CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2017, Exam 2

1. Define the following notions and provide a non-trivial example (2+2 points each).

   - Closed set.
   - Quotient vector space.
   - Dual basis.
   - Riesz map.
   - Abstract measure.

   See the book.

2. State and prove *three* out of the following four theorems (10 points each).

   - Properties of the operation of interior (in $\mathbb{R}^n$).
   - Characterization of a (linear) projection.
   - Relation between rank of a transformation and the rank of its transpose.
   - Properties of an abstract σ-algebra.

   See the book.

3. A "sanity check"...

   (a) Prove that vectors $\boldsymbol{a}_1 = (0, 0, 1), \boldsymbol{a}_2 = (1, 0, 1), \boldsymbol{a}_3 = (1, 1, 1)$ form a basis for $\mathbb{R}^3$.

   (b) Determine the dual basis $\boldsymbol{a}_i^*$.

   (c) Consider the canonical inner product in $\mathbb{R}^3$,

   $$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{3} x_i y_i$$

   and construct the cobasis $\boldsymbol{a}^i$.

   (d) Determine matrix representation of the corresponding Riesz operator with respect basis $\boldsymbol{a}_i$ and its dual $\boldsymbol{a}_i^*$.

   (15 points).

(a) It is sufficient to check linear independence.

$$\sum_{i=1}^{3} \alpha_i \boldsymbol{a}_i = (\alpha_2 + \alpha_3, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3) = \mathbf{0} = (0, 0, 0)$$

implies the homogeneous system of equations for α_i with matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Since det $\boldsymbol{A}$ = 1, the matrix is non-singular and, therefore, the only possible solution is the trivial one.

(b) Expanding an arbitrary vector $\boldsymbol{x}$ in the basis $\boldsymbol{a}_i$,

$$\sum_{i=1}^{3} \alpha_i \boldsymbol{a}_i = (\alpha_2 + \alpha_3, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3) = \mathbf{x} = (x_1, x_2, x_3)$$

gives:

$$\alpha_1 = x_3 - x_1, \quad \alpha_2 = x_1 - x_2, \quad \alpha_3 = x_2$$

The components α_i represent precisely the dual basis $\boldsymbol{a}_i^*$. In terms of canonical dual basis $\boldsymbol{e}_i^*$,

$$\boldsymbol{a}_1^* = \boldsymbol{e}_3^* - \boldsymbol{e}_1^*, \quad \boldsymbol{a}_2^* = \boldsymbol{e}_1^* - \boldsymbol{e}_2^*, \quad \boldsymbol{a}_3^* = \boldsymbol{e}_2^*$$

(c) For the canonical inner product, the Riesz operator is mapping canonical basis $\boldsymbol{e}_i$ into $\boldsymbol{e}_i^*$. Recalling that $\boldsymbol{e}^i = R^{-1}\boldsymbol{e}_i^*$, this immediately implies that $\boldsymbol{e}^i = \boldsymbol{e}_i$, i.e., the canonical basis is orthonormal wrt the canonical inner product. In view of the formulas above, we have

$$\boldsymbol{a}^1 = \boldsymbol{e}_3 - \boldsymbol{e}_1, \quad \boldsymbol{a}^2 = \boldsymbol{e}_1 - \boldsymbol{e}_2, \quad \boldsymbol{a}^3 = \boldsymbol{e}_2$$

(d) We use basis $\boldsymbol{a}_i$ for the original space $\mathbb{R}^3$ and basis $\boldsymbol{a}_i^*$ for its dual. The corresponding matrix representation of the Riesz operator R_{ij} is simply the Gram matrix for the original basis. Indeed,

$$R_{ij} = \langle \boldsymbol{a}_i^{**}, R\boldsymbol{a}_j \rangle = \langle R\boldsymbol{a}_j, \boldsymbol{a}_i \rangle = (\boldsymbol{a}_j, \boldsymbol{a}_i)$$

which gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

4. Another sanity check...

   (a) Demonstrate that
   $$(\boldsymbol{x}, \boldsymbol{y})_w = 2x_1y_1 + x_2y_2 + 3x_3y_3 \tag{0.1}$$
   is an inner product on $\mathbb{R}^3$.

   (b) Consider the map $A : \mathbb{R}^3 \to \mathbb{R}^3$,
   $$A\boldsymbol{x} = (x_2 + x_3, x_3 + x_1, x_1 + x_2) \tag{0.2}$$
   Prove that the map is linear and write down its matrix representation with respect to the canonical basis.

   (c) Determine *adjoints* of map A with respect to the canonical inner product and inner product (0.1).

(15 points).

   (a) Nothing to show really. The function is obviously linear in both $\boldsymbol{x}$ and $\boldsymbol{y}$ and it is symmetric. For $\boldsymbol{y} = \boldsymbol{x}$, we get;
   $$(\boldsymbol{x}, \boldsymbol{x}) = 2x_1^2 + x_2^2 + 3x_3^2 \geq 0$$
   Since $(\boldsymbol{x}, \boldsymbol{x})$ is the sum of non-negative contributions (no cancellation can occur), $(\boldsymbol{x}, \boldsymbol{x}) = 0$ implies that all three terms must be zero, i.e. $\boldsymbol{x} = \mathbf{0}$, i..e. the form is positive-definite.

   (b) Linearity is obvious as all components of $\boldsymbol{x}$ enter the definition in the first power and the summation is a linear operation. The matrix representation in the canonical basis is as follows,
   $$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   (c) To obtain the matrix representtaion of the adjoint of A with respect to the canonical inner product, we only have to transpose the matrix above. Consequently, the adjoint map is:
   $$A^*\boldsymbol{y} = (y_2 + y_3, y_1 + y_3, y_1 + y_2)$$
   The map is *self-adjoint* with respect to the canonical inner product. The adjoint with respect to inner product (0.1) can be computed directly:
   $$\begin{aligned} (A\boldsymbol{x}, \boldsymbol{y}) &= 2(x_2 + x_3)y_1 + (x_1 + x_3)y_2 + 3(x_1 + x_2)y_3 \\ &= x_1(y_2 + 3y_3) + x_2(2y_1 + 3y_3) + x_3(2y_1 + y_2) \\ &= 2x_1[\tfrac{1}{2}(y_2 + 3y_3)] + x_2(2y_1 + 3y_3)] + 3x_3[\tfrac{1}{3}(2y_1 + y_2)] \end{aligned}$$

which gives

$$A^*\boldsymbol{y} = (\frac{1}{2}(y_2 + 3y_3), y_1 + 3y_3, \frac{1}{3}(2y_1 + y_2))$$

The map is *not* self-adjoint with respect to this inner product.

5. Let X, Y be two real finite dimensional vector spaces. Consider two different vector spaces:

   - space $L(X, Y^*)$ of all linear transformations from X into dual Y^*,
   - space $B(X, Y)$ of all bilinear functionals defined on $X \times Y$.

   (a) Argue why the linear transformations and bilinear functionals form vector spaces.

   (b) Select bases $\boldsymbol{e}_j$ and $\boldsymbol{g}_i$ for X and Y respectively, and recall (derive) representations for arbitrary linear transformations and bilinear maps relative to the bases. Argue why the spaces $L(X, Y^*)$ and $B(X, Y)$ are isomorphic.

   (c) Attempt to construct a *canonical*[1] isomorphism between the two spaces.

   (20 points).

   (a) This follows from the fact that a linear combination of linear maps is a linear map, and a linear combination of bilinear functionals is a bilinear map. In other words, both families are closed with respect to the vector space operations for functions.

   (b) For a linear map $B \in L(X, Y^*)$, the corresponding matrix representation B_{ij} with respect to bases $\boldsymbol{e}_j$ and $\boldsymbol{g}_i^*$ is given by the relation:

   $$B_{ij} = \langle \boldsymbol{g}_i^{**}, B\boldsymbol{e}_j \rangle = \langle B\boldsymbol{e}_j, \boldsymbol{g}_i \rangle, \qquad i = 1, \ldots, m, \quad j = 1, \ldots, n$$

   where $n = \text{dim}X$, $m = \text{dim}Y$. The space of $m \times n$ matrices, Mat(m, n), is isomorphic (by construction dicussed in the book and class) to the space $L(X, Y^*)$ of linear transformations.

   Similarly, from the representation formula for bilinear functionals

   $$b(\boldsymbol{x}, \boldsymbol{y}) = b(\sum_{j=1}^{n} x_j \boldsymbol{e}_j, \sum_{i=1}^{m} y_i \boldsymbol{g}_i) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_j y_i \underbrace{b(\boldsymbol{e}_j, \boldsymbol{g}_i)}_{=:b_{ji}}$$

   also implies that the space of bilinear functionals is isomorphic with the space of matrices Mat(m, n).

   Consequently, the two spaces, being isomorphic with the same space, must be isomorphic with each other. In perhaps simpler terms, both spaces are of the same dimension, and all spaces of the same dimension are isomorphic with each other.

[1] Constructed without using any bases

(c) This is deeper... The construction generalizes the concept of Riesz map. Given a bilinear form $b(\boldsymbol{x}, \boldsymbol{y})$, if we fix $\boldsymbol{x}$, we obtain a linear functional on Y,

$$Y \ni \boldsymbol{y} \to b(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}$$

The map,

$$B \ : \ X \ni \boldsymbol{x} \to b(\boldsymbol{x}, \cdot) \in Y^*$$

is a linear map from X into Y^*.

Finally, the map,

$$B(X, Y) \ni b \to B := \{X \ni \boldsymbol{x} \to b(\boldsymbol{x}, \cdot) \in Y^*\} \in L(X, Y^*)$$

establishes a (canonical) isomorphism between the two spaces. Linearity is straightforward. Injectivity follows from the definition of zero vectors in both spaces. Dimensions are the same, so the map must be surjective. Done. Try to remember the fundamental relation:

$$b(x, y) =< Bx, y >$$

By switching x with y, we can obtain a second relation

$$b(x, y) =< x, B^1 y >$$

where $B^1 \in L(Y, X^*)$.