1. A particle is moving in a straight line with an acceleration $a(t) = a(t - t_0)$, where $a$ is a positive constant and $t_0 > 0$ is a specified instant. At the time $t_0$, the corresponding position and velocity are $x_0$ and $v_0$, respectively. Derive the formulas for position $x(t)$ and velocity $v(t)$ at any time $t$ (5 points)

$$
\begin{align*}
\ddot{x} &= a \frac{(t - t_0)}{x_0} \\
\dot{x} &= a \frac{(t - t_0)}{x_0} + c \\
v(t_0) &= v_0 \\
\Rightarrow \quad c &= v_0
\end{align*}
$$

2. At point $r = 1$m, $\theta = \pi/6$, the polar components of a velocity vector for a particle are $v_r = 0, v_\theta = \dot{\theta}$. Calculate the Cartesian components of the velocity vector. (5 points)

$$
\begin{align*}
\vec{v} &= v_r \hat{e}_r + v_\theta \hat{e}_\theta \\
&= \dot{\theta} \hat{e}_\theta \\
&= (-0.5, 0.866) \quad \left[ \frac{m}{s} \right]
\end{align*}
$$
3. Can a particle moving on a curvilinear path have a zero acceleration (vector)? Explain. (5 points)

\[ \ddot{a} = \alpha_t \varepsilon_t + a_n \varepsilon_n \]
\[ a_n = \frac{v^2}{s} \]

Particle is moving \[ \Rightarrow v \neq 0 \]
Path is curved \[ \Rightarrow s \neq \infty \] \[ \Rightarrow a_n \neq 0 \]

No !

4. Derive the formulas for acceleration vector components \( a_r \) and \( a_\theta \) in the polar system of coordinates (5 points)

\[ \varepsilon_r = (\cos \theta, \sin \theta, 0) \quad \frac{d \varepsilon_r}{d \theta} = (-\sin \theta, \cos \theta, 0) = -\varepsilon_\theta \]
\[ \varepsilon_\theta = (-\sin \theta, \cos \theta, 0) \quad \frac{d \varepsilon_\theta}{d \theta} = (-\cos \theta, -\sin \theta, 0) = -\varepsilon_r \]

\[ \ddot{r} = \dot{r} \varepsilon_r + r \frac{d \varepsilon_r}{d \theta} + \ddot{\theta} \varepsilon_\theta + r \dot{\theta} \varepsilon_r = \frac{\dot{r} \varepsilon_r}{v_r} + \frac{\dot{\theta} e_\theta}{v_\theta} \]
\[ \ddot{r} = \frac{(\ddot{r} - \dot{r} \ddot{\theta}) \varepsilon_r + (\ddot{\theta} + 2 \dot{\theta} \dot{r}) \varepsilon_\theta}{\dot{r}} \]

\[ a_r = \frac{\dot{r} \varepsilon_r + \ddot{r} \varepsilon_r + \dot{r} \dot{\theta} \varepsilon_\theta + r \ddot{\theta} \varepsilon_\theta}{\dot{r}} \]

5. A particle moves along a parabola \( y = x^2 \) with a constant speed \( v \). Determine the Cartesian components of the velocity vector as a function of coordinate \( x \). (5 points)

\[ y = x^2 \quad \Rightarrow \quad \dot{y} = 2x \dot{x} \]
\[ x^2 + \dot{y}^2 = v^2 \quad \Rightarrow \quad \dot{x}^2 + (2x \dot{x})^2 = x^2 (1 + 4x^2) = v^2 \]

\[ \begin{cases} \dot{x} = \pm \frac{v}{\sqrt{1 + 4x^2}} \\ \dot{y} = \pm \frac{2xv}{\sqrt{1 + 4x^2}} \end{cases} \]
6. A projectile is launched at 10 m/s from a sloping surface. The angle $\alpha = 80^\circ$. Determine the range $R$. (25 points)

\[ x(t) = 10 \cos 50^\circ \cdot t = 6.428 \, t \, [m] \]
\[ y(t) = -\frac{gt^2}{2} + V_{oy} \, t \]
\[ = -4.905 \, t^2 + 10 \sin 50^\circ \, t \]
\[ = -4.905 \, t^2 + 7.66 \, t \]

At some time $t$:
\[ \frac{y}{x} = -\tan 30^\circ = -0.5774 \]
\[ -4.905 \, t^2 + 7.66 \, t = -0.5774 \left( 6.428 \, t \right) \]
\[ = -3.711 \, t \]
\[ -4.905 \, t = -3.711 - 7.66 = -11.37 \]
\[ t = 2.138 [s] \]

\[ x(t) = 14.90 \, [m] \]
\[ y(t) = -8.60 \, [m] \]

Check: $\frac{y}{x} = -0.5774 \checkmark$

\[ R = \sqrt{x^2 + y^2} = 17.20 \, [m] \]
at \( s = 120 \text{ ft} \), the car is on the first circle, so \( s = r = 120 \text{[ft]} \)

\[
a_n = \frac{v^2}{s} = \frac{5429.2}{120} = 45.24 \text{ [ft/s]}
\]

\[
a = \sqrt{a_n^2 + a_h^2} = 45.9 \text{ [ft/s]}
\]

(b) \( v^2 = 59.3^2 + 2 \cdot 7.97 \cdot 160 = 6067 \text{ [ft}^2/\text{s}^2] \)

at \( s = 160 \text{ ft} \), the car is on the second circle, so \( s = r = 100 \text{[ft]} \)

\[
a_n = \frac{v^2}{s} = \frac{6067}{100} = 60.7 \text{ [ft/s]}
\]

\[
a = \sqrt{a_n^2 + a_h^2} = 61.2 \text{ [ft/s]}
\]
7. The car increases its speed at a constant rate from 40 mi/h at A to 60 mi/h at B. Determine the magnitude of its acceleration when it has traveled along the road a distance (a) 120 ft from A, and (b) 160 ft from A. (25 points)

\[ a_t = \frac{dv}{ds} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{dt} \]

\[ a_t \cdot ds = v \cdot dv \]

\[ S_8 = 80 + 120 \frac{30}{180} \pi + 100 \frac{30}{180} \pi + 80 = 275.2 \text{ [ft]} \]

\[ 40 \text{ mi/h} = \frac{40 \times 5280}{3600} = 59.3 \text{ [ft/s]} \]

\[ 60 \text{ mi/h} = \frac{88.8}{59.3} \]

\[ 1 \text{ mi} = 1600 \text{ m} \times \frac{1 \text{ ft}}{0.305} = 5333 \text{ [ft]} \]

\[ \int_{a_1}^{a_2} dx = \int_{0}^{1} v \cdot dv \]

\[ a_t \cdot 275.2 = \frac{1}{2} \left( 88.8^2 - 59.3^2 \right) \]

\[ a_t = 7.97 \text{ [ft/s}^2] \]

(a) In an arbitrary \( S \), \( \int_{a_1}^{a_2} a_t \cdot ds = \int_{v_1}^{v} v \cdot dv \)

\[ a_t \cdot S = \frac{1}{2} \left( v^2 - v_1^2 \right) \]

So:

\[ v^2 = v_1^2 + 2a_t \cdot S \]

\[ v^2 = 59.3^2 + 2 \cdot 7.97 \cdot 120 = 5429.2 \text{ [ft}^2] \]
8. The hydraulic actuator moves the pin P upward with constant velocity $v = 2j$ (m/s). Determine the acceleration of the pin in terms of polar coordinates and the angular acceleration $\ddot{\theta}$ of the slotted bar when $\theta = 35^\circ$. (25 points)

\[ v_T = v \sin \theta = 1.147 \left[ \frac{m}{s} \right] \]
\[ v_\theta = v \cos \theta = 1.638 \left[ \frac{m}{s} \right] \]

\[ a_T = \omega \dot{\theta} = 0 \]

at $\theta = 35^\circ$  \[ r = \frac{v}{\omega} = 2.442 \left[ m \right] \]

\[ v_T = r \dot{r} = 1.147 \Rightarrow \dot{r} = 1.147 \left[ \frac{m}{s} \right] \]
\[ v_\theta = r \dot{\theta} = 1.638 \Rightarrow \dot{\theta} = 0.67 \left[ \frac{rad}{s} \right] \]

\[ a_\theta = r \ddot{\theta} + 2 \dot{r} \dot{\theta} = 0 \]

\[ \ddot{\theta} = - \frac{2 \dot{r} \dot{\theta}}{r} = - \frac{2 \cdot 1.147 \cdot 0.67}{2.442} = - 0.63 \left[ \frac{rad}{s^2} \right] \]
Switch to the cartesian components

\[ v = v_r \hat{r} + v_\theta \hat{\theta} \]

\[ = v_r (\cos \theta \hat{x} + \sin \theta \hat{y}) + v_\theta (-\sin \theta \hat{x} + \cos \theta \hat{y}) \]

\[ = \left( \frac{v_r \cos \theta - v_\theta \sin \theta}{v_x} \right) \hat{x} + \left( \frac{v_r \sin \theta + v_\theta \cos \theta}{v_y} \right) \hat{y} \]

\[ \begin{align*}
V_x &= -5.76 \quad [\text{m/s}] \\
V_y &= 3.50 \quad [\text{m/s}] 
\end{align*} \]

Verifications:

- Check for \( r \) (cosine law)
  \[ 0.4^2 = 0.532^2 + 0.2^2 - 2 \cdot 0.532 \cdot 0.2 \cdot \cos 40^\circ \]
  \[ 0.16 = 0.16 \quad \text{OK} \]

- Check for speed
  \[ v_r^2 + v_\theta^2 = v_x^2 + v_y^2 \]
  \[ 6.38^2 + 2.16^2 = 5.76^2 + 3.50^2 \]
  \[ 45.42 = 45.42 \quad \text{OK} \]
1. Derive the equation of motion for the center of mass of an arbitrary system of particles. (5 points)

\[ \sum m_i \ddot{r}_i = \sum F_i + \sum R_i \]

2. Define an inertial frame of reference. (5 points)

Any system of coordinates in which the Newton laws of motion hold.
3. If string $A$ is cut, does the tension in string $B$ remain the same as in the stationary case? Explain. (5 points)

\[ \Sigma F_y = 0 \quad T_2 \frac{\sqrt{2}}{2} \frac{mg}{2} = 0 \]

\[ T_2 = \frac{1}{2} mg \]

Dynamic case

\[ mg \cos 45^\circ = T - mg \cos 45^\circ \]

\[ \therefore T = mg \frac{\sqrt{2}}{2} \]

4. What does it mean that a force field $F$ is conservative? Give example of a conservative force field and the corresponding potential energy. (5 points)

\[ F \text{ is conservative if there exists a potential } V = V(x), \text{ such that } F = -\nabla V, \text{ e.g. } \text{ spring force:} \]

\[ F = -k (r-r_0) \frac{r}{r} \]

\[ V = \frac{1}{2} k \Delta r^2 \]

5. Derive the principle of angular momentum for a single particle. (5 points)
6. A 2-kg mass rests on a flat horizontal bar. The bar begins rotating in the vertical plane about O with a constant angular acceleration of 1 rad/s$^2$. The mass is observed to slip relative to the bar when the bar is 30° above the horizontal. What is the static coefficient of friction between the mass and the bar? Does the mass slip toward or away from O? (25 points)

\[ r = 1 \text{ m} = \text{constant} \Rightarrow \dot{r} = \ddot{r} = 0 \]

\[ a_r = \ddot{r} - r \omega^2 = -r \omega^2 = -\frac{\pi}{3} \left[ \frac{a}{s^2} \right] \]

\[ a_\theta = r \alpha + 2 r \dot{\omega} = r \alpha = 1 \text{ m/s}^2 \]

\[ \dot{\omega} = \frac{\frac{\pi}{3}}{r} \]

\[ \omega^2 = \frac{a}{r} = \frac{\pi}{3} \]

\[ \text{friction:} \quad m \dot{\omega} = N - mg \cos 30^\circ \quad \Rightarrow \quad N = m (1 + g \cos 30^\circ) \]

\[ \text{acceleration:} \quad m \ddot{r} = F - mg \sin 30^\circ \]

\[ \Rightarrow \quad F = m \ddot{r} + mg \sin 30^\circ = m \left( -\frac{\pi}{3} + g \frac{\sqrt{3}}{2} \right) > 0 \]

(That indicates that the mass will slide down the bar since $F$ has to oppose the motion)

\[ \mu = \frac{F}{N} = \frac{\mu (0.5 g - \frac{\pi}{3})}{\mu (1 + g \frac{15}{2})} = \frac{3.857}{3.495} = 0.406 \]
7. A 2-kg disk slides on a smooth horizontal table and is connected to an elastic cord whose tension is \( T = 6r \) N, where \( r \) is the radial position of the disk in meters. The disk is at \( r = 1 \) m when it is given an initial velocity of 4 m/s in the transverse direction. What is the maximum value of \( r \) reached by the disk? (25 points)

![Diagram of a disk sliding on a table connected to an elastic cord]

**Conservation of angular momentum about 0**

\[
mlv_{\theta_1} \cdot r_1 = mlv_{\theta_2} \cdot r_{\text{max}}
\]

\[
v_{\theta_2} \cdot r_{\text{max}} = 4 \cdot 1 = 4 \quad \Rightarrow \quad v_{\theta_0} = \frac{4}{r_{\text{max}}}
\]

**Principle of work and energy**

\[
\frac{m v_{\theta_0}^2}{2} = \frac{m v_{\theta_1}^2}{2} + \frac{6}{2} \left( 1^2 - r_{\text{max}}^2 \right)
\]

Work done by the spring

\[
= 16 + 3 - 3r_{\text{max}}^2 = 19 - 3r_{\text{max}}^2
\]

\[
\frac{16}{r_{\text{max}}^2} = 19 - 3r_{\text{max}}^2
\]

\[
\frac{16}{x} = 19 - 3 \cdot x
\]

\[
16 = 19x - 3x^2
\]

\[
3x^2 - 19x + 16 = 0
\]

\[
x_{1,2} = \frac{19 \pm \sqrt{19^2 - 4 \cdot 3 \cdot 16}}{2 \cdot 3} = 5.33
\]

\( x = 1 \) corresponds to the initial position, so

\[
r_{\text{max}} = \sqrt{5.33} = 2.309 \text{ [m]}
\]
8. Bar $AB$ rotates at 4 rad/s in the counterclockwise direction. Determine the velocity of point $C$. (25 points)

\[ \mathbf{v}_C = \mathbf{v}_B + \mathbf{\omega}_{BD} \times \mathbf{r}_C \]

\[ = (-2.4, 1.2, 0) + \frac{\mathbf{\omega}_{BD} (0, 0, 1.246)}{0.584, -1.168, 0} \]

\[ = (-1.816, 0.032, 0) \]

\[ \mathbf{v}_E = \mathbf{v}_D + \mathbf{\omega}_{DE} \times \mathbf{r}_E \]

\[ = (-2.4, 1.2, 0) + \frac{\mathbf{\omega}_{DE} (0, 0, 1.246)}{0.584, -1.168, 0} \]

\[ = \frac{-2.4 + 0.1\mathbf{\omega}_{BD} + 0.5\mathbf{\omega}_{DE}}{1.2 + 0.5\mathbf{\omega}_{BD} + 0.3\mathbf{\omega}_{DE}} = 0 \]

\[ \mathbf{\omega}_{BD} = \begin{bmatrix} 0.1 & 0.5 \\ 0.8 & 0.3 \\ -1.2 & 0.3 \end{bmatrix} \]

\[ \mathbf{\omega}_{DE} = \begin{bmatrix} 0.1 & 2.4 \\ 0.8 & -1.2 \\ -0.37 \end{bmatrix} \]
1. Use integration to compute the product of inertia $I_{xy}$ for a triangle below. (5 points)

$$I_{xy} = \frac{1}{2} \int_0^b \int_0^{h(1-\frac{x}{b})} xy \, dy \, dx = \frac{1}{2} \int_0^b \left( \frac{x^2}{2} - \frac{2x^3}{6b} + \frac{x^4}{4b^2} \right) \, dx$$

$$= \frac{1}{2} \left[ \frac{b^3}{3} \left( 1 - \frac{2}{3} \frac{x^2}{b^2} + \frac{x^4}{4b^2} \right) \right]_0^b = \frac{1}{3} b^3 \left( \frac{1}{2} \frac{x^2}{b^2} + \frac{3}{4} \right) = \frac{1}{3} \cdot \frac{1}{2} b^2 h^2 = \frac{1}{12} \text{mbh}$$

2. System of coordinates $Oxyz$ is rotating with respect to a fixed system of coordinates $OXYZ$ around the $Z = z$ axis with a constant angular velocity $\omega$. Particle $P$ is moving with a constant velocity vector $\mathbf{v} = (v_x, 0, 0)$ in the fixed system of coordinates. Compute the Coriolis acceleration of the particle in frame $Oxyz$, at the instant when the two systems coincide with each other. (5 points)

$$a_c = \frac{2 \times \frac{\partial (0, 0, \omega)}{\partial (v_x, 0, 0)}}{(\omega x, 0, 0)} = \left( \mathbf{\omega} \times \mathbf{v} \right)$$
3. Derive the principle of angular impulse and momentum for an arbitrary system of particles. (5 points)

Consider case: \( \theta \) fixed.

\[
\sum ( \mathbf{I}_i \times \mathbf{v}_i) = \sum \mathbf{I}_i \times \mathbf{v}_i + \mathbf{L}_i + \mathbf{R}_i
\]

\[
\mathbf{H}_0 = \sum \mathbf{I}_i \mathbf{v}_i + \sum \mathbf{F}_i \times \mathbf{r}_i
\]

4. Analyze the two scenarios depicted below. In which case do you expect a smaller initial acceleration of the system? Explain, why. (5 points)

In the second, forces are identical but the inertia is bigger.

5. Derive the formula for the kinetic energy of a rigid body undergoing an arbitrary planar motion. (5 points)

\[
\mathbf{v}_2 = \mathbf{v}_1 + \omega \times \mathbf{r}\]

\[
\mathbf{v}_2 = (v_{ox}, v_{oy}, 0) + \frac{\omega}{2} (0, 0, 0)
\]

\[
\mathbf{v}_2 = (v_{ox} - \omega y, v_{oy} + 2x, 0)
\]

\[
T = \frac{1}{2} \int (v_{ox} + v_{oy})^2 + (v_{oy} + 2x) \frac{\omega}{2} \int \mathbf{r} \times \mathbf{v} dV
\]

\[
= \frac{1}{2} m v_1^2 + \frac{1}{2} I_2 \omega^2
\]
6. The 2-kg slender bar and \( k \)-kg block are released from rest in the position shown. If friction is negligible, what is the block’s acceleration at that instant? (25 points)

\[ I_C = \frac{1}{12} \cdot 2 \cdot 1^2 = 0.167 \text{ [kg}\cdot\text{m}^2] \]

\[ I \alpha = 0.167 \alpha \]

\[ 19.62 \cdot 0.287 + 2(\alpha - 0.41\alpha)(0.41) - 0.167\alpha = 0 \]

\[ 0.82\alpha = -5.631 + 0.165\alpha + 0.336\alpha + 0.167\alpha \]

\[ a = 0.117 \alpha \]

Substituting into (1) \[ 0.096\alpha = -5.631 + 0.668\alpha \]

\[ \alpha = 9.844 \left[ \frac{\text{ms}^2}{\text{s}^2} \right] \]

\[ \Rightarrow a = 1.151 \left[ \frac{\text{ms}^2}{\text{s}^2} \right] \]
7. The slender bar is released from rest with $\theta = 45^\circ$ and falls a distance $h = 1$ m onto the smooth floor. The length of the bar is 1 m and its mass is 2 kg. If the coefficient of restitution of the impact is $e = 0.4$, what is the angular velocity of the bar just after it hits the floor? (25 points)

\[
I_c = \frac{2I}{12} = 0.167 \text{ [kg m}^2]\]

\[
\text{Stage 1: Free fall, use the work and energy principle}
\]

\[
T_1 + U_{k1} = T_2
\]

\[
mgh = \frac{1}{2}mv^2
\]

\[
v = \sqrt{2gh} = 4.43 \text{ [m/s]}
\]

\[
\text{Stage 2: Impact}
\]

\[
\text{C}
\]

\[
\text{A}
\]

\[
\text{impact force}
\]

- Conservation of linear momentum in $x$-direction $\Rightarrow V_{\text{x after}} = 0$

- Conservation of angular momentum at fixed (on the floor) point $A$

Before:

$H_A = H_A$

\[
H_A = AC \times \frac{1}{m} v_C^2 + \frac{1}{2} \omega_c^2 I_c
\]

\[
(0, 0, 3.132) \text{ [m kg m}^2\text{]}\]

After:

$H_a = 0$

\[
H_a = AC \times \frac{1}{m} v_{C, \text{after}}^2 + \frac{1}{2} \omega_c^2 I_c
\]

\[
(-0.353, 0.353, 0)
\]

\[
(0, 0, -0.706 v_{C, \text{after}}^2) + (0, 0, 0.167 C, \text{after})
\]

\[
3.132 = -0.706 v_{C, \text{after}}^2 + 0.167 C, \text{after}\]

\[
0.4 = - \frac{v_{A, \text{after}}}{v_{A, \text{before}}} \Rightarrow v_{A, \text{after}} = -0.4 \cdot (-4.43) = 1.772 \text{ [m/s]}
\]
7 continued

- Kinematics

\[ \mathbf{V}_C = \mathbf{V}_A + \omega \times \mathbf{A}_C = (V_{Ax}, V_{Ay}, 0) + \omega \left( -0.353, 0.353, 0 \right) \]

\[ V_{Cy} = V_{Ay} - 0.353 \omega = 1.772 - 0.353 \omega \]

- Substituting into \( \ast \)

\[ 3.132 = -0.706 \left( 1.772 - 0.353 \omega \text{ after} \right) + 0.167 \omega \text{ after} \]

\[ 3.132 = -1.251 + (0.249 + 0.167) \omega \text{ after} \]

\[ \omega \text{ after} = 10.54 \left[ \frac{\text{rad}}{\text{s}^2} \right] \]
8. The bar shown below is stationary relative to an inertial reference frame when the force \( F = 12 \mathbf{k} \) (N) is applied at the right end of the bar. No other forces or couples act on the bar. Determine the bar's angular acceleration relative to the inertial reference frame. (25 points)

\[ \text{total mass } m = 6 \text{ kg} \]

\[ \begin{align*}
x_c &= \frac{2 \cdot 1}{3} = 0.667 \text{ m} \\
y_c &= \frac{1 \cdot 1.5}{3} = 0.167 \text{ m}
\end{align*} \]

**Step 1: Compute tensor of inertia at C**

\[
I_x = \frac{1}{12} \cdot 2 \cdot 1^2 + 2 \cdot (0.5 - 0.167)^2 = 0.5 \text{ [kg m}^2]\]

\[
I_y = \frac{1}{12} \cdot 4 \cdot 2^2 + 4 \cdot (1 - 0.667)^2 + 0 + 2 \cdot 0.667^2 = 2.67 \text{ [kg m}^2]\]

\[ I_x' = I_x \]

\[ I_y' = I_y \]

\[
I_{xy} = 0 - 4 \cdot (1 - 0.667) \cdot 0.167 + 0 - 2 \cdot 0.167 \cdot (0.5 - 0.167) = -0.322 \text{ - 0.444} = -0.67 \text{ [kg m}^2]\]

\[ I_{xz} = I_{yz} = 0 \]
Step 2: The rotational motion equation:

\[
\frac{d^2 \theta}{dt^2} + \omega \times (\omega \times \mathbf{r}) = \tau_c
\]

\[
\tau_c = \mathbf{C}_A \times \mathbf{F} = \mathbf{F} \times \mathbf{C}_A \begin{pmatrix} 1.333 & -0.167 & 0 \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 12 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} -16 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} \begin{pmatrix} \tau_c \end{pmatrix} [N m]
\]

\[
\begin{pmatrix} 0.5 & 0.67 & 0 \\ 0.67 & 2.67 & 0 \\ 0 & 0 & 3.17 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} -2 \\ -16 \\ 0 \end{pmatrix}
\]

\[
\alpha_z = 0
\]

\[
\alpha_x = \frac{\begin{vmatrix} -2 & 0.67 \\ 0.5 & 2.67 \end{vmatrix}}{\begin{vmatrix} 0.5 & 0.67 \\ 0.67 & 2.67 \end{vmatrix}} = \frac{5.38}{0.886} = 6.07 \left[ \frac{rad}{s^2} \right]
\]

\[
d_y = \frac{\begin{vmatrix} 0.67 & -16 \\ 0.886 & 0.886 \end{vmatrix}}{\begin{vmatrix} 0.5 & -2 \\ 0.67 & 2.67 \end{vmatrix}} = \frac{-6.66}{-0.02} = -7.52 \left[ \frac{rad}{s^2} \right]
\]
1. At the instant shown, the coordinates of the slider $A$ are $x = 1.5$, $y = 1.0$ ft, and its velocity represented in Cartesian coordinates is $v = (0, 5)$ ft/s. Determine the slider’s velocity components in polar coordinates. (5 points)

\[ \sin \Theta = \frac{1}{\sqrt{1.5^2 + 0^2}} = 0.5547 \]
\[ \cos \Theta = \frac{1.5}{\sqrt{1.5^2 + 0^2}} = 0.8321 \]
\[ v_r = 5 \sin \Theta = 2.77 \text{ [ft/s]} \]
\[ v_\Theta = 5 \cos \Theta = 4.16 \text{ [ft/s]} \]

\[ \text{Check: } \sqrt{v_r^2 + v_\Theta^2} = 5.0 \quad \checkmark \]

2. Particles $A$ and $B$ are connected with a rigid link. Explain why the work of reactive forces representing the interaction between the particles is zero. (5 points)

\[ \overrightarrow{F}_{BA} = \overrightarrow{F}_{AB} \]
\[ F_{BA} = -F \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \]

\[ AB^2 = (r_B - r_A)^2 = \text{constant} \]

\[ \therefore \text{ Incremental work } = \]
\[ - F \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \cdot (dr_B - dr_A) + F \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \cdot dr_A \]
\[ = -F \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \cdot (dr_B - dr_A) = 0 \]
3. Is the drag force \( \mathbf{F} = (-cv^2, 0, 0), (c > 0) \) conservative? Explain. (5 points)

\[ \mathbf{F} \quad \rightarrow \quad \mathbf{v} \]

No! Conservative force is expressed in terms of a potential \( \mathbf{F} = -\nabla \mathbf{V} \), where \( \mathbf{V} = \mathbf{V}(x, y, z) \) and, therefore, it may only depend upon the position, not velocity.

(5)

4. The bar is smooth. Determine the minimum velocity the 10-kg slider must have at \( A \) to reach point \( E \). (5 points)

\[ \begin{align*}
\text{The point } &\text{ the slider must make it through point } E \\
\text{with a velocity at } E &> 0, \\
\frac{m \mathbf{v}_A^2}{2} - \frac{mgh}{3} &> 0 \\
\mathbf{v}_A^2 - \frac{mgh}{3} &> 0 \\
\mathbf{v}_A^2 &> \frac{2}{3} \frac{mgh}{3} = 58.86 \\
\mathbf{v}_A &> \sqrt{2gh} = 7.67 \left( \frac{m}{s} \right)
\end{align*} \]

(5)

5. Compute transformation matrix from system \( 0xyz \) to system \( 0x'y'z' \). (5 points)

\[ \mathbf{T} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix} \]

(5)
6. Objects $A$ and $B$ have masses $m_A$ and $m_B$ and velocities $v_A^{\text{before}}, v_B^{\text{before}}$. Show that, if coefficient of restitution $e = 1$, the total kinetic energy is conserved during the impact. (5 points)

$$e = \frac{v_B' - v_A'}{v_A - v_B} = 1$$

$$v_B' - v_A' = v_A - v_B$$

$$v_B' + v_B = v_A' + v_A$$

Conservation of linear momentum \( \Rightarrow \)

$$m_A v_A + m_B v_B = m_A v_A' + m_B v_B'$$

\[ \Rightarrow \]

$$m_A (v_A - v_A') = m_B (v_B' - v_B) \quad \Rightarrow \quad v_B' + v_B = v_A' + v_A$$

Divide by 2 to finish...

7. Determine the instantaneous center (IC) of zero velocity for member $BC$. (5 points)
8. Compute the product of inertia $I_{xy}$ for a quadrant of a homogeneous circle with radius $R$ and mass $m$. (5 points)

$$
\begin{align*}
\text{Parameterization:} \\
&\begin{cases}
x = r \cos \theta & 0 < r < R \\
y = r \sin \theta & 0 < \theta < \frac{\pi}{2}
\end{cases} \\
\text{jacobian} \\
I_{xy} &= \int \int x y \, dA = \int_{0}^{\frac{\pi}{2}} \int_{0}^{R} r \cos \theta \cdot r \sin \theta \cdot r \, dr \, d\theta \\
&= \frac{R^4}{4} \left(1 + \frac{1}{4}\right) = \frac{R^4}{8} = \frac{\pi R^2}{8} = \frac{1}{2} m R^2
\end{align*}
$$

9. Derive the principle of linear momentum for a system of particles (5 points)

$$
\sum_{i} m_i \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i + \mathbf{R}_i \\
\sum_{i} m_i \frac{d\mathbf{v}_i}{dt} = \sum_{i} \mathbf{F}_i + \sum_{i} \mathbf{R}_i \\
\sum_{i} m_i \mathbf{v}_i (t_2) = \sum_{i} \mathbf{F}_i \cdot dt \\
\sum_{i} m_i \mathbf{v}_i (t_1) + \sum_{i} \mathbf{F}_i \cdot dt = \sum_{i} m_i \mathbf{v}_i (t_2)
$$

10. A slender bar with mass $m$ and length $l$ is rotating about its endpoint $A$ with constant angular velocity $\omega = (0, 0, \omega)$. Compute the angular velocity of the bar with respect to point $B$ (half way between point $A$ and center of mass $C$). (5 points)

$$
\text{momentum at } t_1, \quad \text{impulse} \quad \text{momentum at } t_2
$$

Nothing to compute! Angular velocity is an attitude of the rigid body, i.e. it has nothing to do with any point of computation!
The 2-kg mass $m$ is released from rest with the string horizontal. The length of the string is $L = 0.5$ m. Determine the velocity of the mass and the tension in the spring when $\theta = 45^\circ$. (25 points)

11. **Frenet coordinates:**
\[ m a_n = T - mg \cos 45^\circ \]
\[ a_n = \frac{v^2}{L} \implies T = \frac{m v^2}{L} + mg \cos 45^\circ \]  \( \text{\textcircled{7}} \)

**Principle of work and energy:**
\[ T_1 + U_{12} = T_2 \]
\[ \rho mg L \sin \theta = \frac{1}{2} m v^2 \]
\[ v^2 = 2 \rho g L \sin \theta \]
\[ v = 2.63 \text{ m/s} \]  \( \text{\textcircled{5}} \)

\[ T = \frac{m v^2}{L} + mg \cos 45^\circ \]

\[ = 27.67 + 13.87 = 41.54 \text{ N} \]  \( \text{\textcircled{5}} \)
12. The 10-kg slender bar is released from rest in the horizontal position. When it has fallen to the position shown, what are the $x$ and $y$ components of force exerted on the bar by the pin support $A$? (25 points)

Principle of work and energy

\[ T_t + U_{12} = T_f \]

\[ U_{12} = 10 \cdot 9.81 \cdot 3 \cdot \sin 45^\circ = 69.37 \text{ [N m]} \]

\[ T_f \cdot \frac{1}{2} I_A \omega^2 = 6.67 \omega^2 \]

\[ 69.37 = 6.67 \omega^2 \Rightarrow \omega = 3.226 \text{ [rad/s]} \]

Equation of rotational motion

\[ I_A \alpha = M_A \]

13.33 \( \alpha \) = 10 \cdot 9.81 \cdot 1 \cos 45^\circ

\[ \alpha = 5.204 \text{ [rad/s]} \]

Kinematics

\[ \mathbf{r}_c = \mathbf{r}_A + \mathbf{x} \cdot \mathbf{a}_c = (0, 0, 5.204) - 3.226 \cdot \mathbf{v} \cdot (0.707, -0.707) \cdot (0, 0, 0) \]

\[ = (11.04, 3.68, 0) \text{ [m]} \]

Equations of translational motion:

\[ m \cdot a_{c_x} = H_A \Rightarrow H_A = 110.4 \text{ [N m]} \]

\[ m \cdot a_{c_y} = v_A - 10.981 \Rightarrow v_A = 134.9 \text{ [N]} \]
13. The 1-kg sphere $A$ is moving at 10 m/s when it strikes the end of the 3-kg stationary slender bar $B$. If the sphere adheres to the bar, what is the bar’s angular velocity after the impact? (25 points)

Step 1: Center of mass of the combined system:

\[ y = \frac{3 \cdot 1}{1 + 3} = 0.75 \text{ [m]} \]

Step 2: Moment of inertia at center of mass $C$ of the system

\[ I_c = 1 \cdot 0.75^2 + \left( \frac{1}{12} \cdot 3 \cdot 2^2 + 3 \cdot 0.25^2 \right) \]

\[ = 1.75 \text{ [kg m}^2] \]

Step 3: Conservation of angular momentum of the whole system at center of mass $C$

\[ 1 \cdot 10 \cdot 0.75 = I_c \omega_{afm} \]

\[ \therefore \omega_{afm} = 4.29 \text{ [rad/s]} \]
13. The 1-kg sphere $A$ is moving at $10 \text{ m/s}$ when it strikes the end of the 3-kg stationary slender bar $B$. If the sphere adheres to the bar, what is the bar’s angular velocity after the impact? (25 points)

An alternative solution (there are many more...)

- Conservation of linear momentum for the bar in y-direction $\Rightarrow v_c' = 0$

- Conservation of linear momentum for the system in x-direction

$$1 \cdot 10 = 1 \cdot v_A' + 3 \cdot v_c' = 1 \cdot (v_c' + \omega) + 3v_c'$$

$$v_A' = v_c' + \omega \times \vec{r}_A = (v_c', 0, 0) + \frac{(0, 0, \omega)}{(0, 0, 0)}$$

$$= v_c' + \omega$$

- Conservation of angular momentum for the bar at stationary point that coincides with $B$

$$0 = \frac{1}{2} m v_c' (3 v_c', 0, 0) + (0, 0, I_c \omega) = (0, 0, \omega - 3 v_c')$$

$$\Rightarrow \omega = \frac{30}{7} \approx 4.29 \text{ rad/s}$$
14. A slender bar of mass $m$ is released from rest in the position shown. The static and kinetic coefficients of friction at the floor and wall have the same value $\mu$. If the bar slips, what is its angular acceleration at the instant of release? (25 points)

$$J_c = \frac{1}{12}ml^2$$

**Kinematics:**

\[
\begin{align*}
\dot{\alpha}_c &= \dot{\alpha}_A + \alpha \left( (0, 0, \alpha) \right) \\
&= \dot{\alpha}_A + x \frac{\alpha}{A_c} \left( \frac{\dot{A}_C}{2 \sin \theta}, \frac{\dot{A}_C}{2 \cos \theta}, 0 \right) \\
&= \left( A_{AX}, 0, 0 \right) \left( -\frac{1}{2} \dot{A}_C \cos \theta, -\frac{1}{2} \dot{A}_C \sin \theta, 0 \right) \\
\therefore \dot{A}_{AX} &= -\alpha \frac{\dot{A}_C}{2} \sin \theta \\
\dot{\alpha}_c &= \frac{\dot{A}_C}{2} \cos \theta \\
\end{align*}
\]

\[\sum F_x = 0 \]

\[-\mu V_A + H_B - m \alpha \frac{\dot{A}_C}{2} \cos \theta = 0 \]

\[\sum F_y = 0 \]

\[V_A + \mu H_B - m (-\alpha \frac{\dot{A}_C}{2} \sin \theta) - mg = 0 \]

\[
\begin{pmatrix}
-\mu & 1 \\
1 & \mu
\end{pmatrix}
\begin{pmatrix}
V_A \\
H_B
\end{pmatrix}
= 
\begin{pmatrix}
m \alpha \frac{\dot{A}_C}{2} \cos \theta \\
mg - m \alpha \frac{\dot{A}_C}{2} \sin \theta
\end{pmatrix}
\]
\[ V_A = \left| \frac{1}{1 - \frac{m g - m \alpha \frac{1}{2} \sin \theta}{\mu}} \right| \frac{m \alpha \frac{1}{2} \cos \theta \mu}{1 - \mu} \]

\[ = \frac{m \alpha \frac{1}{2} \cos \theta \mu - mg + m \alpha \frac{1}{2} \sin \theta}{\mu^2 - 1} \]

\[ = \frac{mg - m \alpha \frac{1}{2} (\cos \theta \mu + \sin \theta)}{1 + \mu^2} \]

\[ \sum M_B = 0 \]

\[ V_A \sin \theta - \mu V_A \cos \theta \]

\[ - mg \frac{1}{2} \sin \theta - m(-\alpha \frac{1}{2} \sin \theta) \frac{1}{2} \sin \theta \]

\[ - m \alpha \frac{1}{2} \cos \theta \frac{1}{2} \cos \theta - \frac{1}{2} ml \alpha - \frac{1}{2} ml \alpha = 0 \]

\[ \text{(one } \mu \text{ cancels out)} \]

\[ \frac{mg - m \alpha \frac{1}{2} (\cos \theta \mu + \sin \theta)}{1 + \mu^2} \]

\[ \cos \theta \mu \]

\[ - \frac{1}{2} mg \sin \theta + \frac{1}{4} ml \alpha \sin^2 \theta - \frac{1}{2} ml \alpha \cos^2 \theta - \frac{1}{2} ml \alpha = 0 \]

\[ \frac{mg (\sin \theta - \mu \cos \theta)}{1 + \mu^2} - \frac{1}{2} mg \sin \theta \]

\[ + \alpha \left( - \frac{m \frac{1}{2} (\cos \theta \mu + \sin \theta)}{1 + \mu^2} (\sin \theta - \mu \cos \theta) + \frac{1}{4} ml (\sin^2 \theta - \cos^2 \theta) - \frac{1}{2} ml \alpha \right) = 0 \]

\[ \Rightarrow \alpha = \text{(some complex formula ...)} \]
15. The dimensions of the 6-kg plate are $b = 0.9\text{m}$ and $h = 0.6\text{m}$. The plane is supported with a ball-and-socket support at $O$. If the plate is held in the horizontal position and then released from rest, what are the components of its angular acceleration at that instant? (25 points)

\[ \frac{y}{x} = \frac{h}{b} \]

\[ I_x = \int_{0}^{b} \int_{0}^{b} y^2 \, dy \, dx = \int_{0}^{b} \frac{b^3 x^3}{6} \, dx = \frac{8}{3} \frac{b^4}{b} = \frac{8}{3} \frac{h^3}{b^3} \]

\[ = \frac{1}{6} mb^2 = 0.36 \text{ [kg m}^2\text{]} \]

\[ I_y = \int_{0}^{b} \int_{0}^{b} x^2 \, dy \, dx = \int_{0}^{b} x^2 \, dx = \frac{b^4}{4} \frac{h^4}{b} = \frac{8}{4} \frac{h^4}{b^4} \]

\[ = \frac{1}{2} mb^2 = 2.43 \text{ [kg m}^2\text{]} \]
\[ I_{xy} = \int_{0}^{b} \int_{0}^{x_g} xy \, dy \, dx \]
\[ = \int_{0}^{b} x \left( \frac{b^2 - x^2}{2} \right) \, dx \]
\[ = \frac{b}{2} \int_{0}^{b} \frac{x^2}{2} \, dx = \frac{b^2}{2} \left[ \frac{x^4}{4} \right]_{0}^{b} \]
\[ = \frac{1}{8} b^4 \]
\[ I_z = I_x + I_y = 2.79 \text{ [kg m}^2\text{]} \]
\[ I_{yz} = I_{xz} = 0 \]
\[ \alpha_{0} = 0.36 \begin{pmatrix} 0.6 & 0.2 & 0 \\ -0.81 & 2.43 & 0 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} -11.77 \\ 17.66 \\ 0 \end{pmatrix} \text{ [N m]} \]
\[ M_{a0} = \begin{pmatrix} 0.36 & -0.81 & 0.26 \\ -0.81 & 2.43 & -11.77 \\ 0.26 & -11.77 & 35.32 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0 \\ -0.81 & 2.43 & 0 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0 \\ -0.81 & 2.43 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0.219 \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} -11.77 \\ 17.66 \\ 0 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0 \\ -0.81 & 2.43 & 0 \end{pmatrix} \begin{pmatrix} 0.6 & 0.2 & 0 \\ -0.81 & 2.43 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0.219 \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} -3.181 \\ 0.219 \\ 14.52 \end{pmatrix} \text{ [N m]} \]
16. The homogeneous disk weighs 80 lb and its radius is $R = 1$ ft. It rolls on the plane surface. The spring constant is $k = 100$ lb/ft and the damping constant is $c = 3$ lb-s/ft. Determine the frequency of small vibrations of the disk relative to its equilibrium position. (25 points)

\[ \dot{\alpha} = \ddot{x} \]
\[ \ddot{x} = -\frac{1}{R} \dot{x} \]

\[ \sum M_A = 0 \]

\[ m\ddot{R} - \left(-\frac{1}{2} mR^2 \ddot{x}\right) + c\dot{R} + kR = 0 \]

\[ \frac{3}{2} m \dddot{x} + c \dot{x} + kx = 0 \]

\[ \omega = \frac{c}{3m} = \frac{3}{3 \times 32.2} = 0.4025 \quad \left[ \frac{16 \text{ s}}{16 \text{ s}^2} \right] = \frac{1}{s} \]

\[ \omega_0 = \frac{2c}{3m} = 26.83 \quad \left[ \frac{16}{16} \text{ s}^2 \right] = \frac{1}{s} \]

\[ \omega = \sqrt{\omega_0^2 - \gamma^2} = 5.16 \left[ \frac{1}{s} \right] \]
17. (bonus) A satellite can be modeled as an 1000-kg cylinder 4m in length and 2m in diameter. If the nutation angle is \( \theta = 20^\circ \), and the spin rate \( \dot{\phi} \) is one revolution per second, what is the satellite's precession rate \( \dot{\psi} \) in revolutions per second? (25 points)
\[ I_x = \int \int \int (r^2 \sin^2 \theta + z^2) \, dV \]

\[ = \frac{S}{2} \left( \frac{R^4}{4} \pi h + \frac{R^2}{2} \cdot \frac{1}{2} \cdot \frac{h^3}{24} \right) = \frac{1}{24} \pi R^2 h \left( \frac{R^2}{4} + \frac{h^2}{8} \right) \]

\[ \cos^2 \theta - \sin^2 \theta = \cos 2\theta \]

\[ \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \]

\[ I_x = \int \int \int (r^2 \sin^2 \theta + r^2 \cos^2 \theta) \, dV \]

\[ = \frac{1}{2} \int_0^{2\pi} \int_0^\frac{\pi}{2} \int_0^R (r^4 \sin^2 \theta + r^2 \cos^2 \theta) \, dr \, d\theta \, d\phi \]

\[ = \frac{1}{2} \cdot 1000 \cdot 1^2 = 500 \, [\text{kg} \cdot \text{m}^2]\]

\[ I_x = \frac{1}{12} \cdot 1000 \cdot (6 + 16) = 916.7 \, [\text{kg} \cdot \text{m}^2]\]
Equation of rotational motion

\[ \dot{H}_c = M_c = 0 \]
\[ (I_c \dot{\omega}) = 0 \]
\[ I_c \ddot{\omega} + \gamma \times (I_c \omega) = 0 \]

Tensor of inertia at \( C \)

\[ I_c = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{pmatrix} \]

\[ \times \begin{pmatrix} 0, \sin \theta \dot{\psi}, \cos \theta \dot{\psi} \\ 0, \cos \theta \psi, \sin \theta \psi \end{pmatrix} \]

\[ I_c \dot{\omega} = (0, I_x \sin \theta \psi, I_z (\cos \theta \dot{\psi} + \dot{\phi})) \]

\[ (I_z - I_x) \cos \theta \sin \theta \psi \dot{x} + I_z \dot{\phi} \sin \theta \psi \dot{y} = 0 \]

\[ \dot{\psi} = \frac{I_z}{(I_x - I_z) \cos \theta} \dot{\phi} \]

\[ \psi \approx \frac{500}{416.7 \cos 20^\circ} \cdot 1.28 \cdot \left[ \frac{\text{rev}}{s} \right] \]