CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2016, Exam 3

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - Lebesgue measure of an open set $G \subset \mathbb{R}^n$ (through partition of G),
   - Lebesgue integral of a non-negative function (can you provide the corresponding definition for a sign changing function as well ?),
   - weaker and stronger topologies,
   - topological subspace,
   - compact set (in an arbitrary topological space).

   See the book.

2. State and prove *three* out of the four theorems (10 points each).

   - Properties of an abstract measure.
   - Properties of Borel sets.
   - Properties of the operation of interior (in an arbitrary topological space).
   - Properties of compact sets (in an arbitrary topological (Hausdorff) space.

   See the book.

3. Let $f_n, \varphi \in L^p(\Omega), p \in [1, \infty)$ such that

   (a) $|f_n(x)| \leq \varphi(x)$ a.e. in Ω, and

   (b) $f_n(x) \to f(x)$ a.e. in Ω.

   Prove that

   (i) $f \in L^p(\Omega)$, and

   (ii) $\|f_n - f\|_p \to 0$.

   (15 points).

(i) We have: $|f_n(x)|^p \le |\varphi(x)|^p, \quad |f_n(x)|^p \to |f(x)|^p$ and, therefore, by Lebesgue Dominated Convergence Theorem,

$$\int |f_n(x)|^p \to \int |f(x)|^p \le \int |\varphi|^p$$

(ii) It is sufficient to show a summable function dominating $|f_n - f|^p$, e.g.,

$$|f_n - f|^p \le (|f_n| + |f|)^p \le (2|\varphi|)^p$$

Note that the theorem is false for $p = \infty$. Counterexample:

$$f_n(x) = \begin{cases} 1 & x \in (0, \frac{1}{n}) \\ 0 & x \in [\frac{1}{n}, 1) \end{cases}$$

Functions $f_n \to 0$ a.e. in $(0,1)$, are bounded by unity, but $\|f_n - 0\|_\infty = 1$.

4. A topology "sanity check". Verify that the family $\mathcal{X} = \{X, \emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$ of subsets of $X = \{a,b,c,d\}$, satisfies the axioms for open sets. Consider then the corresponding topology in X (introduced through the open sets).

   (i) Identify the closed sets in this topology.

   (ii) What is the closure of $\{a\}$, of $\{a,b\}$?

   (iii) Determine the interior of $\{a,b,c\}$ and the filter (system) of neighborhoods of b.

   (10 points).

   Empty set and X are in $\mathcal{X}$, and union and intersection of any subfamily of $\mathcal{X}$ is in $\mathcal{X}$.

   The corresponding family of closed sets may be determined by taking complements of open sets:

$$\{\emptyset, X, \{b,c,d\}, \{a,c,d\}, \{c,d\}, \{d\}, \{c\}\}$$

   Closure of a set A is the smallest closed set containing A and interior of a set A is the largest open set contained in A. Consequently,

$$\overline{\{a\}} = \{a,c,d\}, \quad \overline{\{a,b\}} = X, \quad \mathrm{int}\{a,b,c\} = \{a,b,c\}$$

   Base of open neighborhoods of b includes the singleton of b. Consequently, all subsets of X containing b are neighborhoods of the point.

5. Prove that the interior of a set A, $\mathrm{int}A$ is the *largest open* set contained in A, and the closure of A, $\overline{A}$, is the *smallest closed* set containing A (10 points).

   We know that int A is an open subset of A. Let $G \subset A$ be now an open set. Then $G =$ int $G \subset$ int A, so int A *is* indeed the largest open subset of A.

   Same reasoning with closed sets. Closure $\overline{A}$ is closed and it contains A. Let $F \supset A$ be a closed set. Then $F = \overline{F} \supset \overline{A}$, so $\overline{A}$ *is* the smallest closed set containing A.

6. We say that a topology has been introduced in a set X through *closed sets* if we have introduced a family $\mathcal{Y} \subset \mathcal{P}(X)$ that satisfies the following three axioms.

   (i) $\emptyset, X \in \mathcal{Y}$.

   (ii) $F_\iota \in \mathcal{Y}$, $\iota \in I$ implies $\bigcap_{\iota \in I} F_\iota \in \mathcal{Y}$.

   (iii) $F_1, \ldots, F_n \in \mathcal{Y}$ implies $F_1 \cup \ldots \cup F_n \in \mathcal{Y}$.

   (a) Define open sets $\mathcal{X}$ as complements of sets from $\mathcal{Y}$ and verify that they satisfy the axioms for open sets.

   (b) Use the open sets to introduce a topology (through open sets) in X. Identify the corresponding closed sets $\mathcal{Y}_1$ in that topology, and show that $\mathcal{Y}_1 = \mathcal{Y}$.

   (c) Conversely, introduce a topology in X by selecting a family $\mathcal{X} \subset \mathcal{P}(X)$ that satisfies the axioms for open sets. Proceed with the usual constructions to identify the corresponding closed sets $\mathcal{Y}$. Use then the closed sets to introduce a (potentially) different topology in X through the closed sets $\mathcal{Y}$ and identify the corresponding family $\mathcal{X}_1$ of complements of sets from $\mathcal{Y}$. Prove then that $\mathcal{X} = \mathcal{X}_1$.

   (15 points)

   (a) This follows immediately from the de Morgan's laws.

   (b) The whole point here is the duality principle. Sets from $\mathcal{Y}$ are complements of sets from $\mathcal{X}$ by construction, At the same time, sets from $\mathcal{Y}_1$ are complements of sets from $\mathcal{X}$ by the duality principle (proved in class and book). So, $\mathcal{Y}_1 = \mathcal{Y}$.

   (c) Same reasoning.