1. Define the following notions and provide a non-trivial example (2+2 points each).

- union of an arbitrary (possibly infinite) family of sets,
- quotient set,
- Cartesian product of two functions,
- a chain in a partially ordered set,
- cardinal number.

See the book.

2. State and prove three out of the following four theorems (10 points each).

- Characterization of a bijection.
- Properties of the direct image.
- Properties of countable sets.
- Comparability of cardinal numbers.

See the book.

3. A function \( f : X \to Y \) is called left invertible if there exists a function \( g : Y \to X \) such that \( g \circ f = id_X \). Prove that \( f \) is left-invertible if and only if \( f \) is a injection. Is the left-inverse \( g \) unique? (10 points).

See the book.

4. Let \( f : X \to Y \) be a bijection and \( f^{-1} \) its inverse. Show that:

\[
(f^{-1})(B) = f^{-1}(B)
\]

(direct image of \( B \) through inverse \( f^{-1} \)) = (inverse image of \( B \) through \( f \))

(10 points).

Let \( x \in (f^{-1})(B) \). By definition of the direct image, there exists \( y \in B \) such that \( f^{-1}(y) = x \). Since \( f(x) = f(f^{-1}(y)) = y \), this implies that \( x \in f^{-1}(B) \). Conversely, assume that \( x \in f^{-1}(B) \). By the definition of the inverse image, \( y = f(x) \in B \). But \( f \) is invertible so \( x = f^{-1}(y) \). Consequently, \( x \in (f^{-1})(B) \).
5. Let $f : X \to Y$ be a function. Prove that, for an arbitrary set $C \subset Y$,

$$f^{-1}(\mathcal{R}(f) \cap C) = f^{-1}(C)$$

(10 points).

Use the property

$$f^{-1}(D \cap C) = f^{-1}(D) \cap f^{-1}(C)$$

with $D = \mathcal{R}(f)$. Notice that $f^{-1}(\mathcal{R}(f)) = X$.

6. Prove that if $A$ is infinite, then $\#(A \times \{1, 2, \ldots, m\}) = \#A$. You may use the fact that if $A$ is infinite, and $B$ is finite then $\#(A \cup B) = \#A$. Follow the steps:

(a) Argue why the result is true for a denumerable set $A$.

Let $A = \{a_1, a_2, a_3, \ldots\}$. Define map $T$:

$$T(n) = \begin{cases} 
(a_k, 1) & \text{if } n = m(k - 1) + 1 \\
(a_k, 2) & \text{if } n = m(k - 1) + 2 \\
\vdots & \text{if } n = m(k - 1) + m = mk 
\end{cases}$$

The map is a bijection from $\mathbb{N}$ onto $A \times \{1, 2, \ldots, m\}$.

(b) Define a family $\mathcal{F}$ of couples $(X, T_X)$, where $X \subset A$ is infinite and $T_X : X \times \{1, 2, \ldots, m\} \to X$ is a bijection. Prove that the following relation is a partial ordering in $\mathcal{F}$.

$$(X_1, T_{X_1}) \leq (X_2, T_{X_2}) \text{ iff } X_1 \subset X_2 \text{ and } T_{X_2} \text{ is an extension of } T_{X_1}$$

Standard reasoning. The relation is reflexive since $X \subset X$ and $T_X$ is an extension of itself. It is antisymmetric. Indeed, $(X_1, T_{X_1}) \leq (X_2, T_{X_2})$ and $(X_2, T_{X_2}) \leq (X_1, T_{X_1})$ imply that $X_1 = X_2$ and, consequently, $T_{X_1} = T_{X_2}$. Similar arguments are used to prove transitiveness.

(c) Use the Kuratowski-Zorn lemma to conclude that $\mathcal{F}$ has a maximal element $(X, T_X)$.

Let $(X_\tau, T_{X_\tau}), \tau \in I$ be a chain. Define,

$$X = \bigcup_{\tau \in I} X_\tau, \quad T_X(x) = T_{X_\tau}(x), \text{ where } x \in X_\tau$$

Then map $T_X$ is well defined, $T_X$ is a bijection from $X$ onto $X \times \{1, 2\}$ so, the pair $(X, T_X)$ is an upper bound for the chain.

Consequently, by the K-Z lemma, there exists a maximal element $(X, T_X)$.
(d) Use the existence of the maximal element to conclude the theorem.

This is the tricky part... Two cases are possible.

**Case:** $X \sim A$. We have then

$$A \sim X \sim X \times \{1, 2, \ldots, m\} \sim A \times \{1, 2, \ldots, m\}$$

since $X \sim A$ implies $X \times \{1, 2, \ldots, m\} \sim A \times \{1, 2, \ldots, m\}$.

**Case:** $\#X < \#A$. The difference $A - X$ cannot be finite as then $X \sim A$. In this case, we are able to find a denumerable subset $Y \subset A$. By step (a), there exists a bijection $T_Y$ from $Y$ onto $Y \times \{1, \ldots, m\}$. Consequently, the pair

$$(X \cup Y, T_X \cup T_Y)$$

is in the family and provides a strict bound for $(X, T_X)$, a contradiction with $(X, T_X)$ being maximal.

**Question:** Where do we need step (a)?

To assure that family $\mathcal{F}$ is non-empty, and in the last step.

(20 points).