1. (a) State the Sturm–Liouville theorem (5 points).

Consider differential operator,

\[ Ly = -(a(x)y')' + c(x)y, \quad x \in (0, l) \]

accompanied with a combination of any of the boundary conditions:

- **Dirichlet BC:**
  \[ y = 0 \]

- **Neumann BC:**
  \[ y' = 0 \]

- **Robin (Cauchy) BC:**
  \[ \alpha y + \beta y' = 0 \]

- **Finite energy condition:**
  \[ y, y' \text{ finite} \]

or the periodic case:

\[ a(0) = a(l), \quad c(0) = c(l), \quad y(0) = y(l), \quad y'(0) = y'(l) \]

The operator is then self-adjoint and possesses a sequence of real eigenvalues

\[ \lambda_1 < \lambda_2 < \ldots < \lambda_n \rightarrow \infty \]

with the corresponding eigenvectors \( y_n \) providing an \( L^2 \)-orthogonal basis for space \( L^2(0, l) \).

(b) Consider the problem:

\[ y'' + \lambda y = 0, \quad y(0) = y'(0), \quad y(1) = 0 \]

Is this a Sturm–Liouville eigenproblem? Explain (5 points).

Yes, it is. \( Ly = -y'' \), we have Robin BC at \( x = 0 \) and Dirichlet BC at \( x = 1 \).
(c) Determine the eigenfunctions for the problem deriving an appropriate transcendental equation for the eigenvalues. (15 points).

The operator $-y''$ with the BCs is positive-definite, so we can assume $\lambda = k^2, k > 0$. This gives

$$y = A \sin kx + B \cos kx$$

Due to the simpler BC at $x = 0$, it is convenient to shift the origin to $x = 1$ and consider the general solution in the form:

$$y = A \sin k(x - 1) + B \cos k(x - 1)$$

Then BC $y(1) = 0$ implies $B = 0$. BC $y(0) = y'(0)$ implies condition:

$$- \sin k = k \cos k$$

$\cos k$ must be different from zero. Indeed, if $\cos k = 0$ then $\sin k \neq 0$ and the equation cannot be satisfied. Dividing by $\cos k$, we get a transcendental equation,

$$\tan k = -k$$

with a sequence of roots $k_n \in (n\pi, n\pi + \frac{\pi}{2}), n = 1, \ldots$. The corresponding eigenvectors,$$
y_n = \sin k_n(x - 1), \quad n = 1, \ldots$$

form an $L^2$-orthogonal basis in $L^2(0, 1)$, i.e. any function $f \in L^2(0, 1)$ can be expanded into the (non-standard) Fourier series:

$$f(x) = \sum_{n=1}^{\infty} f_n \sin k_n(x - 1)$$

where

$$f_n = \frac{\int_0^1 f(x) \sin k_n(x - 1) \, dx}{\left[ \int_0^1 \sin^2 k_n(x - 1) \, dx \right]^{1/2}}$$

\[^1\]Sine and cosine functions never vanish simultaneously.
2. (a) Consider wave equation in three space dimensions,

$$\Delta u = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

Represent the equation in standard spherical coordinates (5 points.)

A good starting point is the formula for the gradient,

$$\nabla u = \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial u}{\partial \psi} e_{\psi} + \frac{1}{r \sin \psi} \frac{\partial u}{\partial \theta} e_{\theta}$$

Integration by parts (jacobian = $r^2 \sin \psi$) yields then

$$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin \psi} \frac{\partial}{\partial \psi} (\sin \psi \frac{\partial u}{\partial \psi}) + \frac{1}{r^2 \sin^2 \psi} \frac{\partial^2 u}{\partial \theta^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}$$

(b) Assume that the solution is point-symmetric (depends only upon radial coordinate $r$ and time $t$).

Provide a classification for second order PDEs and classify the equation (5 points).

Equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \text{lower order terms} = f$$

is elliptic if $d = AC - B^2 > 0$, parabolic if $d = 0$ and hyperbolic if $d < 0$. In our case $A = 1, C = 1/a^2$, so the equation is hyperbolic.

(c) Demonstrate that the equation may be reexpressed as

$$\frac{\partial^2}{\partial r^2} (ru) = \frac{1}{a^2} \frac{\partial^2}{\partial t^2} (ru)$$

and use the d’Alembert method or any other possible approach to derive the general solution to the problem (15 points).

The first part is just simple algebra. Substituting $v = ru$ then, we get the 1D wave equation for which the d’Alembert solution is

$$v(x) = f(x - at) + g(x + at)$$

where $f$, $g$ are arbitrary functions. This yields

$$u(r) = \frac{1}{r} [f(r - at) + g(r + at)]$$
3. (a) Solve the following 2D boundary-value problem (25 points).

Recalling Laplacian in polar coordinates,

\[-\Delta u = \frac{1}{r}(ru_r)_r - \frac{1}{r^2}u_{\theta\theta}\]

we seek the solution in the form:

\[u = v(r, \theta) + 50\]

trading the non-homogeneous BC in \(u\) at \(\theta = \alpha\), for a non-homogeneous BC in \(v\) at \(r = a\),

\[v(a, \theta) = -50\]

Seeking \(v = R(r)T(t)\), we get

\[\frac{r(rR')'}{R} = -\frac{\Theta''}{\Theta} = \lambda = k^2 > 0, \quad k > 0\]

Notice that we have obtained a Sturm-Liouville problem in \(\theta\) and that operator \(-\Theta''\) with Dirichlet BCs is positive-definite, so we can assume that \(\lambda\) is real and positive. Solving for \(\Theta\), we get,

\[\Theta_n = \cos k\theta, \quad k = k_n = \frac{\pi}{2\alpha} + n\frac{\pi}{\alpha}, \quad n = 1, 2, \ldots\]

which gives now the Cauchy-Euler equation for \(R(r)\),

\[r(rR')' + k^2R = 0\]

with

\[R(r) = r^{\pm k}\]

Rejecting the singular solution, we get by superposition,

\[v(r, \theta) = \sum_{n=1}^{\infty} A_n r^{k_n} \cos k_n \theta, \quad k_n = \frac{\pi}{2\alpha} + n\frac{\pi}{\alpha}, \quad n = 1, 2, \ldots\]

Constants \(A_n\) are determined from the boundary condition at \(r = a\),

\[\sum_{n=1}^{\infty} A_n a^{k_n} \cos k_n \theta = -50\]

\[A_n = a^{-k_n} \int_{0}^{\alpha} (-50) \cos k_n \theta \left[\int_{0}^{\alpha} \cos^2 k_n \theta\right]^{1/2} d\theta\]
4. (a) Define characteristics for a single first order equation,

\[ a(x, y, z)u_x + b(x, y, z)u_y + c(x, y, z)u_z = 0 \]

and discuss the relation with the concept of prime integrals for a system of ODEs. (5 points).

Characteristics are curves that satisfy the differential equation:

\[ \frac{dx}{a(x, y, z)} = \frac{dy}{b(x, y, z)} = \frac{dz}{c(x, y, z)} \]

or, in a parametric form,

\[ \frac{dx}{dt} = a, \quad \frac{dy}{dt} = b, \quad \frac{dz}{dt} = c \]

Solution of the original equation is constant along the characteristics, i.e. it is its prime integral.

(b) Determine the general solution of the equation:

\[ u_x + u_y + 3u_z = 0 \]

(10 points).

The equations for the characteristics yield,

\[ \frac{dx}{1} = \frac{dy}{1} \implies y = x + c \]

and

\[ \frac{dx}{1} = \frac{dz}{3} \implies z = 3x + d \]

Consequently, \( y - x \) and \( z - 3x \) are two LI prime integrals, and the general solution of the original equation is given by,

\[ u = f(y - x, z - 3x) \]

where \( f \) is an arbitrary function.

(c) Determine the solution of the equation above in the first octant: \( x, y, z > 0 \) with initial conditions:

\[ u(x, y, 0) = xy^2; \quad u(x, 0, z) = zx^2; \quad u(0, y, z) = yz^2 \]

(10 points).

It is easier to work with the parametric representation of the characteristic

\[ x = t + A, \quad y = t + B, \quad z = 3t + C \]

We can scale parameter \( t \) to depend upon a point of interest \( x_0, y_0, z_0 \) in such a way that,

\[ x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0 \]
This gives,

\[ x = t + x_0, \quad y = t + y_0, \quad z = 3t + z_0 \]

Depending upon the location of \((x_0, y_0, z_0)\), the characteristics will pass through one of the coordinate planes first, and the IC corresponding to that plane has to be used. For instance, if \(x_0 < y_0, z_0/3\) then the characteristics issued at \((x_0, y_0, z_0)\), will intersect plane \(x = 0\) first, at the point

\[ y = y_0 - x_0, \quad z = z_0 - 3x_0 \]

and the value of the solution at \(x = 0\) will be equal to the value at \((x_0, y_0, z_0)\),

\[ u(x_0, y_0, z_0) = (y_0 - x_0)(z_0 - 3x_0)^2 \]

Same reasoning applies to the two remaining cases. Note that the resulting solution is discontinuous.