CSE386M/EM386M
FUNCTIONAL ANALYSIS IN THEORETICAL MECHANICS
Fall 2023, Exam 1

1. Define the following notions and provide a non-trivial example (2+2 points each):

   - union of an arbitrary (possibly infinite) family of sets,
   - supremum of a subset of a partially ordered set,
   - Cartesian product of two functions,
   - interior of a set in $\mathbb{R}^n$,
   - closed set in $\mathbb{R}^n$.

   See the book.

2. State and prove *three* out of the following four theorems (10 points each).

   - Principle of mathematical induction.
   - Bijection between the family of all equivalence classes and the family of all partitions for a set.
   - Comparability of cardinal numbers.
   - Relation between open and closed sets (duality principle).

   See the book.

3. Let $\mathcal{P}(A)$ denote the power class of a set A. Show that $\mathcal{P}(A)$ is partially ordered by the inclusion relation. Does $\mathcal{P}(A)$ have the smallest and greatest elements? (10 points)

   Solution:

   - Reflexivity: $X \subset X$.
   - Antisymmetry: $X \subset Y, Y \subset X \Rightarrow X = Y$.
   - Transitivity: $X \subset Y,\ Y \subset Z \Rightarrow X \subset Z$.

   The empty set $\emptyset$ is the smallest element, and the whole set A is the biggest element in $\mathcal{P}(A)$.

4. A function $f : X \to Y$ is called *right-invertible* if there exists a function $g : Y \to X$ such that $f \circ g = id_Y$. Prove that f is right-invertible if and only if f is a surjection. Is the right-inverse g unique ? (10 points).

   Solution: See the book.

5. Let $f : X \to Y$ be a function. Prove that $f^{-1}(D \cap C) = f^{-1}(D) \cap f^{-1}(C)$ for $D, C \subset Y$. Use the result to conclude that

$$f^{-1}(\mathcal{R}(f) \cap C) = f^{-1}(C)$$

(10 points)

Solution: The first part was proved in class. Use $D = Y$ to conclude the second part.

6. Prove that if A is infinite, $A \times A \sim A$.

Hint: Use the following steps:

(i) Recall that $N \times N \sim N$.

(ii) Define a family $\mathcal{F}$ of couples (X, T_X) where X is an infinite subset of A and $T_X : X \to X \times X$ is a bijection. Introduce a relation $\leq$ in $\mathcal{F}$ defined as

$$(X_1, T_{X_1}) \leq (X_2, T_{X_2})$$

iff $X_1 \subset X_2$ and T_{X_2} is an extension of T_{X_1}.

(iii) Prove that $\leq$ is a partial ordering of $\mathcal{F}$.

(iv) Show that family $\mathcal{F}$ with its partial ordering $\leq$ satisfies the assumptions of the Kuratowski–Zorn Lemma and conclude the existence of a maximal element.

(v) Use the existence of a maximal element to show that $X \sim X \times X$. You may use here results of the previous, related exercises in the text.

Question: Why do we need the first step?

Solution: Follow precisely the lines in Exercise 1.12.3 to arrive at the existence of a maximal element (X, T_X) in the family. Let $Y = A - X$. We will consider two cases.

Case: $\#Y \leq \#X$. By Exercise 1.12.5, $A \sim X$. i.e. $\#A = \#X$. We have then,

$$\#A = \#X = \#(X \times X) = \#(A \times A)$$

Case: $\#X < \#Y$. In this case, we can split Y into two disjoint subsets, $Y = Y_1 \cup Y_2$ with $Y_1 \sim X$. We claim that

$$Y_1 \sim (X \times Y_1) \cup (Y_1 \times X) \cup (Y_1 \times Y_1)$$

Indeed, since $\#Y_1 = \#X$, we have

$$\#\left((X \times Y_1) \cup (Y_1 \times X) \cup (Y_1 \times Y_1)\right) = \#\left((X \times X) \times \{1, 2, 3\}\right) = \#(X \times X) = \#X$$

But this contradicts that (X, T_X) is the maximal element. Indeed, by the equivalence above, there exists a bijection from Y_1 onto $(X \times Y_1) \cup (Y_1 \times X) \cup (Y_1 \times Y_1)$. We can use it then to extend $T_X : X \to X \times X$ to a bijection from $X \cup Y_1$ onto

$$(X \cup Y_1) \times (X \cup Y_1) = (X \times X) \cup (X \times Y_1) \cup (Y_1 \times X) \cup (Y_1 \times Y_1)$$

Step (i) was necessary to assure that family $\mathcal{F}$ is nonempty.