1. Define the following notions and provide a non-trivial example (2+2 points each).

- Lebesgue measurable sets.
- Essential supremum.
- Stronger and weaker topologies.
- Topological subspace.
- Compact topological space.

See the book.

2. State and prove three out of the following four theorems (10 points each).

- Lebesgue Dominated Convergence Theorem (for non-negative functions).
- Hölder inequality.
- Properties of compact sets.
- The Weierstrass Theorem (for a general topological space)

See the book.

3. Let \( f_n, \varphi \in L^p(\Omega), p \in [1, \infty) \) such that

(a) \( |f_n(x)| \leq \varphi(x) \) a.e. in \( \Omega \), and
(b) \( f_n(x) \to f(x) \) a.e. in \( \Omega \).

Prove that

(i) \( f \in L^p(\Omega) \), and
(ii) \( \|f_n - f\|_p \to 0 \).

Is the result true for \( p = \infty \)? (10 points)

(i) We have: \( |f_n(x)|^p \leq |\varphi(x)|^p \), \( |f_n(x)|^p \to |f(x)|^p \) and, therefore, by Lebesgue Dominated Convergence Theorem,

\[
\int |f_n(x)|^p \to \int |f(x)|^p \leq \int |\varphi|^p
\]
(ii) It is sufficient to show a summable function dominating \(|f_n - f|^p\), e.g.,

\[ |f_n - f|^p \leq (|f_n| + |f|)^p \leq (2|\varphi|)^p \]

The theorem is false for \(p = \infty\). Counterexample:

\[ f_n(x) = \begin{cases} 1 & x \in (0, \frac{1}{n}) \\ 0 & x \in \left[ \frac{1}{n}, 1 \right) \end{cases} \]

Functions \(f_n \to 0\) a.e. in \((0, 1)\), are bounded by unity, but \(\|f_n - 0\|_\infty = 1\).

4. Let \(f : X \to Y\). Prove that the following conditions are equivalent to each other:

a) \(f\) is globally continuous.

b) \(\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})\) for every \(B \subset Y\).

c) \(f(A) \subset \overline{f(A)}\) for every \(A \subset X\).

(20 points).

[a] \(\iff\) [b]

Recall that \(f\) is continuous iff the inverse image of every closed set is closed. Assume that \(f\) is continuous and pick an arbitrary set \(B \subset Y\). The closure \(\overline{B}\) is closed, so the inverse image \(f^{-1}(\overline{B})\) must be closed. It also contains \(f^{-1}(B)\). Since the closure of a set is the smallest closed set including the set, we have

\[ \overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \text{ (closed)} \]

Conversely, assume that the condition is satisfied. Pick any closed set \(B = \overline{B} \subset Y\). Then

\[ \overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) = f^{-1}(B) \]

which implies that set \(f^{-1}(B)\), being equal to its closure, is closed.

[c] \(\iff\) [b]

\(\Rightarrow\) Set \(A = f^{-1}(B)\). Then

\[ f(\overline{f^{-1}(B)}) \subset \overline{ff^{-1}(B)} = \overline{B} \]

Taking the inverse image on both sides, we get

\[ \overline{f^{-1}(B)} = f^{-1}(\overline{f^{-1}(B)}) \subset f^{-1}(\overline{B}) \]

\(\Leftarrow\) Set \(B = f(A)\). Then

\[ \overline{A} \subset f^{-1}(\overline{f(A)}) \subset f^{-1}(f(A)) \]

Taking the direct image on both sides, we get

\[ f(\overline{A}) \subset ff^{-1}(\overline{f(A)}) = \overline{f(A)} \]
5. We say that a topology has been introduced in a set $X$ through the *operation of interior*, if we have introduced operation (of taking the interior) $P(X) \ni A \to \text{int}^*A \in P(X)$ with $\text{int}^*A \subset A$

that satisfies the following four properties:

(i) $\text{int}^*X = X$

(ii) $A \subset B$ implies $\text{int}^*A \subset \text{int}^*B$

(iii) $\text{int}^*(\text{int}^*A) = \text{int}^*A$

(iv) $\text{int}^*(A \cap B) = \text{int}^*A \cap \text{int}^*B$

Sets $G$ such that $\text{int}^*G = G$ are identified then as open sets.

(a) Prove that the open sets defined in this way satisfy the usual properties of open sets (empty set and the whole space are open, unions of arbitrary families, and intersections of finite families of open sets are open).

(b) Use the identified family of open sets to introduce a topology (through open sets) in $X$ and consider the corresponding interior operation $\text{int}$ with respect to the new topology. Prove then that the original and the new operations of taking the interior coincide with each other, i.e.,

$$\text{int}^*A = \text{int}A$$

for every set $A$.

(c) Conversely, assume that a topology was introduced by open sets $\mathcal{X}$. The corresponding operation of interior satisfies then properties listed above and can be used to introduce a (potentially different) topology and corresponding (potentially different) open sets $\mathcal{X}'$. Prove that families $\mathcal{X}$ and $\mathcal{X}'$ must be identical.

(20 points).

(a) By assumption, $\text{int}^*\emptyset \subset \emptyset$. Conversely, the empty set is a subset of every set, so $\text{int}^*\emptyset = \emptyset$. The whole space $X$ is open by axiom (i). Let $G_\iota, \iota \in I$, be now an arbitrary family of sets such that $\text{int}^*G_\iota = G_\iota$. By assumption,

$$\text{int}^*\bigcup_{\iota \in I} G_\iota \subset \bigcup_{\iota \in I} G_\iota = \bigcup_{\iota \in I} \text{int}^*G_\iota$$

Conversely,

$$G_\iota \subset \bigcup_{\kappa \in I} G_\kappa, \quad \forall \iota \in I$$
Axiom (ii) implies that,
\[ \text{int}^* G_\iota \subset \text{int}^* \bigcup_{\kappa \in I} G_\kappa, \quad \forall \iota \in I \]
and, consequently,
\[ \bigcup_{\iota \in I} \text{int}^* G_\iota \subset \text{int}^* \bigcup_{\kappa \in I} G_\kappa \]

Finally, by induction, axiom (iv) implies that interior of a common part of a finite family of sets is equal to the common part of their interiors.

(b) Let \( A \) be an arbitrary set. By axiom (iii), \( \text{int}^* A \in \mathcal{X} \) and, by definition, \( \text{int}^* A \subset A \). Recall (comp. Exercise ??) that the interior of a set is the largest open subset of the set. As \( \text{int}^* A \) is open and contained in \( A \), we have \( \text{int} A \subset \text{int}^* A \). On the other side, since \( \text{int}^* A \in \mathcal{X} \), \( \text{int}(\text{int}^* A) = \text{int}^* A \). Consequently, \( \text{int}^* A = \text{int}(\text{int}^* A) \subset \text{int} A \) since \( \text{int}^* A \subset A \).

(c) This is trivial. Open sets \( A \) in both families are identified through the same property: \( \text{int}^* A = A \).