp}\|_U \leq \gamma_{hp}^{-1} \|l\|_{V'_{hp}}
\]

and convergence

\[
\|u - u_{hp}\|_U \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in \tilde{u}_D + U_{hp}} \|u - w_{hp}\|_U
\]
If $U = V$, and the bilinear (sesquilinear) form is coercive,

$$b(u, u, ) \geq \alpha \|u\|_V^2$$

Then both continuous and discrete stability constants are bounded below by $\alpha$,

$$\gamma, \gamma_{hp} \geq \alpha \implies \frac{1}{\gamma_{hp}} \leq \frac{1}{\alpha}$$

Thus, for coercive problems, stability is guaranteed automatically.
FE classics:

If the bilinear form is symmetric (hermitian) and positive-definite, \[ b(u,v) = b(v,u), \]
\[ b(v,v) > 0 \]
for \( u, v \in \text{a Hilbert space} \ V \),

then

\[ \{ u \in V | J(u) := \frac{1}{2} b(u,u) - l(u) \rightarrow \min \} \]

\[ \{ u \in V | b(u,v) = l(v), v \in V \} \]

and, Bubnov-Galerkin method delivers the best approximation error in the energy norm,

\[ u_h \in V_h \subset V | b(u_h,v) = l(v), v \in V_h \rightarrow \min \]

where \( \| v \|_E = b(v,v) \).

You cannot do better! (in energy norm...)
FE classics:

- If the bilinear form is symmetric (hermitian) and positive-definite,

\[ b(u, v) = \overline{b(v, u)}, \quad b(v, v) > 0 \]

\[ u, v \in \text{a Hilbert space } V, \]
FE classics:

- If the bilinear form is symmetric (hermitian) and positive-definite,
  \[ b(u, v) = \overline{b(v, u)}, \quad b(v, v) > 0 \]
  \( u, v \in \text{a Hilbert space } V, \)

- then
  \[
  \begin{cases}
    u \in V \\
    J(u) := \frac{1}{2} b(u, u) - l(u) \rightarrow \min
  \end{cases}
  \Rightarrow
  \[
  \begin{cases}
    u \in V \\
    b(u, v) = l(v), \quad v \in V
  \end{cases}
  \]
Ritz and Bubnov-Galerkin Methods

**FE classics:**

- If the bilinear form is symmetric (hermitian) and positive-definite,

\[ b(u, v) = \overline{b(v, u)}, \quad b(v, v) > 0 \]

\[ u, v \in \text{a Hilbert space } V, \]

- then

\[ \begin{cases} u \in V \\ J(u) := \frac{1}{2} b(u, u) - l(u) \to \min \end{cases} \iff \begin{cases} u \in V \\ b(u, v) = l(v), \; v \in V \end{cases} \]

- and, Bubnov-Galerkin method delivers the *best approximation error* in the energy norm,

\[ \begin{cases} u_h \in V_h \subset V \\ b(u_h, v_h) = l(v_h), \; v_h \in V_h \end{cases} \iff \begin{cases} u_h \in V_h \\ \| u - u_h \|_E \to \min \end{cases} \]

where \[ \| v \|^2_E = b(v, v). \]
If the bilinear form is symmetric (hermitian) and positive-definite,
\[ b(u, v) = \overline{b(v, u)}, \quad b(v, v) > 0 \]

\( u, v \in \text{a Hilbert space } V, \)

then
\[
\left\{ \begin{array}{l}
  u \in V \\
  J(u) := \frac{1}{2} b(u, u) - l(u) \rightarrow \min
\end{array} \right. \Leftrightarrow \left\{ \begin{array}{l}
  u \in V \\
  b(u, v) = l(v), \quad v \in V
\end{array} \right.
\]

and, Bubnov-Galerkin method delivers the best approximation error in the energy norm,
\[
\left\{ \begin{array}{l}
  u_h \in V_h \subset V \\
  b(u_h, v_h) = l(v_h), \quad v_h \in V_h
\end{array} \right. \Leftrightarrow \left\{ \begin{array}{l}
  u_h \in V_h \\
  \|u - u_h\|_E \rightarrow \min
\end{array} \right.
\]

where \( \|v\|_E^2 = b(v, v). \)

You cannot do better! (in energy norm...)
1910 — (Bubnov) Galerkin method

1954 — numerical flux of P. Lax
1959 — Petrov–Galerkin method
1964 — Cea’s lemma
1969 — Mikhlin’s asymptotic stability
1971 — Babuska’s theorem
1974 — Brezzi’s theory
1980 — Barrett and Morton use Petrov–Galerkin to symmetrize
1981 — SUPG method of Brooks and Hughes, stabilized methods
1985 — D and Oden use PG to change the norm of convergence
1986 — Franca and Russo – bubble methods
1989 — DPG method of Cockburn and Shu

2009 — D and Gopalakrishnan – DPG method with optimal test functions
The supremum in the inf-sup condition defines an equivalent, problem-dependent energy (residual) norm,

\[ \|u\|_E := \sup_{\|v\|=1} |b(u, v)| = \|Bu\|_{V'}, \]

For the energy norm, \( M = \gamma = 1 \). Recalling that the Riesz operator is an isometry form \( V \) into \( V' \), we may characterize the energy norm in an equivalent way as

\[ \|u\|_E = \|v_u\|_V \]

where \( v_u \) is the solution of the variational problem,

\[
\begin{aligned}
& v_u \in V \\
& (v_u, \delta v)_V = b(u, \delta v) \quad \forall \delta v \in V
\end{aligned}
\]
Optimal Test Functions

Select your favorite trial basis functions: \( e_j, j = 1, \ldots, N \). For each function \( e_j \), introduce a corresponding optimal test (basis) function \( \bar{e}_j \in V \) that realizes the supremum,

\[
|b(e_j, \bar{e}_j)| = \sup_{\|v\|_V = 1} |b(e_j, v)|
\]
i.e. it solves the variational problem,

\[
\begin{cases}
\bar{e}_j \in V \\
(\bar{e}_j, \delta v)_V = b(e_j, \delta v) \quad \forall \delta v \in V
\end{cases}
\]

Define the discrete test space as \( \bar{V}_{hp} := \text{span}\{\bar{e}_j, j = 1, \ldots, N\} \subset V \). It follows from the construction of the optimal test functions that the discrete inf-sup constant

\[
\inf_{\|u_{hp}\|_E = 1} \sup_{\|v_{hp}\| = 1} |b(u_{hp}, v_{hp})| = 1
\]
Consequently, Babuška’s Theorem

\[ \| u - u_{hp} \|_E \leq \frac{M}{\gamma_{hp}} \inf_{w_{hp} \in U_{hp}} \| u - w_{hp} \|_E \]

implies that

\[ \| u - u_{hp} \|_E \leq \inf_{w_{hp} \in U_{hp}} \| u - w_{hp} \|_E \]

i.e., the method delivers the best approximation error in the energy norm.
\[ b(e_i, \bar{e}_j) = (\bar{e}_i, \bar{e}_j)_V = (\bar{e}_j, \bar{e}_i)_V = b(e_j, \bar{e}_i) \]
Energy Norm of FE Error \( e_{hp} = u - u_{hp} \)

can be computed \textit{without} knowing the exact solution.

\[
\begin{aligned}
\left\{ \begin{array}{l}
\quad v_{e_{hp}} \in V \\
(v_{e_{hp}}, \delta v)_V = b(u - u_{hp}, \delta v) = l(\delta v) - b(u_{hp}, \delta v) \quad \forall \delta v \in V
\end{array} \right.
\end{aligned}
\]

We have then

\[
\|e_{hp}\|_E = \|v_{e_{hp}}\|_V
\]

We shall call \( v_{e_{hp}} \) the \textit{error representation function}.

\textbf{Note:} No need for an a-posteriori error estimation.
Rewrite the variational problem in the operator form:

\[ Bu = l, \quad B : U \rightarrow V', \quad < Bu, v >= b(u, v) \]
Relation with Least Squares

Rewrite the variational problem in the operator form:

\[ Bu = l, \quad B : U \rightarrow V', \quad < Bu, v > = b(u, v) \]

Precondition with inverse of the Riesz operator \( R_V \),

\[ R_V^{-1} Bu = R_V^{-1} l, \quad R_V^{-1} B : U \rightarrow V \]
Rewrite the variational problem in the operator form:

\[ Bu = l, \quad B : U \to V', \quad < Bu, v > = b(u, v) \]

Precondition with inverse of the Riesz operator \( R_V \),

\[ R_V^{-1} Bu = R_V^{-1} l, \quad R_V^{-1} B : U \to V \]

Apply the least squares method

\[ \| R_V^{-1} Bu_{hp} - R_V^{-1} l \|_V \to \min \]
Rewrite the variational problem in the operator form:

\[ Bu = l, \quad B : U \to V', \quad < Bu, v > = b(u, v) \]

Precondition with inverse of the Riesz operator \( R_V \),

\[ R_V^{-1} Bu = R_V^{-1} l, \quad R_V^{-1} B : U \to V \]

Apply the least squares method

\[ \| R_V^{-1} Bu_{hp} - R_V^{-1} l \|_V \to \min \]

This is exactly our DPG method
Petrov-Galerkin Method with Optimal Test Functions.

Ultraweak variational formulation and the DPG method for convection-dominated diffusion.

1D analysis. Adaptivity.

Wave propagation as an example of a complex-valued problem.

Systematic choice of test norms. Robustness.

Convergence proofs.
A reminder:

How does the usual Bubnov–Galerkin method perform for 1D Convection?

\[
\begin{align*}
- \epsilon u'' + u' &= 0 \quad \text{in } (0, 1) \\
u(0) &= 1, \quad u(1) = 0
\end{align*}
\]
\[ \epsilon = 10^{-1} \]
Bubnov-Galerkin Method

\[ \epsilon = 10^{-2} \]
\[ \epsilon = 10^{-3} \]
Ultraweak Variational Formulation and DPG Method for 2D Confusion Problem
\[ \begin{aligned}
\frac{1}{\epsilon} \sigma - \nabla u &= 0 \quad \text{in } \Omega \\
-\operatorname{div}(\sigma - \beta u) &= f \quad \text{in } \Omega \\
u &= u_0 \quad \text{on } \partial \Omega
\end{aligned} \]
DPG Method

Elements: $K$
Edges: $e$
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma^0_h = \Gamma_h - \partial \Omega$
Take an element $K$. Multiply the equations with test functions $\tau \in H(\text{div}, K), v \in H^1(K)$:

\[
\begin{align*}
&\frac{1}{\varepsilon} \sigma \cdot \tau - \nabla u \cdot \tau = 0 \\
&-\text{div}(\sigma - \beta u)v = fv
\end{align*}
\]
Integrate over the element $K$:

\[
\begin{align*}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau - \nabla u \cdot \tau &= 0 \\
- \int_K \text{div}(\sigma - \beta u)v &= fv
\end{align*}
\]
Integrate by parts (relax) both equations:

\[
\begin{align*}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau + \int_K u \, \text{div} \tau - \int_{\partial K} u \, \tau_n &= 0 \\
\int_K (\sigma - \beta u) \cdot \nabla v - \int_{\partial K} q \, \text{sgn}(n) \, v &= \int_K f v
\end{align*}
\]

where \( q = (\sigma - \beta u) \cdot n_e \) and

\[
\text{sgn}(n) = \begin{cases} 
1 & \text{if } n = n_e \\
-1 & \text{if } n = -n_e
\end{cases}
\]
Declare traces and fluxes to be independent unknowns:

\[
\begin{align*}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau + \int_K u \, \text{div} \tau - \int_{\partial K} \hat{u} \, \tau_n &= 0 \\
- \int_K (\sigma - \beta u) \cdot \nabla v + \int_{\partial K} \hat{q} \, \text{sgn}(n)v &= \int_K f v
\end{align*}
\]
Use BC to eliminate known traces

\[
\begin{align*}
\int_K \frac{1}{\varepsilon} \sigma \cdot \tau + \int_K u \, \text{div} \tau - \int_{\partial K - \partial \Omega} \hat{u} \, \tau_n &= \int_{\partial K \cap \partial \Omega} u_0 \, \tau_n \\
- \int_K (\sigma - \beta u) \cdot \nabla v + \int_{\partial K} \hat{q} \, \text{sgn}(n) v &= \int_K f v
\end{align*}
\]
Trace and Flux Spaces

\[ \Gamma_h := \bigcup_K \partial K \quad \text{(skeleton)} \]
\[ \Gamma_0^h := \Gamma_h - \partial \Omega \quad \text{(internal skeleton)} \]
\[ \tilde{H}^{1/2}(\Gamma_0^h) := \{ V|_{\Gamma_0^h} : V \in H^1_0(\Omega) \} \]
with the minimum extension norm:
\[ \| v \|_{\tilde{H}^{1/2}(\Gamma_0^h)} := \inf \{ \| V \|_{H^1} : V|_{\Gamma_0^h} = v \} \]
\[ H^{-1/2}(\Gamma_h) := \{ \sigma_n|_{\Gamma_h} : \sigma \in H(\text{div}, \Omega) \} \]
with the minimum extension norm:
\[ \| \sigma_n \|_{H^{-1/2}(\Gamma_h)} := \inf \{ \| \sigma \|_{H(\text{div}, \Omega)} : \sigma n|_{\Gamma_h} = \sigma_n \} \]
DPG Method, a summary

\[
\left\{ \begin{array}{l}
\int_K \frac{1}{\epsilon} \sigma \cdot \tau + \int_K u \text{div}\tau - \int_{\partial K \setminus \partial \Omega} \hat{u} \tau_n = \int_{\partial K \cap \partial \Omega} u_0 \tau_n \\
- \int_K \sigma \cdot \nabla v + \int_{\partial K} \hat{q} \text{sgn}(n) v = \int_K f v
\end{array} \right.
\]

Main points:

- **Both** equations have been integrated by parts (relaxed).
- **Traces** $\hat{u} \sim u$ and **fluxes** $\hat{q} \sim (\sigma - \beta u) \cdot n_e$ are **independent unknowns** (DPG is a hybrid method).
- **Boundary conditions** have been built in.
- **Test functions** are **discontinuous** (come from “broken” Sobolev spaces). This is critical to enable the idea of using optimal test functions.
Group variables:
Solution $\mathbf{U} = (u, \sigma, \hat{u}, \hat{q})$:

\begin{align*}
&u, \sigma_1, \sigma_2 \in L^2(\Omega_h) \\
&\hat{u} \in \tilde{H}^{1/2}(\Gamma_0^h) \\
&\hat{q} \in H^{-1/2}(\Gamma_h)
\end{align*}

Test function $\mathbf{V} = (\tau, v)$:

\begin{align*}
&\tau \in H(\text{div}, \Omega_h) \\
v \in H^1(\Omega_h)
\end{align*}

Variational problem:

$$b(\mathbf{U}, \mathbf{V}) = l(\mathbf{V}), \quad \forall \mathbf{V}$$
\[
\begin{aligned}
\left\{ \frac{1}{\epsilon} (\sigma, \tau)_{\Omega} + (u, \text{div} \tau)_{\Omega_h} - <\hat{u}, \tau_n>_{\Gamma_0}^{h} \right. & \quad = <u_0, \tau_n>_{\partial \Omega}^{h} \\
& \left. - (\sigma, \nabla v)_{\Omega_h} - <\hat{q}, v>_{\Gamma_h} ight. \right. &= (f, v)_{\Omega} \\
& \ \ \\
& \left. b((u, \sigma, \hat{u}, \hat{q}), (\tau, v)) \right. &= (u, \text{div} \tau + \beta \cdot \nabla v)_{\Omega_h} + (\sigma, \frac{1}{\epsilon} \tau - \nabla v)_{\Omega_h} \\
& \left. - <\hat{u}, \tau_n>_{\Gamma_0}^{h} - <\hat{q}, v>_{\Gamma_h} \right. \right.
\end{aligned}
\]
DPG Method with Optimal Test Functions
If the test norm is localizable, i.e. 
\[(v, \delta v)_V \] 
\[V^K = \sum (v, \delta v)_V^K \]
where \((v, \delta v)_V^K\) defines an inner product for test functions over element \(K\),
then the determination of the optimal test functions is done locally. Given trial functions \(e_i\), we compute on the fly corresponding optimal test functions \(\hat{e}_i\) by solving element variational problems,
\[
\hat{e}_i \in V(K) \\
(\hat{e}_i, \delta v)_V = b(e_i, \delta v), \forall \delta v \in V(K)
\]
Solution of the local problem above can still be only approximated using an "enriched space" and standard Bubnov-Galerkin method.
If the test norm is **localizable**, i.e.

\[(v, \delta v)_V = \sum_K (v, \delta v)_{V_K}\]

where \((v, \delta v)_{V_K}\) defines an inner product for test functions over element \(K\),
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\[
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\hat{e}_i \in V(K) \\
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\end{cases}
\]
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\[
\begin{cases}
\hat{e}_i \in V(K) \\
(\hat{e}_i, \delta v)_V = b(e_i, \delta v), \quad \forall \delta v \in V(K)
\end{cases}
\]

Solution of the local problem above can still be only approximated using an “enriched space” and standard Bubnov-Galerkin method.
Challenges

▶ If the optimal test functions are not well approximated, some nice properties are lost.

*Crucial for $h$-refinements*
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- If the optimal test functions are not well approximated, some nice properties are lost.
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- How do we prove that the stability of the continuous problem is mesh independent*? √

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Challenges

- If the optimal test functions are not well approximated, some nice properties are lost.
- How do we prove that the continuous (hybrid) problem is well-posed? √
- How do we prove that the stability of the continuous problem is mesh independent? * √
- How do we choose the test norm so the method delivers results (is robust) in a norm we want? √

*Crucial for $h$-refinements
Nov 09  Proved: mesh independence for any 1D problem
Dec 09  Proved: robustness for 1D confusion with special weighted test norm
Jan 10  Developed: 1D and 2D hp-adaptive codes for the confusion problem and broke solvability records; $\epsilon = 10^{-11}$ for 1D, and $\epsilon = 10^{-7}$ for 2D problems.
Mar 10  Discovered: concept of optimal and (practical) quasi-optimal test norm.
Jul 10  Solved: 1D Burgers and N-S eqns with $\epsilon = 10^{-11}$ and $\epsilon = 10^{-10}$.
Aug 10  Proved: mesh independence and well-posedness (but not robustness) for nD confusion.
Jun 11  Developed a strategy for constructing robust DPG methods, and proved robustness for nD confusion.
Sample test norm(s)

Mathematician’s test norm:
\[
\|(v, \tau)\|_1^2 := \|v\|^2 + \|\nabla v\|^2 + \|\tau\|^2 + \|\text{div}\tau\|^2
\]

Weighted norm:
\[
\|(v, \tau)\|_2^2 := \|v\|_w^2 + \|\nabla v\|_w^2 + \|\tau\|_w^2 + \|\text{div}\tau\|_w^2
\]

Quasi-optimal test norm:
\[
\|(v, \tau)\|_3^2 := \|v\|^2 + \left\| \frac{1}{\epsilon} \tau + \nabla v \right\|^2 + \|\text{div}\tau - \beta \cdot \nabla v\|^2
\]

Weighted norm revisited:
\[
\|(v, \tau)\|_4^2 := \epsilon \|v\|^2 + \|\beta \cdot \nabla v\|_w^2 + \epsilon \|\nabla v\|_w^2 + \|\tau\|_w^2 + \|\text{div}\tau\|_w^2
\]
Residual equals energy norm of the error:

\[ \| u - u_h \|_E^2 = \| Bu_h - l \|_{V'}^2 = \| \underbrace{R_V^{-1}(Bu_h - l)}_{:= \psi} \|_V^2 = \sum_K \| \psi_K \|_{V_K}^2 \]

where the element error representation function \( \psi_K \) is determined by solving,

\[
\begin{aligned}
\psi_K & \in V_K \\
(\psi_K, \delta v)_{V_K} & = b(u - u_h, \delta v) = l(\delta v) - b(u_h, \delta v), \quad \delta v \in V_K
\end{aligned}
\]
Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.
1D Confusion Problem:
Ultraweak variational formulation and the DPG method
1D analysis and adaptivity
1D model problem:

\[
\begin{align*}
    u(0) &= u_0, & u(1) &= 0 \\
    \frac{1}{\epsilon} \sigma - u' &= 0 \\
    -\sigma' + u' &= f
\end{align*}
\]
DPG Method for 1D Confusion

Pick an element:

\[ 0 \quad x_{k-1} \quad x_k \quad 1 \]

Multiply the equations with test functions:

\[ \frac{1}{\epsilon} \sigma \tau - u' \tau = 0 \]
\[ -\sigma' v + u' v = f v \]
DPG Method for 1D Confusion

Pick an element:

0 \quad x_{k-1} \quad x_k \quad 1

Integrate over the element:

\frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma \tau - \int_{x_{k-1}}^{x_k} u' \tau = 0

- \int_{x_{k-1}}^{x_k} \sigma' v + \int_{x_{k-1}}^{x_k} u' v = \int_{x_{k-1}}^{x_k} f v
DPG Method for 1D Confusion

Pick an element:

\[ 0 \quad x_{k-1} \quad x_k \quad 1 \]

Integrate by parts:

\[
\frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma \tau + \int_{x_{k-1}}^{x_k} u \tau' - [u \tau]_{x_{k-1}}^{x_k} = 0
\]

\[
\int_{x_{k-1}}^{x_k} \sigma v' - [\sigma v]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u v' + [uv]_{x_{k-1}}^{x_k} = \int_{x_{k-1}}^{x_k} f v
\]
DPG Method for 1D Confusion

Pick an element:

\[ 0 \quad x_{k-1} \quad x_k \quad 1 \]

Declare fluxes to be independent unknowns:

\[
\frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma \tau + \int_{x_{k-1}}^{x_k} u \tau' - [\hat{\tau}]_{x_{k-1}}^{x_k} = 0
\]

\[
\int_{x_{k-1}}^{x_k} \sigma v' - [\hat{v}]_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u v' + [\hat{u} v]_{x_{k-1}}^{x_k} = \int_{x_{k-1}}^{x_k} f v
\]
Pick an element:

\[ 0 \]
\[ \underline{x_{k-1}} \quad \underline{x_k} \]

For elements adjacent to the boundary use the BC's to move known fluxes to the RHS:

\[
\frac{1}{\epsilon} \int_{x_0}^{x_1} \sigma \tau + \int_{x_0}^{x_1} u \tau' - \hat{u}(x_1) \tau(x_1) = -u_0 \tau(0)
\]

\[
\int_{x_0}^{x_1} \sigma \nu' - [\hat{\sigma} \nu]|_{x_0}^{x_1} - \int_{x_0}^{x_1} uv' + \hat{u}(x_1)v(x_1) = \int_{x_0}^{x_1} f v + u_0 v(x_0)
\]
DPG Method for 1D Confusion

Pick an element:

\[ 0 \hspace{1cm} x_{k-1} \hspace{1cm} x_k \hspace{1cm} 1 \]

Sum up over elements:

\[
\frac{1}{\epsilon} \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \sigma \tau + \int_{x_j}^{x_0} u \tau' - \sum_{j=1}^{N-1} \hat{u}(x_j)[\tau(x_j)] - \hat{u}(x_N)\tau(x_N) = -u_0\tau(0)
\]

\[
\int_{x_0}^{x_1} \sigma v' - [\hat{\sigma} v]_{x_0}^{x_1} - \int_{x_1}^{x_0} u v' + \hat{u}(x_1)v(x_1) = \int_{x_{k-1}}^{x_k} f v + u_0
\]
For each \( k = 1, \ldots, N \),

\[
\begin{align*}
\frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma_k \tau 
&+ \int_{x_{k-1}}^{x_k} u_k \tau' 
- (\hat{u}\tau)|_{x_{k-1}}^{x_k} 
= 0 \\
\int_{x_{k-1}}^{x_k} \sigma_k v' 
&+ (\hat{\sigma}v)|_{x_{k-1}}^{x_k} 
- \int_{x_{k-1}}^{x_k} u_k v' 
+ (\hat{u}v)|_{x_{k-1}}^{x_k} 
= \int_{x_{k-1}}^{x_k} f v
\end{align*}
\]

for every optimal test function \( \tau, v \). For \( k = 1 \), \( \hat{u}(0) = u_0 \) is known and is moved to the right-hand side. Similarly, \( \hat{u}(1) = 0 \) in the last equation for \( k = N \).
Optimal Test Functions

\[
\begin{align*}
(\tau_k, \delta \tau)_k &= \frac{1}{\epsilon} \int_{x_{k-1}}^{x_k} \sigma_k \delta \tau + \int_{x_{k-1}}^{x_k} u_k \delta \tau' - (\hat{u} \delta \tau)|_{x_{k-1}}^{x_k} \quad \forall \delta \tau \\
(u_k, \delta u)_k &= \int_{x_{k-1}}^{x_k} \sigma_k u' - (\hat{\sigma} \delta u)|_{x_{k-1}}^{x_k} - \int_{x_{k-1}}^{x_k} u_k u' + (\hat{u} \delta u)|_{x_{k-1}}^{x_k} \quad \forall \delta u
\end{align*}
\]

where \((\cdot, \cdot)_k\) is an inner product for \(k\)-th element.

Optimal test function corresponding to flux \(\hat{u}(x_k) = 1\) and

\[
(u, v)_k = \int_{x_{k-1}}^{x_k} u' v' \, dx + u(x_k)v(x_k)
\]
Approximate Optimal Test Functions

**Practical approach:**
Solve for the optimal test functions in an *enriched space*:

$$\mathcal{P}^{p+\Delta p}(K)$$

with a globally defined $\Delta p$.

**Warning:**
This should not be confused with using $\mathcal{P}^{p+\Delta p}(K)$ for the test space. The optimal test functions constitute only a proper subspace of $\mathcal{P}^{p+\Delta p}(K)$.
Globally and Locally Optimal Test Functions in 1D (Issue: mesh independence)
Formulation with continuous test functions:

\[
\begin{align*}
\sigma, u &\in L^2(0,1), \hat{\sigma}(0), \hat{\sigma}(1) \in \mathbb{R} \\
\frac{1}{\epsilon} \int_0^1 \sigma \tau + \int_0^1 u\tau' &= u_0 \tau(0) \quad \forall \tau \in H^1(0,1) \\
\int_0^1 \sigma v' + [\hat{\sigma} v]_{0}^{1} - \int_0^1 uv' &= u_0 v(0) \quad \forall v \in H^1(0,1)
\end{align*}
\]

requires no interelement fluxes but leads to a global problem for the optimal test functions:

\[
\begin{align*}
\tau, v &\in H^1(0,1) \\
\int_0^1 \tau' \delta \tau' + \tau \delta \tau &= \frac{1}{\epsilon} \int_0^1 \sigma \delta \tau + \int_0^1 u \delta \tau' \quad \forall \delta \tau \in H^1(0,1) \\
\int_0^1 v' \delta v' + v \delta v &= \int_0^1 \sigma \delta v' + [\hat{\sigma} \delta v]_{0}^{1} - \int_0^1 u \delta v' \quad \forall \delta v \in H^1(0,1)
\end{align*}
\]

The resulting stiffness matrix is full but the resulting energy norm is mesh independent!
Q: A relation between the globally and locally optimal test functions?
\[
\begin{align*}
-\tau'' + \tau &= \frac{1}{\epsilon} \sigma - u' \quad \text{in } \mathcal{D}'(0, 1) \\
-v'' + v &= (-\sigma - u)' \quad \text{in } \mathcal{D}'(0, 1)
\end{align*}
\]
Globally Optimal Test Functions

Equivalently,

\[
\begin{align*}
-\tau'' + \tau &= \frac{1}{\epsilon} \sigma - u' & \text{in} \ (x_{k-1}, x_k), \ k = 1, \ldots, N \\
[\tau' - u] &= 0 & \text{at} \ x_k, \ k = 1, \ldots, N - 1 \\
-v'' + v &= (-\sigma - u)' & \text{in} \ (x_{k-1}, x_k), \ k = 1, \ldots, N \\
[v' - \sigma + u] &= 0 & \text{at} \ x_k, \ k = 1, \ldots, N - 1
\end{align*}
\]
With boundary conditions,

\[
\begin{align*}
-\tau'' + \tau &= \frac{1}{\epsilon} \sigma - u' \quad \text{in } (x_{k-1}, x_k), \ k = 1, \ldots, N \\
[\tau' - u] &= 0 \quad \text{at } x_k, \ k = 1, \ldots, N - 1 \\
\tau' - u &= 0 \quad \text{at } x_0, x_N \\
-v'' + v &= (-\sigma - u)' \quad \text{in } (x_{k-1}, x_k), \ k = 1, \ldots, N \\
[v' - \sigma + u] &= 0 \quad \text{at } x_k, \ k = 1, \ldots, N - 1 \\
v' - \sigma + u &= \hat{\sigma} \quad \text{at } x_0, x_N
\end{align*}
\]
Multiply with discontinuous test functions $\delta \tau, \delta v$ and integrate over individual elements,

\[
\begin{align*}
\int_{x_{k_1}}^{x_k} (-\tau'' + \tau) \delta \tau &= \int_{x_{k_1}}^{x_k} \left( \frac{1}{\epsilon} \sigma - u' \right) \delta \tau \\
\int_{x_{k_1}}^{x_k} (-\nu'' + \nu) \delta \nu &= \int_{x_{k_1}}^{x_k} (-\sigma + u)' \delta \nu
\end{align*}
\]
Integrate by parts,

\[
\begin{align*}
\int_{x_{k_1}}^{x_k} \tau' \delta \tau' + \tau \delta \tau &= \int_{x_{k_1}}^{x_k} \frac{1}{\epsilon} \sigma \delta \tau + ud\tau' + (\tau' - u) \delta \tau \bigg|_{x_{k-1}}^{x_k} \\
\int_{x_{k_1}}^{x_k} \nu' \delta \nu' + \nu \delta \nu &= \int_{x_{k_1}}^{x_k} (\sigma - u) \delta \nu' + (\nu' - \sigma + u) \delta \nu \bigg|_{x_{k-1}}^{x_k}
\end{align*}
\]
Globally Optimal Test Functions

Sum up over elements using interface and boundary conditions

\[
\begin{align*}
\sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} \tau' \delta \tau' + \tau \delta \tau &= \sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} \frac{1}{\epsilon} \sigma \delta \tau + u \delta \tau' \\
+ \sum_{k=1}^{N-1} (\tau' - u)[\delta \tau](x_k) \\
\sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} \nu' \delta \nu' + \nu \delta \nu &= \sum_{k=1}^{N} \int_{x_{k_1}}^{x_k} (\sigma - u) \delta \nu' \\
+ \sum_{k=1}^{N-1} (\nu' - \sigma + u)[\delta \nu](x_k) + (\hat{\sigma} \delta \nu)|_0^1
\end{align*}
\]
Conclusion:
The globally optimal test function corresponding to an \( hp \) trial shape function \((\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1))\) is a linear combination of the corresponding locally optimal test function corresponding to the same trial function and locally optimal test functions corresponding to fluxes \((\tau' - u), (v' - \sigma + u)\) at interelement boundaries \(x_k\).

Remark: The result is true for any 1D problem but it does not generalize to multidimensions where the globally optimal test functions can only be approximated with the locally optimal ones.
**Theorem**

Test space corresponding to formulation with globally conforming test functions is contained in the DPG test space. Consequently, the FE solutions corresponding to both formulations are identical. Part of the energy norm corresponding to the DPG formulation and unknowns \((\sigma, u, \hat{\sigma}(0), \hat{\sigma}(1))\) coincides with the energy norm corresponding to the globally optimal test functions and, therefore, is mesh independent.
A related result:

**Theorem**
The error representation function corresponding to the DPG formulation is globally conforming (continuous).
(A great check for the control of round off error...)
Notation:

\[ \mathcal{U} = (\sigma, u, \hat{\sigma}, \hat{u}) \]  exact solution

\[ \mathcal{U}_{hp} \]  approximate solution

\[ (x_{k-1}, x_k), (x_k, x_{k+1}) \]  neighboring elements

\[ (\tau_{\hat{u}_k}, v_{\hat{u}_k}) \]  optimal test function corresponding to flux \( \hat{u}_k \)

Orthogonality condition for the error function \( \mathcal{E}_{hp} := \mathcal{U} - \mathcal{U}_{hp} \):

\[
\begin{align*}
    b(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k})) \\
    = b_k(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k})) + b_{k+1}(\mathcal{U} - \mathcal{U}_{hp}, (\tau_{\hat{u}_k}, v_{\hat{u}_k})) = 0
\end{align*}
\]

where \( b_k \) denotes \( k \)-th element contribution to the global bilinear form.
Let \((\phi_k, \psi_k)\) be the error representation function for the \(k\)-th element,
\[
((\phi_k, \psi_k), (\delta \phi, \delta \psi))_k = b_k(\mathcal{E}_{hp}, (\delta \phi, \delta \psi), \forall (\delta \phi, \delta \psi)
\]
The error orthogonality condition implies then
\[
((\phi_k, \psi_k), (\tau \hat{u}_k, v \hat{u}_k))_k + ((\phi_{k+1}, \psi_{k+1}), (\tau \hat{u}_k, v \hat{u}_k))_{k+1} = b(\mathcal{U} - \mathcal{U}_{hp}, (\tau \hat{u}_k), v \hat{u}_k) = 0
\]
On the other side, it follows from the definition of optimal test functions that
\[
((\tau \hat{u}_k, v \hat{u}_k), (\delta \phi, \delta \psi))_k = \delta \psi(x_k), \quad \forall (\delta \phi, \delta \psi)
\]
and
\[
((\tau \hat{u}_k, v \hat{u}_k), (\delta \phi, \delta \psi))_{k+1} = -\delta \psi(x_k), \quad \forall (\delta \phi, \delta \psi)
\]
Selecting \((\delta \phi, \delta \psi) = (\phi_k, \psi_k)\) and \((\phi_{k+1}, \psi_{k+1})\) above, and summing up the two equations, we get

\[
\psi_k(x_k) - \psi_{k+1}(x_k) = 0
\]

In the same way we prove continuity of \(\phi\).

Important consequence: solution of the global problem

\[
\begin{cases}
(\phi, \psi) \in H^1(0, 1) \\
((\phi, \psi), (\delta \phi, \delta \psi)) = b(E_{hp}, (\delta \phi, \delta \psi)) \forall (\delta \phi, \delta \psi) \in H^1(0, 1)
\end{cases}
\]

where \((\phi, \psi) = \sum_{k=1}^{N}(\phi, \psi)_k\), leads to the same error representation function.

**Conclusion:** If \((\phi, \psi)\) is mesh independent then so is the energy norm of the FE error. Consequently, both \(h\) and \(p\)-refinements must lead to the decrease of the energy error.
Consider the spectral (one element) case and two norms for test functions

\[ \|v\|_1^2 = \int_0^1 |v'|^2 w(x) \, dx + |v(1)|^2 \]
\[ \|v\|_2^2 = \int_0^1 (|v'|^2 + |v|^2) w(x) \, dx \]

where \( w(x) \) is a weight function. Under appropriate conditions on \( w(x) \), the two norms are equivalent with order 1 equivalence constants. The energy norm corresponding to the first norm can be computed analytically

\[ \| (\sigma, \hat{\sigma}(0), \hat{\sigma}(1), u) \|_E^2 = \]
\[ \| \frac{1}{\epsilon} \int_x^1 \sigma + u \|_{L^2_{1/w}}^2 + \| \frac{1}{\epsilon} \int_0^1 \sigma \|_{L_{1/w}}^2 + \| \sigma - u - \hat{\sigma}(0) \|_{L^2_{1/w}}^2 + |\hat{\sigma}(0) - \hat{\sigma}(1)|^2 \]

The second test norm is localizable.
**Theorem**

Let

\[ w(x) = \max\{x, \epsilon\} \]

Then there exists an order one constant \( C \) such that:

\[ \|\sigma\|_{L^2}, \|u\|_{L^2} \leq C\|(\sigma, \hat{\sigma}(0), \hat{\sigma}(1), u)\|_E \]

By the equivalence of the two test norms, the result holds also for the energy norm corresponding to the localizable test norm.
\[
\|\sigma - \sigma_{hp}\|_{L^2}, \quad \|u - u_{hp}\|_{L^2} \lesssim \|(\sigma - \sigma_{hp}, u - u_{hp}, \hat{\sigma} - \hat{\sigma}_{hp}, \hat{u} - \hat{u}_{hp})\|_E = \inf_{(\sigma_{hp}, u_{hp}, \hat{\sigma}_{hp}, \hat{u}_{hp})} \|(\sigma - \sigma_{hp}, u - u_{hp}, \hat{\sigma} - \hat{\sigma}_{hp}, \hat{u} - \hat{u}_{hp})\|_E
\]

estimate needed
Set $\alpha = 0.5$
Do while $\alpha < 0.1$
    Solve the problem on the current mesh
    For each element $K$ in the mesh
        Compute element error contribution $e_K$
    end of loop through elements
    For each element $K$ in the mesh
        if $e_K > \alpha^2 \max_K e_K$ then
            if new $h \leq \epsilon$ then
                $h$-refine the element
            elseif new $p \leq p_{max}$ then
                $p$-refine the element
            endif
        endif
    end of loop through elements
if (new $N_{dof} = \text{old } N_{dof}$) reset $\epsilon = \epsilon/2$
end of loop through mesh refinements
Convergence History for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{\text{max}} = 4$

SCALES: log(nr dof), log(error)
Resolution of boundary layer for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{max} = 4$
Error representation function $\phi$ for $\epsilon = 10^{-3}$

$\Delta p = 4$, $p_{max} = 4$
Convergence History for $\epsilon = 10^{-6}$

$\Delta p = 4, p_{max} = 4$

![Graph showing convergence history with scale log(nrdof), log(error)]
Resolution of boundary layer for $\epsilon = 10^{-6}$

$\Delta p = 4\,\text{, } p_{max} = 4$
Error representation function $\phi$ for $\epsilon = 10^{-6}$

$\Delta p = 4$, $p_{max} = 4$
For $\epsilon = 10^{-7}$ the method falls apart...
A Remedy

Use a rescaled inner product:

\[(v, \delta v) = \int_{x_{k-1}}^{x_k} (h_k v' \delta v' + v \delta v) w(x) \, dx\]

With the rescaled inner product, convergence is no longer guaranteed to be monotone (theory, in practice is...).
Convergence History for $\epsilon = 10^{-6}$ and Rescaled Inner Product
Rescaled Inner Product and $\epsilon = 10^{-6}$

Increment in order to solve local problems $\Delta p = 4$, $p_{max} = 4$
\( \phi \) for \( \epsilon = 10^{-6} \) and Rescaled Inner Product

Increment in order to solve local problems \( \Delta p = 4, \ p_{max} = 4 \)
With the rescaled inner product, we can solve the problem for $\epsilon = 10^{-11}$.

It is possible to work with $h^\theta$, $1 < \theta < 2$ in the rescaled norm but not with $\theta = 2$ (produces wrong refinements).
2D Confusion Problem
\[ \begin{aligned}
\frac{1}{\varepsilon} \sigma - \nabla u &= 0 \quad \text{in } \Omega \\
- \text{div}\sigma + \text{div}(\beta u) &= f \quad \text{in } \Omega \\
u &= u_0 \quad \text{on } \partial\Omega
\end{aligned} \]
2D Convection-Dominated Diffusion

Problem definition.
DPG Formulation

\[
\begin{aligned}
\frac{1}{\epsilon} \int_K \sigma \tau + \int_K u \text{div} \tau - \int_{\partial K} \hat{u} \tau_n &= 0 \\
\int_K \sigma \nabla v - \int_{\partial K} \hat{\sigma} v - \int_K u \beta \cdot \nabla v + \int_{\partial K} \hat{u} \beta v &= \int_K f v
\end{aligned}
\]

Energy setting:

\[\tau \in H_w(\text{div}, K), \; v \in H^1_w(K),\]

\[\sigma \in L^2_{1/w}(K), \; u \in L^2_{1/w}(K),\]

\[\hat{u} \in \tilde{H}^{1/2}(\Gamma^0_h), \hat{\sigma}_n \in H^{-1/2}(\Gamma_h)\]
Flux (Trace) Spaces

\[ \Gamma := \bigcup_K \partial K \quad \text{(skeleton)} \]

\[ \Gamma_0 := \Gamma - \partial \Omega \quad \text{(internal skeleton)} \]

\[ H^{1/2}(\Gamma_0) := \{ V|_{\Gamma_0} : V \in H^1_0(\Omega) \} \]

with the minimum extension norm:

\[ \| v \|_{H^{1/2}(\Gamma_0)} := \inf \{ \| V \|_{H^1} : V|_{\Gamma_0} = v \} \]

\[ H^{-1/2}(\Gamma) := \{ \sigma_n|_{\Gamma} : \sigma \in H(\text{div}, \Omega) \} \]

with the minimum extension norm:

\[ \| \sigma_n \|_{H^{-1/2}(\Gamma)} := \inf \{ \| \sigma \|_{H(\text{div}, \Omega)} : \sigma n|_{\Gamma} = \sigma_n \} \]
Stability Result

Let $w = 1$ (no weight).

**Theorem** [D, Gopalakrishnan, Sep 2010]
The DPG variational formulation for 2D or 3D confusion problems is well-posed with the inf-sup constant independent of mesh.

**Corollary 1:** There exists $C > 0$:

$$
\| \sigma - \sigma_{hp} \|_{L^2(\Omega)} + \| u - u_{hp} \|_{L^2(\Omega)} \\
+ \| \hat{\sigma}_n - \hat{\sigma}_{n, hp} \|_{H^{-1/2}(\Gamma)} + \| \hat{u} - \hat{u}_{hp} \|_{H^{1/2}(\Gamma_0)} \\
\leq C \inf_{\sigma_{hp}, u_{hp}, \hat{\sigma}_{n, hp}, \hat{u}_{hp}} [...]
$$

Robustness requires use of weighted norms and appropriate norms for fluxes (in progress...)
discretization

triangles:

\[ \sigma_i, u \in \mathcal{P}^p(K), \quad \hat{\sigma}_n, \hat{u} \in \mathcal{P}^{p_e}(e) \]

quadrilaterals:

\[ \sigma_i, u \in Q^{(p,q)}(K) := \mathcal{P}^p(K) \otimes \mathcal{P}^q(K), \quad \hat{\sigma}_n, \hat{u} \in \mathcal{P}^{p_e}(e) \]

Max rule for determining approximation for fluxes:

triangles: \( p_e = \max\{p_1, p_2, p_3\} + 1 + \Delta p_e \)

quadrilaterals: \( p_e = \max\{q_1, q_2, q_3\} + \Delta p_e \) (accounting for directionality)

(piecewise polynomials used for 2-1 edges)

Convergence result indicates that we should use

\[ \Delta p_e = 1 \]
Assume $w = 1$ and uniform $h$-refinements.

**Theorem**

For elements of order $p$ and fluxes of order $p + 1$,

\[
\| \sigma - \sigma_{hp} \|_{L^2(\Omega)} + \| u - u_{hp} \|_{L^2(\Omega)} \\
+ \| \hat{\sigma}_n - \hat{\sigma}_{n, hp} \|_{H^{-1/2}(\Gamma)} + \| \hat{u} - \hat{u}_{hp} \|_{H^{1/2}(\Gamma_0)} \\
\leq C h^p
\]
Norm for Test Functions

\[ \| (\tau, v) \|_K^2 = \int_K \left\{ \left| \frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} \right|^2 + \tau_1^2 + \tau_2^2 \\
+ \left| \frac{\partial v}{\partial x_1} \right|^2 + \left| \frac{\partial v}{\partial x_2} \right|^2 + v^2 \right\} w(x) \, dx \]

Definition of weight function

\[ w = 1.0 \]

\[ w = 0.1 \]
Computation of Anisotropy Factor

Computation of error function

\[ \begin{cases} (\tau, v) \in V_K \\ ((\tau, v), (\delta \tau, \delta v)) = b_K(U_{hp}, (\delta \tau, \delta v)) - l_K((\delta \tau, \delta v)) \forall (\delta \tau, \delta v) \in V_K \end{cases} \]

\[ c_1 = \int_K (|\tau_1|^2 + |\frac{\partial v}{\partial x_1}|^2) w(x) \, dx \quad c_2 = \int_K (|\tau_2|^2 + |\frac{\partial v}{\partial x_2}|^2) w(x) \, dx \]

Refinement flag = \[ \begin{cases} 10 & \text{if } c_1 \geq 10c_2 \\ 01 & \text{if } c_2 \geq 10c_1 \\ 11 & \text{otherwise} \end{cases} \]
\[ \epsilon = 10^{-3}, \text{ Triangles} \]
$\epsilon = 10^{-3}$, Triangles

Final mesh after 21 refinements
$\epsilon = 10^{-3}$, Triangles

$10^{-2} \times$ zoom on upper boundary
$\epsilon = 10^{-3}$, Triangles

$10^{-2} \times$ zoom on north-east corner
$\epsilon = 10^{-3}$, Triangles

Solution $u$
$\epsilon = 10^{-3}$, Triangles

$10^2 \times$ zoom on upper boundary. Solution $u$ with the mesh
$\epsilon = 10^{-3}$, Triangles

$10^2 \times$ zoom on upper boundary. Solution $u$ without the mesh
\( \epsilon = 10^{-3} \), Triangles

10^2 \times zoom on north-east corner. Solution \( u \) with the mesh
$\epsilon = 10^{-3}$, Triangles

$10^2 \times$ zoom on north-east corner. Solution $u$ without the mesh
Limit for Triangles

\[ \epsilon = 10^{-4}, \text{ almost 1M d.o.f.} \]

(4 years old IBM Think Pad, 1Gb memory, frontal solver for a symmetric problem, no pivoting)
$\epsilon = 10^{-4}$, Quads

Convergence history
Limit for Quads with Standard Inner Product

$\epsilon = 10^{-5}$, almost 0.5M d.o.f.

with the following limitations:

- $L^2$-contribution scaled with factor 10
- aspect ratio $h_1/h_2 \leq 100$
- $p_{max} = 4$
\[ \| (\tau, v) \|_K^2 = \int_K \left\{ \left| \sqrt{h_1} \frac{\partial \tau_1}{\partial x_1} + \sqrt{h_2} \frac{\partial \tau_2}{\partial x_2} \right|^2 + |\tau_1|^2 + |\tau_2|^2 + h_1 \left| \frac{\partial v}{\partial x_1} \right|^2 + h_2 \left| \frac{\partial v}{\partial x_2} \right|^2 + |v|^2 \right\} w(x) \, dx \]
$\epsilon = 10^{-7}$

Convergence history for the redefined norm
$\epsilon = 10^{-7}$

Optimal $h_p$ mesh after 45 mesh refinements.
$\epsilon = 10^{-7}$

Optimal $hp$ mesh after 45 mesh refinements. Zoom $\times 10$ on the north-east corner.
\( \epsilon = 10^{-7} \)

Optimal \( h_p \) mesh after 45 mesh refinements. Zoom \( \times 100 \) on the north-east corner.
\( \epsilon = 10^{-7} \)

Optimal \( h_p \) mesh after 45 mesh refinements. Zoom \( \times 1000 \) on the north-east corner.
$\epsilon = 10^{-7}$

Optimal $hp$ mesh after 45 mesh refinements. Zoom $\times 10000$ on the north-east corner.
Optimal $h_p$ mesh after 45 mesh refinements. Zoom $\times 10^5$ on the north-east corner.
\[ \epsilon = 10^{-7} \]

Velocity \( u \).
\( \epsilon = 10^{-7} \)

Velocity \( u \). Zoom \( \times 10^5 \) on the north-east corner.
Velocity $u$. Zoom $\times 10^6$ on the north-east corner with the mesh.
Velocity $u$. Zoom $\times 10^6$ on the north-east corner w/o the mesh. OK, is not ideal yet...
Limitations

- aspect ratio $h_1/h_2 \leq 10000$
- $p_{max} = 4$
Petrov-Galerkin Method with Optimal Test Functions.

Ultraweak variational formulation and the DPG method for convection-dominated diffusion.

1D analysis. Adaptivity.

Wave propagation as an example of a complex-valued problem.

Systematic choice of test norms. Robustness.

Convergence proofs.
Ultraweak Variational Formulation and DPG Method for Linear Acoustics
DPG method

Linear acoustics in frequency domain:

\[
\begin{align*}
 i\omega u + \nabla p &= 0 \\
 i\omega p + \text{div} u &= 0
\end{align*}
\]

with, e.g. hard boundary condition:

\[ u_n = g \]
Elements: $K$
Edges: $e$
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma_0 = \Gamma_h - \partial \Omega$
Take an element $K$. Multiply the equations with test functions $v \in H(\text{div}, K), q \in H^1(K)$:

\[
\begin{aligned}
i \omega u \cdot v + \nabla p \cdot v &= 0 \\
i \omega p q + \text{div} u q &= 0
\end{aligned}
\]
Integrate over the element $K$:

\[
\begin{align*}
&i\omega \int_{K} \mathbf{u} \cdot \mathbf{v} + \int_{K} \nabla p \cdot \mathbf{v} = 0 \\
&i\omega \int_{K} p q + \int_{K} \text{div} \mathbf{u} \ q = 0
\end{align*}
\]
Integrate by parts (relax) \textit{both} equations:

\[
\begin{align*}
    i\omega \int_K u \cdot v - \int_K p \cdot \text{div} v + \int_{\partial K} pv_n &= 0 \\
    i\omega \int_K p q - \int_K u \cdot \nabla q + \int_{\partial K} u_n q \text{sgn}(n) &= 0
\end{align*}
\]

where \( u_n = u \cdot n_e \) and

\[\text{sgn}(n) = \begin{cases} 
1 & \text{if } n = n_e \\
-1 & \text{if } n = -n_e
\end{cases}\]
Declare traces and fluxes to be independent unknowns:

\[
\begin{align*}
    i\omega \int_K u \cdot v - \int_K p \cdot \text{div}v + \int_{\partial K} \hat{p}v_n &= 0 \\
    i\omega \int_K pq - \int_K u \cdot \nabla q + \int_{\partial K} \hat{u}_n q \text{sgn}(n) &= 0
\end{align*}
\]
Use BCs to eliminate known fluxes

\[
\begin{align*}
    i\omega \int_K \mathbf{u} \cdot \mathbf{v} - \int_K p \cdot \text{div}\mathbf{v} + \int_{\partial K} \hat{p} v_n &= 0 \\
    i\omega \int_K pq - \int_K \mathbf{u} \cdot \nabla q + \int_{\partial K - \Gamma} \hat{u}_n q \text{sgn}(n) &= \int_{\partial K \cap \Gamma} g q
\end{align*}
\]
Sum up over all elements and replace $\nu, q$ with $\bar{\nu}, \bar{q}$ to comply with the sesquilinear forms setting,

\[
\left\{
\begin{array}{l}
\iota \omega (u, v)_\Omega - (u, \text{div} v)_{\Omega_h} + < \hat{p}, v_n >_{\Gamma_h} = 0 \\
\iota \omega (p, q)_\Omega - (u, \nabla q)_{\Omega_h} + < \hat{u}_n, q >_{\Gamma_h^0} = < g, q >_{\Gamma}
\end{array}
\right.
\]
Trace and Flux Spaces

\[ \Gamma_h := \bigcup_K \partial K \quad \text{(skeleton)} \]

\[ \Gamma_0^h := \Gamma_h - \partial \Omega \quad \text{(internal skeleton)} \]

\[ H^{1/2}(\Gamma_h) := \{ q|_{\Gamma_h} : q \in H^1(\Omega) \} \]

with the minimum extension norm:

\[ \| q \|_{H^{1/2}(\Gamma_h)} := \inf \{ \| Q \|_{H^1} : Q|_{\Gamma_h} = q \} \]

\[ \tilde{H}^{-1/2}(\Gamma_0^h) := \{ v_n|_{\Gamma_h} : v \in H_0(\text{div}, \Omega) \} \]

with the minimum extension norm:

\[ \| v_n \|_{\tilde{H}^{-1/2}(\Gamma_0^h)} := \inf \{ \| V \|_{H_0(\text{div}, \Omega)} : V \cdot n|_{\Gamma_0^h} = \sigma_n \} \]
Functional Setting

Group variables:
Solution \( \mathbf{U} = (u, p, \hat{u}_n, \hat{p}) \):

\[
\begin{align*}
    u_1, u_2, p &\in L^2(\Omega_h) \\
    \hat{u}_n &\in \tilde{H}^{-1/2}(\Gamma_0^h) \\
    \hat{p} &\in \tilde{H}^{1/2}(\Gamma_h)
\end{align*}
\]

Test function \( \mathbf{V} = (v, q) \):

\[
\begin{align*}
    v &\in H(\text{div}, \Omega_h) \\
    q &\in H^1(\Omega_h)
\end{align*}
\]

Sesquilinear form
\[
\begin{align*}
b(\mathbf{U}, \mathbf{V}) &= -(u, i\omega v + \nabla q)_{\Omega_h} - (p, i\omega q + \text{div} v)_{\Omega_h} \\
    &\quad + <\hat{u}_n, q>_{\Gamma_0^h} + <\hat{p}, v_n>_{\Gamma_h}
\end{align*}
\]
Local invertibility of Riesz operator
Due to the use of “broken” Sobolev spaces (discontinuous test functions), the Riesz operator is inverted elementwise! Given any (linear) problem, and any trial shape functions, we compute the corresponding optimal test functions on the fly.

Approximate optimal test functions
The locally determined optimal test functions still need to be approximated. This is done using standard Bubnov-Galerkin method and an enriched space. If polynomials of order $p$ are used to approximate the unknown velocity and pressure, the approximate optimal test functions are determined using polynomials of order:

$$p + \Delta p$$
Quasi-optimal test norm

Trial norm:
\[
\| (u, p, \hat{u}_n, \hat{p}) \|_U^2 = \| u \|_{L^2}^2 + \| p \|_{L^2}^2 + \| \hat{u} \|_?^2 + \| \hat{p} \|_?^2
\]

Optimal test norm (unfortunately, non-local):
\[
\| (v, q) \|_{opt}^2 = \| i\omega v + \nabla q \|_{\Omega_h}^2 + \| i\omega q + \text{div} v \|_{\Omega_h}^2 \\
+ \sup_{\hat{u}_n, \hat{p}} \frac{|<\hat{u}_n, q> + <\hat{p}, v_n>|}{(\| \hat{u}_n \|_?^2 + \| \hat{p} \|_?^2)^{1/2}}
\]

Quasi-optimal test norm (local):
\[
\| (v, q) \|_{opt}^2 = \| i\omega v + \nabla q \|_{\Omega_h}^2 + \| i\omega q + \text{div} v \|_{\Omega_h}^2 + \| v \|^2 + \| q \|^2
\]
Robust stability result

**Theorem:** (Gopalakrishnan, Muga, D, Zitelli, 2011)
Assume: Ω contractable, impedance BC
Use: the quasi-optimal norm to define the minimum energy extension norms for fluxes $\hat{u}_n$ and traces $\hat{p}$.

Then

$$\|(v, q)\|_{opt}^2 \approx \|(v, q)\|_{qopt}^2$$  \hspace{1cm} (uniformly in $k$ and mesh)

Consequently, we get the robust stability in the *desired norm*:

$$\left(\|u - u_h\|^2 + \|p - p_h\|^2 + \|\hat{u}_n - \hat{u}_{n,h}\| + \|\hat{p} - \hat{p}_h\|^2\right)^{\frac{1}{2}}$$

$$\lesssim \|(u, p, \hat{u}_n, \hat{p}) - (u_h, p_h, \hat{u}_{n,h}, \hat{p}_h)\|_E$$

$$= \text{BAE of } (u, p, \hat{u}_n, \hat{p}) \text{ in energy norm}$$

$$\lesssim \text{BAE of } (u, p, \hat{u}_n, \hat{p}) \text{ in desired norm}$$
No pollution in 1D case

In 1D, traces and fluxes and just numbers. Thus, the BAE of fluxes and traces is zero. We get,

\[
\left( \| u - u_h \|^2 + \| p - p_h \|^2 + \| \hat{u}_n - \hat{u}_{n,h} \|^2 + \| \hat{p} - \hat{p}_h \|^2 \right)^{\frac{1}{2}} \lesssim \inf_{w_h, r_h} \left( \| u - w_h \|^2 + \| p - r_h \|^2 \right)^{\frac{1}{2}}
\]

The BAE of \( u, p \) in \( L^2 \)-error is pollution free.
NUMERICAL EXPERIMENTS
Ansatz in time $e^{i\omega t}$, 
Exact solution: $u = p = e^{-ikx}$ (going to the right) 
BCs: 
hard boundary at $x = 0$: $u(0) = 1$ 
impedance BC at $x = 1$: $u(1) = p(1)$ 
enriched space: $\Delta p = 6$
The standard $H^1$ conforming solution $p_{hp}$ quickly exhibits excessive phase error; it is reduced but still present in $p_{\text{blended}}$.
Adhering to a fixed number of elements per wavelength is sufficient to control error
One quartic element per wavelength

Adhering to a fixed number of elements per wavelength is sufficient to control error
Discretization:

- field variables are discretized using isoparametric $L^2$-conforming quads of order $p$,
  $$u_1, u_2, p \in \mathcal{P}^p \otimes \mathcal{P}^p,$$
- traces are discretized using $H^1$-conforming elements of order $p + 1$,
- fluxes are discretized using $L^2$-conforming elements of order $p + 1$
- optimal test functions are approximated with polynomials of order $p + 1 + \Delta p$, i.e.
  $$v \in (\mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p}) \times (\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p+1}),$$
  $$q \in \mathcal{P}^{p+\Delta p+1} \otimes \mathcal{P}^{p+\Delta p+1}$$
2D experiment A

Exact solution: horizontal plane wave
Enriched space: $\Delta p = 2$. 

![Diagram of impedance BC]
Ratio of $L^2$ discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.
2D experiment B

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$. 
Ratio of $L^2$ discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.
2D experiment C

Exact solution: plane wave along diagonal
Enriched space: $\Delta p = 2$. 
Ratio of $L^2$ discretization error vs BAE as a function of wave number. DPG vs standard FEs and Ainsworth-Wajid underintegration scheme.
2D experiment D

Exact solution: outgoing cylindrical wave (Hankel functions...)
Enriched space: $\Delta p = 2$.

Boundary conditions, real part of pressure, initial mesh for $k = 4\pi$. 
Discretization error as a function of wave number.
Pekeris problem, $k = 50$ (8 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 50$ (8 wavelengths)

Classical FEs, four \textit{biquadratic} elements per wavelength.
Pekeris problem, $k = 50$ (8 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 50$ (8 wavelengths)

Ainsworth-Wajid quadrature, four bi-quadratic elements per wavelength.
Pekeris problem, $k = 50$ (8 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 50$ (8 wavelengths)

DPG method, four bilinear elements per wavelength.
Pekeris problem, $k = 50$ (8 wavelengths)

Error for the DPG method.
Pekeris problem, $k = 100$ (16 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 100$ (16 wavelengths)

Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.
Pekeris problem, $k = 100$ (16 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 100$ (16 wavelengths)

DPG method, four bilinear elements per wavelength.
Pekeris problem, $k = 100$ (16 wavelengths)

Error for the DPG method.
Pekeris problem, $k = 200$ (32 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, $k = 200$ (32 wavelengths)

Ainsworth-Wajid quadrature, four biquadratic elements per wavelength.
Pekeris problem, \( k = 200 \) (32 wavelengths)

Exact solution (real part of pressure).
Pekeris problem, \( k = 200 \) (32 wavelengths)

DPG method, four bilinear elements per wavelength.
Pekeris problem, $k = 200$ (32 wavelengths)

Error for the DPG method.
Pressurized cylindrical cavity problem with PML layer. Radial component of velocity.
Pressurized cylindrical cavity problem with PML layer. Comparison of relative $L^2$ error for standard FEs and DPG with the BAE for increasing wave numbers.
2D acoustics (electromagnetics) cloaking problem

Exact solution (pressure or magnetic field)
2D acoustics (electromagnetics) cloaking problem

An $h_p$ mesh (4 bilinear elements per wavelength)
2D acoustics (electromagnetics) cloaking problem

Numerical solution (pressure or magnetic field)
Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.
A Recipe:
How to Construct a Robust DPG Method
for the Confusion Problem
(and Any Other Linear Problem as Well)
We want the $L^2$ robustness in $u$:

$$\|u\| \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E$$

($a \lesssim b$ means that there exists a constant $C$, independent of $\epsilon$ such that $a \leq Cb$). This implies

$$\|u - u_h\| \lesssim \|(u - u_h, \sigma - \sigma_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E$$

$$= \inf_{(u_h, \sigma_h, \hat{u}_h, \hat{q}_h)} \|(u - u_h, \sigma - \sigma_h, \hat{u} - \hat{u}_h, \hat{q} - \hat{q}_h)\|_E$$

**Best Approximation Error (BAE)**

$$\leq C(\epsilon) h^p$$
Step 2: Select a special test function...

\[ b((u, \sigma, \hat{u}, \hat{q}), (v, \tau)) = (\sigma, \frac{1}{\epsilon} \tau + \nabla v)_{\Omega_h} + (u, \text{div} \tau - \beta \cdot \nabla v)_{\Omega_h} \]

\[ - < \hat{u}, \tau_n >_{\Gamma_h^0} - < \hat{q}, v >_{\Gamma_h} \]

Choose a test function \((v, \tau)\) such that

\[
\begin{cases}
    v \in H^1_0(\Omega), \tau \in H(\text{div}, \Omega) \\
    \frac{1}{\epsilon} \tau + \nabla v = 0 \\
    \text{div} \tau - \beta \cdot \nabla v = u
\end{cases}
\]

Then

\[
\|u\|^2 = b((u, \sigma, \hat{u}, \hat{q}), (v, \tau)) = \frac{b((u, \sigma, \hat{u}, \hat{q}), (v, \tau))}{\| (v, \tau) \|_V} \| (v, \tau) \|_V 
\]

\[
\leq \sup_{(v, \tau)} \frac{b((u, \sigma, \hat{u}, \hat{q}), (v, \tau))}{\| (v, \tau) \|_V} \| (v, \tau) \|_V = \| (u, \sigma, \hat{u}, \hat{q}) \|_E \| (v, \tau) \|_V
\]
... and request stability of the adjoint problem

Consequently, we need to select the test norm in such a way that

\[ \|(v, \tau)\|_V \lesssim \|u\| \]

This gives,

\[ \|u\|^2 \lesssim \|(u, \sigma, \hat{u}, \hat{q})\|_E \|u\| \]

Dividing by \(\|u\|\), we get what we wanted.

**The point:** Construction of a robust DPG reduces to the classical stability analysis for the adjoint equation!
Step 3: Study the stability of the adjoint equation

**Theorem** (Generalization of Erickson-Johnson Theorem) (Heuer, D., 2011)

\[
\left\{ \| \mathbf{\beta} \cdot \nabla v \|_w, \sqrt{\epsilon} \| \nabla v \|, \| \text{div} \tau \|_{w + \epsilon}, \frac{1}{\epsilon} \| \mathbf{\beta} \cdot \tau \|_w, \frac{1}{\sqrt{\epsilon}} \| \tau \| \right\} \lesssim \| u \|
\]

where \( w = O(1) \) is a weight vanishing on the inflow boundary that satisfies some “mild” assumptions.

The terms on the left-hand side are our “Lego” blocks with which we can build different test norms.
Step 4: Construct test norm(s)

Quasi-optimal test norm:

\[ \|(v, \tau)\|_1^2 := \|v\|^2 + \frac{1}{\epsilon} \|\tau + \nabla v\|^2 + \|\text{div}\tau - \beta \cdot \nabla v\|^2 \]

Weighted norm:

\[ \|(v, \tau)\|_2^2 := \epsilon \|v\|^2 + \|\beta \cdot \nabla v\|_w^2 + \epsilon \|\nabla v\|^2 + \|\tau\|_{w+\epsilon}^2 + \|\text{div}\tau\|_{w+\epsilon}^2 \]

Remark: Both choices imply also $L^2$-robustness in $\sigma$, as well as in traces and fluxes measured in special energy norms.
Estimates for $\sigma, \hat{u}, \hat{q}$

Same methodology can be used to design a test norm that will imply,

$$\|\sigma\| \lesssim \|(\sigma, u, \hat{u}, \hat{q})\|_E$$

In fact both quasioptimal and weighted norms imply the robust estimate for $\sigma$. They also imply a robust estimate for traces and fluxes measured in a minimum extension norm implied by the problem,

$$(\ast) \quad \|(\hat{u}, \hat{q})\|^2 := \left\| \frac{1}{\epsilon} \Sigma - \nabla U \right\|^2 + \left\| - \text{div}\Sigma + \beta \cdot \nabla U \right\|^2$$

where $\Sigma, U$ are extensions of $\hat{u}, \hat{q}$ from mesh skeleton to the whole domain,

$$U = \hat{u} \text{ on } \Gamma_h^0, \quad (\Sigma - \beta U) \cdot n_e = \hat{q} \text{ on } \Gamma_h$$

that minimize the right hand side of ($\ast$).
Pros and cons for both test norms

- The quasi-optimal test norm produces strong boundary layers that need to be resolved, also in 1D,

Left: \( \tau \) and \( v \) components of the optimal test function corresponding to trial function \( u = 1 \) and element size \( h = 0.25 \), along with the optimal \( hp \) subelement mesh. Right: 10 \( \times \) zoom on the left end of the element. Determining optimal test functions is expensive.

- The weighted test norm produces no boundary layers. Solving for the optimal test functions is inexpensive.

- Quasi-optimal test norm yields better estimates for the best approximation error measured in the corresponding energy norm.
1D: Quasi-Optimal Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

Left: convergence in energy error. Right: convergence in relative $L^2$-error for the field variables (in percent of their $L^2$-norm).
1D: Quasi-Optimal Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

Ratio of $L^2$ and energy norms.
1D: Weighted Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

Left: convergence in energy error. Right: convergence in relative $L^2$-error for the field variables (in percent of their $L^2$-norm).
1D: Weighted Test Norm, $\epsilon = 10^{-2}, 10^{-3}, 10^{-4}$

Ratio of $L^2$ and energy norms.
2D: Model problem of Erickson and Johnson

\[ \Omega = (0, 1)^2, \quad \beta = (1, 0), \quad f = 0, \quad u_0 = \begin{cases} \sin \pi y & \text{on } x = 0 \\ 0 & \text{otherwise} \end{cases} \]

The problem can be solved analytically using separation of variables.

Velocity \( u \) and "stresses" \( \sigma_x, \sigma_y \) (using scale for \( \sigma_y \)) for \( \epsilon = 0.01 \).
Weight: $w = x$.

Ratio of energy and $L^2$ errors (left), energy vs $L^2$ error (right) for 29 $hp$-adaptive meshes. Relative $L2$-error range $12.6 - 0.00068\%$. 
2D: Weighted norm, $\epsilon = 10^{-4}$

Ratio of energy and $L^2$ errors (left), energy vs $L^2$ error (right) for 23 $hp$-adaptive meshes. Relative $L2$-error range 13.5 - 0.24 %.
2D: Weighted norm, $\epsilon = 10^{-5}$

Ratio of energy and $L^2$ errors (left), energy vs $L^2$ error (right) for 27 $hp$-adaptive meshes. Relative $L2$-error range 13.5 - 0.21 %.
2D: Weighted norm, $\epsilon = 10^{-2}$

Optimal $hp$ mesh corresponding to 0.006 % $L^2$ error and the corresponding $u$ component of the solution.
2D: Weighted norm, $\epsilon = 10^{-2}$

$\sigma_x$ and $\sigma_y$ components of the solution.
2D: Quasi-optimal norm, $\epsilon = 10^{-1}$

Ratio of energy and $L^2$ errors (left), energy vs $L^2$ error (right) for 5 $h$-adaptive meshes. Relative $L2$-error range 4.3 - 0.0267 %. Optimal test functions obtained with $\Delta p = 6$. 
Ratio of energy and $L^2$ errors (left), energy vs $L^2$ error (right) for 6 $h$-adaptive meshes.

Relative $L2$-error range 1.3 - 0.6 %. Optimal test functions obtained with Shishkin meshes and $\Delta p = 2$. The non-monotone behavior of the energy error indicates a significant error in the resolution of optimal test functions.
2D: Eye-ball norm comparison for $\epsilon = 10^{-4}$

Velocity $u$ on the initial mesh of four quadratic elements for quasi-optimal (left) and weighted (right) norms.
Lectures

- Petrov-Galerkin Method with Optimal Test Functions.
- Ultraweak variational formulation and the DPG method for convection-dominated diffusion.
- 1D analysis. Adaptivity.
- Wave propagation as an example of a complex-valued problem.
- Systematic choice of test norms. Robustness.
- Convergence proofs.
Convergence Analysis in Multidimensions
Poisson Problem

\[
\begin{aligned}
&\begin{cases}
  u = u_0 & \text{on } \partial \Omega \\
  -\nabla \cdot (\alpha \nabla u) + \beta \cdot \nabla u = f & \text{in } \Omega
\end{cases}
\end{aligned}
\]

For a moment $\beta = 0$. 
First order system:
\[
\begin{aligned}
\alpha^{-1}\sigma - \nabla u &= 0 \quad \text{in } \Omega \\
\nabla \cdot \sigma &= f \quad \text{in } \Omega \\
u &= u_0 \quad \text{on } \partial\Omega
\end{aligned}
\]
Elements: $K$
Edges: $e$
Skeleton: $\Gamma_h = \bigcup_K \partial K$
Internal skeleton: $\Gamma^0_h = \Gamma_h - \partial \Omega$
Take an element $K$. Multiply the equations with test functions $\tau \in H(\text{div}, K), v \in H^1(K)$:

\[
\begin{cases}
(\alpha^{-1}\sigma) \cdot \tau - (\nabla u) \cdot \tau = 0 \\
(\nabla \cdot \sigma)v = f v
\end{cases}
\]
Integrate over the element $K$:

\[
\begin{cases}
\int_K (\alpha^{-1} \sigma) \cdot \tau - \int_K (\nabla u) \cdot \tau &= 0 \\
\int_K (\nabla \cdot \sigma) v &= \int_K f v
\end{cases}
\]
Integrate by parts (relax) both equations:

\[
\begin{cases}
\int_{K} (\alpha^{-1} \sigma) \cdot \tau + \int_{K} u \text{div} \tau - \int_{\partial K} u \tau_n &= 0 \\
- \int_{K} \sigma \cdot \nabla v + \int_{\partial K} q \text{sgn}(n)v &= \int_{K} f v
\end{cases}
\]

where \( q = \sigma n_e \) and

\[
\text{sgn}(n) = \begin{cases}
1 & \text{if } n = n_e \\
-1 & \text{if } n = -n_e
\end{cases}
\]
Declare fluxes to be independent unknowns:

\[
\begin{align*}
\int_K (\alpha^{-1}\sigma) \cdot \tau + \int_K u \, \text{div} \tau - \int_{\partial K} \hat{u} \, \tau_n &= 0 \\
-\int_K \sigma \cdot \nabla v + \int_{\partial K} \hat{q} \, \text{sgn}(n)v &= \int_K f v
\end{align*}
\]

where \( q = \sigma n_e \) and

\[
\text{sgn}(n) = \begin{cases} 
1 & \text{if } n = n_e \\
-1 & \text{if } n = -n_e
\end{cases}
\]
Use BCs to eliminate known fluxes

\[
\begin{align*}
\int_K (\alpha^{-1} \sigma) \cdot \tau + \int_K u \text{ div} \tau - \int_{\partial K - \partial \Omega} \hat{u} \tau_n &= + \int_{\partial K \cap \partial \Omega} u_0 \tau_n \\
- \int_K \sigma \cdot \nabla v + \int_{\partial K} \hat{q} \text{ sgn}(n)v &= \int_K f v
\end{align*}
\]
Trace and Flux Spaces

\[ \Gamma_h := \bigcup_K \partial K \] (skeleton)

\[ \Gamma^0_h := \Gamma_h - \partial \Omega \] (internal skeleton)

\[ \tilde{H}^{1/2}(\Gamma^0_h) := \{ V|_{\Gamma^0_h} : V \in H^1_0(\Omega) \} \]

with the minimum extension norm:

\[ \| v \|_{\tilde{H}^{1/2}(\Gamma^0_h)} := \inf \{ \| V \|_{H^1} : V|_{\Gamma^0_h} = v \} \]

\[ H^{-1/2}(\Gamma_h) := \{ \sigma_n|_{\Gamma_h} : \sigma \in H(\text{div}, \Omega) \} \]

with the minimum extension norm:

\[ \| \sigma_n \|_{H^{-1/2}(\Gamma_h)} := \inf \{ \| \sigma \|_{H(\text{div}, \Omega)} : \sigma n|_{\Gamma_h} = \sigma_n \} \]
Functional Setting

Group variables:
Solution $\mathbf{U} = (u, \sigma, \hat{u}, \hat{q})$:

\[
\begin{align*}
&u, \sigma_1, \sigma_2 \in L^2(\Omega_h) \\
&\hat{u} \in \tilde{H}^{1/2}(\Gamma^0_h) \\
&\hat{q} \in H^{-1/2}(\Gamma_h)
\end{align*}
\]

Test function $\mathbf{V} = (\tau, v)$:

\[
\begin{align*}
&\tau \in H(\text{div}, \Omega_h) \\
&v \in H^1(\Omega_h)
\end{align*}
\]

Variational problem:

\[
b(\mathbf{U}, \mathbf{V}) = l(\mathbf{V}), \quad \forall \mathbf{V}
\]
Simple facts

- Form $b$ is continuous
- $b(U, V) = 0, \forall V$ implies $U = 0$.

In operator terms,

$$b(U, V) = \langle BU, V \rangle = \langle U, B^*V \rangle$$

$B$ is injective, $B, B^*$ are well-defined and continuous.
**Theorem 1**
The DPG variational formulation is well-posed with a mesh-independent inf-sup constant.

**Theorem 2**
There exists a mesh-independent $C > 0$:

$$
\| u - u_{hp} \|_{L^2(\Omega)} + \| \sigma - \sigma_{hp} \|_{L^2(\Omega)} \\
+ \| \hat{u} - \hat{u}_{hp} \|_{\overline{H}^{1/2}(\Gamma^0_h)} + \| \hat{q} - \hat{q}_{hp} \|_{H^{-1/2}(\Gamma_h)} \\
\leq C \inf_{\sigma_{hp}, u_{hp}, \hat{q}_{hp}, \hat{u}_{hp}} [...]
$$

where $u_{hp}, \sigma_{hp}, \hat{u}_{hp}, \hat{q}_{hp}$ is the DPG FE solution.
Define:

\[ \| V \|_o = \| B^* V \| = \sup_U \frac{|b(U, V)|}{\| U \|} \]

\[ = \sup_{u, \sigma, \hat{u}, \hat{q}} \frac{(u, -\text{div} \tau)_\Omega + (\sigma, \alpha^{-1} \tau - \nabla v)_\Omega + \langle \hat{u}, \tau_n \rangle_{\Gamma_0} + \langle v, \hat{q} \rangle_{\Gamma_h}}{(\| u \|^2 + \| \sigma \|^2 + \| \hat{u} \|^2 + \| \hat{q} \|^2)^{1/2}} \]

\[ = \left( \| \text{div} \tau \|^2 + \| \alpha^{-1} \tau - \nabla v \|^2 + \| [v] \|^2_{\Gamma_0} + \| [\tau_n] \|^2_{\Gamma_h} \right)^{1/2} \]

where

\[ \| [v] \|^2_{\Gamma_0} = \sup_{w \in H(\text{div}, \Omega)} \frac{\langle v, w_n \rangle_{\Gamma_h}}{\| w \|_{H(\text{div}, \Omega)}} \]

\[ \| [\tau_n] \|^2_{\Gamma_h} = \sup_{w \in H^1_0(\Omega)} \frac{\langle w, \tau_n \rangle_{\Gamma_0}}{\| w \|_{H^1(\Omega)}} \]
Equivalence of Norms

We will show that the standard and optimal norms are equivalent, i.e.

\[ \|V\| \leq C \|V\|_o \quad \text{and} \quad \|V\|_o \leq C \|V\| \]

The second inequality is straightforward, we will focus on the first one.

Conclusions:

- $B^*$ is injective,
- $b$ satisfies the inf-sup condition ($B$ is bounded below).

Consequently, Nečas - Babuška (Generalized Lax-Milgram, Lions, Banach Closed Range) Theorem implies that the variational problem is well-posed. Theorem 2 follows.
Proof

Take $\tau \in H(\text{div}, \Omega_h), v \in H^1(\Omega_h)$. Denote

$$\alpha^{-1} \tau - \nabla v =: f$$
$$\text{div} \tau =: g$$

Need to show the bounds:

$$\| \tau \|_{H(\text{div}, \Omega_h)} + \| v \|_{H^1(\Omega_h)} \leq C \left( \| f \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} + \|[v]\|_{\Gamma^0_h} + \|[[\tau_n]]\|_{\Gamma_h} \right)$$

Step 1: $f = 0, g = 0$.
Consider the weighted Helmholtz decomposition:

$$\tau = \alpha \nabla \psi + \nabla \times z, \quad \psi \in H^1_0(\Omega), \ z \in H(\text{curl}, \Omega)$$

Potentials $\psi, \tau$ are unique, orthogonal in the weighted $(\alpha^{-1} \cdot, \cdot) L^2$-product, and depend continuously upon $\tau$. 
Step 1: $f, g = 0$

$$\| \tau \|_{\alpha^{-1}}^2 = (\alpha^{-1} \tau, \tau) = (\alpha^{-1} \tau, \alpha \nabla \psi + \nabla \times z)_{\Omega_h}$$
Step 1: $f, g = 0$

\[
\|\tau\|_{\alpha^{-1}}^2 = (\alpha^{-1}\tau, \tau) = (\alpha^{-1}\tau, \alpha\nabla\psi + \nabla \times z)_{\Omega_h} \\
= (\tau, \nabla\psi)_{\Omega_h} + (\nabla v, \nabla \times z)_{\Omega_h}
\]
Step 1: $f, g = 0$

\[
\|\tau\|_{\alpha-1}^2 = (\alpha^{-1} \tau, \tau) = (\alpha^{-1} \tau, \alpha \nabla \psi + \nabla \times z)_{\Omega_h}
\]

\[
= (\tau, \nabla \psi)_{\Omega_h} + (\nabla v, \nabla \times z)_{\Omega_h}
\]

\[
= -(\text{div} \tau, \psi)_{\Omega_h} + <\psi, \tau_n>_{\Gamma_h} + <v, (\nabla \times z) \cdot n>_{\Gamma_h^0}
\]
Step 1: $f, g = 0$

$$\|\tau\|_{\alpha-1}^2 = (\alpha^{-1} \tau, \tau) = (\alpha^{-1} \tau, \alpha \nabla \psi + \nabla \times z)_{\Omega_h}$$

$$= (\tau, \nabla \psi)_{\Omega_h} + (\nabla v, \nabla \times z)_{\Omega_h}$$

$$= -(\text{div} \tau, \psi)_{\Omega_h} + <\psi, \tau_n >_{\Gamma_h} + < v, (\nabla \times z) \cdot n >_{\Gamma_0}$$

$$= \frac{<\psi, \tau_n >_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{< v, (\nabla \times z) \cdot n >_{\Gamma_0}}{\|\nabla \times z\|_{H(\text{div}, \Omega)}} \|\nabla \times z\|_{L^2(\Omega)}$$
Step 1: $f, g = 0$

\[
\|\tau\|_{\alpha^{-1}}^2 = (\alpha^{-1} \tau, \tau) = (\alpha^{-1} \tau, \alpha \nabla \psi + \nabla \times z)_{\Omega_h}
\]

\[
= (\tau, \nabla \psi)_{\Omega_h} + (\nabla v, \nabla \times z)_{\Omega_h}
\]

\[
= -(\text{div} \tau, \psi)_{\Omega_h} + <\psi, \tau_n>_{\Gamma_h} + <v, (\nabla \times z) \cdot n>_{\Gamma_0^h}
\]

\[
= \frac{<\psi, \tau_n>_{\Gamma_h}}{\|\psi\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \frac{<v, (\nabla \times z) \cdot n>_{\Gamma_0^h}}{\|\nabla \times z\|_{H(\text{div}, \Omega)}} \|\nabla \times z\|_{L^2(\Omega)}
\]

\[
\leq \sup_{w \in H_0^1(\Omega)} \frac{<w, \tau_n>_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\text{div}, \Omega)} \frac{<v, w_n>_{\Gamma_0^h}}{\|w\|_{H(\text{div}, \Omega)}} \|\nabla \times z\|_{L^2(\Omega)}
\]
Step 1: $f, g = 0$

\[
\|\tau\|^{2}_{\alpha^{-1}} = (\alpha^{-1} \tau, \tau) = (\alpha^{-1} \tau, \alpha \nabla \psi + \nabla \times z)_{\Omega_h}
\]

\[
= (\tau, \nabla \psi)_{\Omega_h} + (\nabla v, \nabla \times z)_{\Omega_h}
\]

\[
= - (\text{div} \tau, \psi)_{\Omega_h} + \langle \psi, \tau_n \rangle_{\Gamma_h} + \langle v, (\nabla \times z) \cdot n \rangle_{\Gamma_0^h}
\]

\[
\leq \sup_{w \in H^1_0(\Omega)} \frac{\langle w, \tau_n \rangle_{\Gamma_h}}{\|w\|_{H^1(\Omega)}} \|\psi\|_{H^1(\Omega)} + \sup_{w \in H(\text{div}, \Omega)} \frac{\langle v, w_n \rangle_{\Gamma^0_h}}{\|w\|_{H(\text{div}, \Omega)}} \|\nabla \times z\|_{L^2(\Omega)}
\]

\[
\leq C \left( \|v\|_{\Gamma_0^h} + \|\tau_n\|_{\Gamma_h} \right) \|\tau\|_{\alpha^{-1}}
\]
Step 1: \( f, g = 0 \)

Consequently,

\[
\| \nabla v \|_{L^2(\Omega_h)} \leq C \left( \|[v]\|_{\Gamma_h^0} + \|\tau_n\|_{\Gamma_h} \right)
\]

as well.

**Discrete Poincaré Inequality:**

\[
\| v \|_{\Omega_h} \leq C \left( \| \nabla v \|_{\Omega_h} + \|[v]\|_{\Gamma_h^0} \right)
\]

gives

\[
\| v \|_{H^1(\Omega_h)} \leq C \left( \|[v]\|_{\Gamma_h^0} + \|\tau_n\|_{\Gamma_h} \right)
\]
Step 2: $f, g \neq 0$

Let $\tau_1 \in H(\text{div}, \Omega), v_1 \in H^1_0(\Omega)$ such that

$$\begin{cases} 
\alpha^{-1} \tau_1 - \nabla v_1 &= f \\
\text{div} \tau_1 &= g 
\end{cases}$$

Brezzi’s Theory implies

$$\|\tau_1\|_{H(\text{div}, \Omega), \|v_1\|_{H^1(\Omega)} \leq C(\|f\| + \|g\|)}$$

Final step: replace $\tau, v$ with $\tau - \tau_1, v - v_1$ and use Step 1 result. Note that jump terms for $\tau - \tau_1, v - v_1$ are controlled by the original jump terms and norms of $\tau_1, v_1$. 
In Step 1, use the decomposition:

\[ \tau = (\alpha \nabla \psi + \beta \psi) + \nabla \times z, \quad \psi \in H^1_0(\Omega), \ z \in H(\text{curl}, \Omega) \]
Numerical Experiments

Test problems:
  ▶ Square domain with $u(x, y) = \sin(\pi x) \sin(\pi y)$,
  ▶ L-shape domain with $u(r, \theta) = r^{2/3} \sin \left( \frac{2}{3} \left( \theta + \frac{\pi}{2} \right) \right)$
Uniform $h$-convergence rates

(a) The square case

(b) The case of the L-shaped domain

Figure: $h$-convergence rates for the two examples
Uniform $p$-convergence rates

(a) Results from the square domain

(b) Results from the L-shaped domain

Figure: $p$-convergence rates for the two examples
Adaptivity

(a) Comparison of convergence of adaptive schemes

(b) Energy error estimator vs. $L^2$-error

Figure: Convergence curves from adaptive schemes
Adaptivity - cont.

(a) Effect of varying $\delta p$

(b) Effect of varying $\delta p_F$

Figure: Convergence curves from adaptive schemes
Figure: Left: The $hp$ mesh found by the $hp$-adaptive algorithm after 15 refinements. (Color scale represents polynomial degrees.) Right: The corresponding solution $u$. (Color scale represent solution values.)
Thank You!


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DPG Code
A General Variational Problem

\[
\begin{aligned}
&u_1, \ldots, u_N \in L^2(\Omega), \quad f_1, \ldots, f_L \in H^{1/2}(\Gamma_h), \quad g_1, \ldots, g_M \in H^{-1/2}(\Gamma_h) \\
&\int_K (\sum_{j=1}^N a_{ij} u_j) \, \text{div} q_i + \int_{\partial K} f_i \, q_{in} + \int_K (\sum_{j=1}^N b_{ij} u_j) \cdot q_i \\
&\quad = \int_K A_i \, \text{div} q_i + \int_{\partial K} F_i \, q_{in} + \int_K B_i \cdot q_i \\
&\qquad q_i \in H(\text{div}, K), \ i = 1, \ldots, L \\
&\int_K (\sum_{j=1}^N c_{ij} u_j) \, \nabla v_i + \int_{\partial K} g_i \, v_i + \int_K (\sum_{j=1}^N d_{ij} u_j) \, v_i \\
&\quad = \int_K C_i \, \nabla v_i + \int_{\partial K} G_i \, v_i + \int_K d_i \, v_i \\
&\qquad v_i \in H^1(K), \ i = 1, \ldots, M
\end{aligned}
\]

Number of (field) unknowns equals number of (scalar) equations,

\[N = 2L + M\]
\[
\| (q_1, \ldots, q_L; v_1, \ldots, v_M) \|^2 \\
= \sum_{j=1}^{N} \int_{K} \left| \sum_{i=1}^{L} a_{ij} \text{div} q_i + \sum_{i=1}^{L} b_{ij} \cdot q_i + \sum_{i=1}^{M} c_{ij} \cdot \nabla v_i + \sum_{i=1}^{M} d_{ij} v_i \right|^2 \\
+ \sum_{l=1}^{L} \int_{K} e_{l} |q_l|^2 + \sum_{m=1}^{M} \int_{K} f_{m} |v_m|^2
\]