CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2016, Final Exam, 9-noon, Thu, Dec 8, ACES 6.304

1. A linear algebra "sanity check". Let A be a counterclockwise rotation around the x_3-axis in $\mathbb{R}^3$ about an angle α.

   (a) Define a linear map from a vector space into itself and argue why the rotation is a linear map from $\mathbb{R}^3$ into itself. Write down an explicit formula defining the rotation (3 points).

   (b) Write down the matrix representation for the rotation in the canonical basis (2 points).

   (c) Explain why all linear maps from $\mathbb{R}^3$ into itself form a vector space. What is the dimension of the space ? (3 points)

   (d) Do the rotations about the x_3-axis (about different angles) form a vector subspace ? Explain why yes or why not (2 points).

   (e) Define adjoint for a linear operator in a general setting (2 points).

   (f) Compute the adjoint of the rotation with respect to the canonical inner product (3 points).

   (g) Define an orthonormal matrix (2 points).

   (h) Is matrix representation of the rotation an orthonormal matrix ? Explain, why (3 points).

   Answers:

   (a) Let V be a vector space. A map $A\, V \to V$ is linear if it is homogeneous and additive, i.e.

   $$A(\alpha x) = \alpha A(x) \quad A(x+y) = A(x) + A(y)$$

   The rotation satisfies both of these conditions (you may want to illustrate it with a picture). The explicit formula for the transformation is:

   $$\begin{cases} y_1 = \cos\alpha\, x_1 - \sin\alpha\, x_2 \\ y_2 = \sin\alpha\, x_1 + \cos\alpha\, x_2 \\ y_3 = x_3 \end{cases}$$

   (b) The matrix representation is:

   $$\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) Functions defined on any set (in our case $\mathbb{R}^3$) with values in a vector space (in our case $\mathbb{R}^3$) equipped with pointwise addition and scalar multiplication, form a vector space. One has only to argue that the linear maps form a subset closed with respect to the vector space operations and, therefore, form a vector subspace of all functions defined on $\mathbb{R}^3$. This follows from the fact that a linear combination of linear maps is a linear map itself. Dimension of $L(X, Y)$ is always equal to the product of dim $X = n$ and dim $Y = m$ (in our case = 9). This follows from the isomorphism between $L(X, Y)$ and $m \times n$ matrices.

(d) No. Sum of two rotations is *not* a rotation. Note, we are not talking about compositions of rotations here but merely linear combinations.

(e) See the book.

(f) Trivial, transpose the matrix.

(g) A matrix whose inverse equals its transpose.

(h) Yes. The transpose represents rotation by angle $-\alpha$ which is exactly the inverse map.

2. An integration exercise.

(a) State the Lebesgue Dominated Convergence Theorem (5 points).

(b) Let $\gamma > 0$ be a positive constant. Prove that the integral

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\gamma+n\cos\theta}}{\sqrt{(\gamma+n\cos\theta)^2+(n\sin\theta)^2}}\, nd\theta$$

converges to zero as $n \to \infty$ (15 points).

Answers:

(a) See the book.

(b) Rewrite the integral in the form,

$$\int_{\pi/2}^{3\pi/2} \frac{e^{\gamma+n\cos\theta}}{\sqrt{(\gamma/n+\cos\theta)^2+(\sin\theta)^2}}\, d\theta$$

For $\theta \in (\pi/2, 3\pi/2)$, the denominator converges to one, whereas the numerator converges to zero (exponential with a negative exponent), as $n \to \infty$. Consequently the integrand converges a.e. to zero. In order to apply the Lebesgue Dominated Convergence Theorem, we need to show only that the integrand is dominated by an integrable function, for all n. The numerator is bounded by e^γ. For the denominator, we have

$$\begin{aligned}\left(\frac{\gamma}{n}+\cos\theta\right)^2+\sin^2\theta &= \frac{\gamma^2}{n^2}+\frac{2\gamma}{n}\cos\theta+1\\ &\geq \frac{\gamma^2}{n^2}-\frac{2\gamma}{n}+1\\ &= \left(\frac{\gamma}{n}-1\right)^2\end{aligned}$$

Thus, for sufficiently large n, the denominator is bounded below by a positive number (independent of angle θ).

3. Let $f : X \to Y$ where X and Y are arbitrary topological spaces. Prove that f is continuous iff $\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for every $B \subset Y$ (20 points).

   Recall that f is continuous iff the inverse image of every closed set is closed. Assume that f is continuous and pick an arbitrary set $B \subset Y$. The closure $\overline{B}$ is closed, so the inverse image $f^{-1}(\overline{B})$ must be closed. It also contains $f^{-1}(B)$. Since the closure of a set is the smallest closed set including the set, we have

   $$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \text{ (closed)}$$

   Conversely, assume that the condition is satisfied. Pick any closed set $B = \overline{B} \subset Y$. Then

   $$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) = f^{-1}(B)$$

   which implies that set $f^{-1}(B)$, being equal to its closure, is closed.

4. Consider $\mathbb{R}^N$.

   - Define the l^1 norm $\|\boldsymbol{x}\|_1$ in $\mathbb{R}^N$ and demonstrate that indeed it satisfies the three defining conditions for a norm (2 points).
   - State the Weierstrass Theorem for an arbitrary topological space (3 points). See the book.
   - Let $\|\boldsymbol{x}\|$ be now any other norm defined on $\mathbb{R}^n$.
     (i) Show that there exists a constant $C > 0$ such that,

     $$\|\boldsymbol{x}\| \leq C\|\boldsymbol{x}\|_1 \quad \forall \boldsymbol{x} \in \mathbb{R}^N$$

     (ii) Use (i) to demonstrate that function

     $$\mathbb{R}^N \ni \boldsymbol{x} \to \|\boldsymbol{x}\| \in \mathbb{R}$$

     is continuous in l^1-norm.

(iii) Use the Weierstrass Theorem to conclude that there exists a constant $D > 0$ such that

$$\|\boldsymbol{x}\|_1 \leq D\|\boldsymbol{x}\| \quad \forall \boldsymbol{x} \in \mathbb{R}^N$$

Conclude that the l_1 norm is equivalent to any other norm on $\mathbb{R}^N$. Explain why the result implies that *any two norms* defined on an arbitrary finite-dimensional vector space must be equivalent (15 points).

(i) Let $\boldsymbol{e}_i$ denote the canonical basis in $\mathbb{R}^n$. Then

$$\|\boldsymbol{x}\| = \|\sum_{i=1}^{n} x_i \boldsymbol{e}_i\| \leq \sum_{i=1}^{n} |x_i| \, \|\boldsymbol{e}_i\| \leq C \sum_{i=1}^{n} |x_i|$$

where

$$C = \max\{\|\boldsymbol{e}_1\|, \ldots, \|\boldsymbol{e}_n\|\}$$

(ii) This follows immediately from the fact that

$$|\|\boldsymbol{x}\| - \|\boldsymbol{y}\|| \leq \|\boldsymbol{x} - \boldsymbol{y}\|$$

and property (i).

(iii) The l_1 unit ball is compact. Consequently, norm $\|\cdot\|$ attains a minimum on the l_1 unit ball, i.e.,

$$C \leq \|\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_1}\| \quad \forall \boldsymbol{x}$$

Positive definitness of the norm implies that $C > 0$. Multiplying by $\|\boldsymbol{x}\|_1 / C$, we get

$$\|\boldsymbol{x}\|_1 \leq C^{-1}\|\boldsymbol{x}\|$$

Take now two arbitrary norms. As each of them is equivalent to norm $\|\cdot\|_1$, they must be equivalent with each other as well.

5. Consider the Fredholm integral equation:

$$f(x) = \phi(x) + \lambda \int_0^1 K(x,y) f(y) \, dy$$

where $\phi \in L^2(0,1)$, kernel $K \in L^2((0,1)^2)$, and λ is a complex constant.

- State Banach Contractive Map Theorem (5 points).
- Use the theorem to establish an upper bound for $|\lambda|$ such that the problem has a unique solution $f \in L^2(0,1)$.

  (15 points).

Consider the map from the space $L^2(0,1)$ into itself:

$$(Af)(x) = \phi + \lambda \int_0^1 K(x,y) f(y)\, dy$$

Clearly, f solves the equation iff it is a fixed point of map A. It follows that

$$(Af_1 - Af_2)(x) = \lambda \int_0^1 K(x,y)[f_1(y) - f_2(y)]\, dy$$

We estimate using Cauchy-Schwarz inequality,

$$\begin{aligned}
\|Af_1 - Af_2\|^2_{L^2(0,1)} &= |\lambda|^2 \int_0^1 |\int_0^1 K(x,y)[f_1(y) - f_2(y)]\, dy|^2\, dx \\
&\leq |\lambda|^2 \int_0^1 \int_0^1 |K(x,y)|^2 \int_0^1 |f_1(y) - f_2(y)|^2\, dy\, dx \\
&= \lambda|^2 \int_0^1 \int_0^1 |K(x,y)|^2\, \|f_1 - f_2\|^2_{L^2(0,1)}\, dy\, dx \\
&= |\lambda|^2 \|K\|^2_{L^2((0,1))^2}\, \|f_1 - f_2\|^2_{L^2(0,1)}
\end{aligned}$$

Consequently, map A is a contraction for

$$|\lambda| < \|K\|^{-1}_{L^2((0,1))}$$