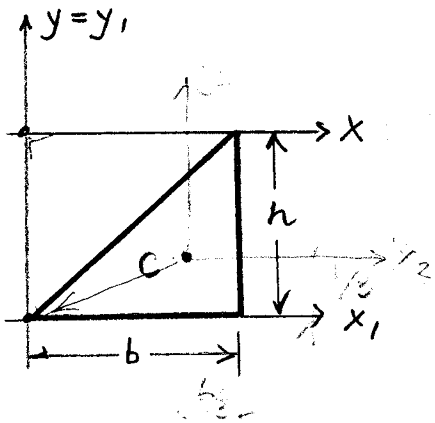
EM311M - Dynamics
Final Exam
UTC 3.102 - 3.104 Monday, December 15, 2008, 2-5 PM

1. The homogeneous thin plate has mass m and dimensions b and h. The product of inertia with respect axes $x_1 y_1$ is $I_{x_1 y_1} = \frac{1}{4}mbh$. Use the Parallel Axes Theorem to compute the product of inertia I_{xy} with respect to axes x, y shown below. *Hint:* Do not let me fool you. (5 points)



$I_{xy} = I_{x_2 y_2} + m\left(-\frac{2}{3}b\right)\left(\frac{2}{3}h\right)$

$= I_{x_1 y_1} - m\left(\frac{2}{3}b\right)\left(\frac{1}{3}h\right) + m\left(-\frac{2}{3}b\right)\left(\frac{2}{3}h\right)$

$= \frac{1}{4}mbh - \frac{2}{9}mbh - \frac{4}{9}mbh$

$= -\frac{5}{12}mbh$

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2. Derive the principle of angular momentum and impulse for a rigid body undergoing *an arbitrary planar motion* with respect to a fixed point. You may use the principle for an arbitrary system of particles as a starting point. (5 points)

$\dot{H}_O = M_O \qquad H_O^{initial} + \int M_O \, dt = H_O^{final}$

$H_O = \int_V \rho (OP \times v_P) \, dV = OC \times m v_C + I_C \omega$

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$v_P = v_C + \omega \times CP \qquad OP \times v_P = (OC + CP) \times (v_C + \omega \times CP)$

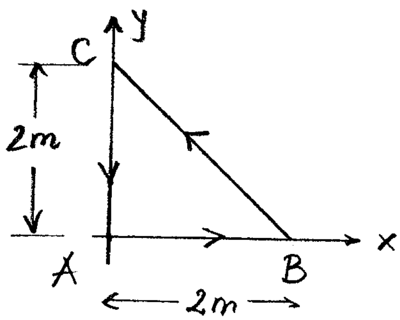
$OP = OC + CP \qquad = OC \times v_C + CP \times v_C + OC \times (\omega \times CP) + CP \times (\omega \times CP)$

$\int_V \rho \, OC \times v_C = OC \times \int_V \rho \, v_C = OC \times m v_C \; / \; \int_V \rho \, CP \times v_C = \int_V \rho \, CP \times v_C = 0$

$\int_V \rho \, OC \times (\omega \times CP) = OC \times \omega \times \int_V \rho \, CP = 0 \qquad \int_V \rho (CP \times (\omega \times CP)) = I_C \omega$

$= (0, 0, I_C \omega)$

3. Compute the work of force field $\boldsymbol{F} = (y, 2x)$ [N] along the closed curve $ABCA$ shown below. (5 points)



$\overline{AB}: \begin{cases} x = t \\ y = 0 \end{cases} \quad \frac{dr}{dt} = (1, 0), \quad 0 < t < 2$

$\int_{AB} \underset{\sim}{F} \cdot \underset{\sim}{dr} = \int_0^2 (y \cdot 1 + 2x \cdot 0)\, dt = 0$

$\overline{BC}: \begin{cases} x = 2 - t \\ y = t \end{cases} \quad \frac{dr}{dt} = (-1, 1) \quad 0 < t < 2$

$\int_{BC} F \cdot dr = \int_0^2 (y(-1) + 2x \cdot 1)\, dt$, with $y = t$, $2x = 2(2-t)$

$= \int_0^2 (-t + 4 - 2t)\, dt = \int_0^2 (4 - 3t)\, dt$

$= \left(4t - 3\frac{t^2}{2}\right)\Big|_0^2 = 8 - 6 = 2$ [N·m]

$\overline{CA}: \begin{cases} x = 0 \\ y = 2 - t \end{cases} \quad \frac{dr}{dt} = (0, -1) \quad 0 < t < 2$

$\int_{CA} F \cdot dr = \int_0^2 (2 - t \cdot 0 + 2x \cdot (-1))\, dt = 0$

Total work $= \int_{AB} + \int_{BC} + \int_{CA} = 2$ [N·m]

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4. Is the force from the previous problem is conservative? Explain. (5 points)

No! If it were, the work from the previous problem would have been $= 0$.

Double check

$\nabla \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times F(y, 2x, 0)$

$(0, 0, 2 - 1) \neq 0$

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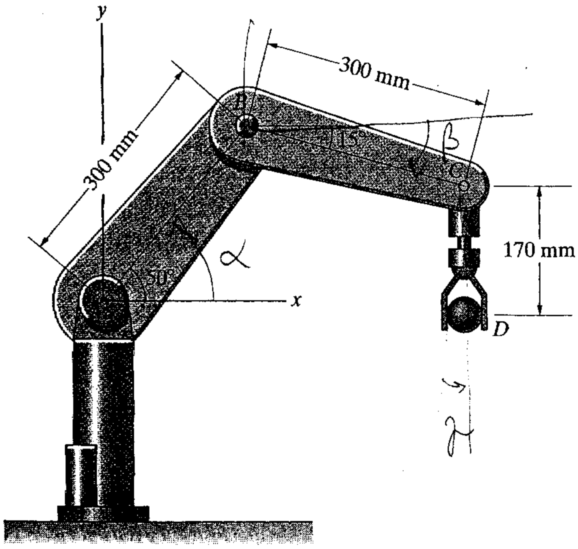
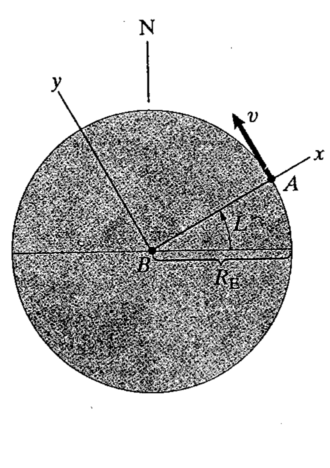
5. How many degrees of freedom does the system depicted below have (treat it as a planar system) ? Explain. (5 points)



Three! It takes three angles (α, β, γ) [illegible] to describe uniquely position of the system.

6. A car A at north latitude L drives north on a north-south highway with constant speed v. The Earth's radius is R and the Earth's angular velocity is ω (directed south to north). The coordinate system xyz is Earth-fixed. Compute the car's velocity wit respect to (a) system xyz, (b) a fixed system of coordinates with the origin at the center of the Sun, (c) a non-rotating system of coordinates with origin at the center of the Earth. At the instant shown, the three systems of coordinates are parallel to each other. (5 points)



Velocity wrt Earth-bound system:

$\underset{\sim}{V} = (0, v, 0)$ (1)

Velocity wrt the non-rotating system centered at the Earth's center.

$$\underset{\sim}{V} = (0, v, 0) + \begin{array}{c} \underset{\sim}{\omega}\,(\omega \sin L,\ \omega \cos L,\ 0) \\ \times\ \underset{\sim}{r}\,(R,\ 0,\ 0) \\ \hline (0,\ 0,\ -\omega R \cos L) \end{array}$$

$= (0, v, -\omega R \cos L)$ (2)

Velocity wrt the system centered at the Sun

$\underset{\sim}{V} = \underset{\sim}{V}_C + (0, v, -\omega R \cos L)$

↑ velocity of Earth's center (2)

7. A rigid body is rotating about its center of mass C with an angular velocity vector $\boldsymbol{\omega} = (1, 1, 1)$ [rad/s] with the components specified in a system of coordinates centered at C. The corresponding moments and products of inertia are $I_x = I_y = 10, I_z = 20, I_{xy} = I_{yz} = 0, I_{xz} = -10$ [kg m^2]. Compute the angular momentum of the body with respect to point C. (5 points)

$$\underset{\sim}{H}_C = \begin{pmatrix} 10 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 20 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 20 \\ 10 \\ 30 \end{pmatrix} \quad [kg \tfrac{m^2}{s}]$$

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8. Suppose the same rigid body as in the previous problem is undergoing an arbitrary motion, its mass is $m = 20$ kg, $\boldsymbol{v}_C = (1, 0, 2)$ m/s, and $\boldsymbol{\omega}$ is the same as above. Compute the angular momentum of the body with respect to a (fixed) point O with coordinates $(1, 2, 0)$ in the system x, y, z centered at the center of mass. (5 points)

$$H_O = \rho \times m v_C + \underset{\sim}{I}_C\, \Omega$$

$$\frac{(-1, -2, 0) \times 20(1, 0, 2)}{(-4, 2, 2)} + (20, 10, 30)$$

$(-60, 50, 70)$ $[kg \frac{m^2}{s}]$

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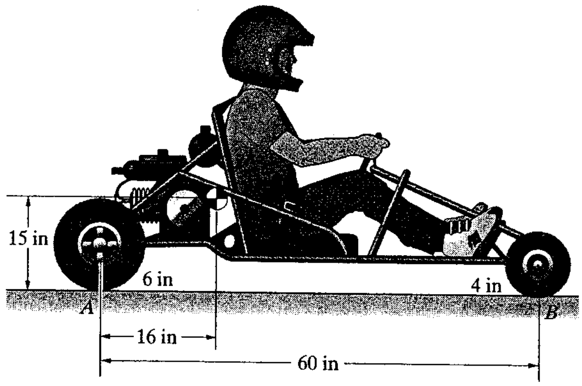
9. The total weight of the go-cart is 240 lb. The location of their combined center of mass is shown. The rear drive wheels together exert a 24 lb horizontal force on the track. Neglect the horizontal forces exerted on the front wheels. (a) What is the magnitude of the go-cart's acceleration ? (b) What normal forces are exerted on tires at A and B ? (25 points)



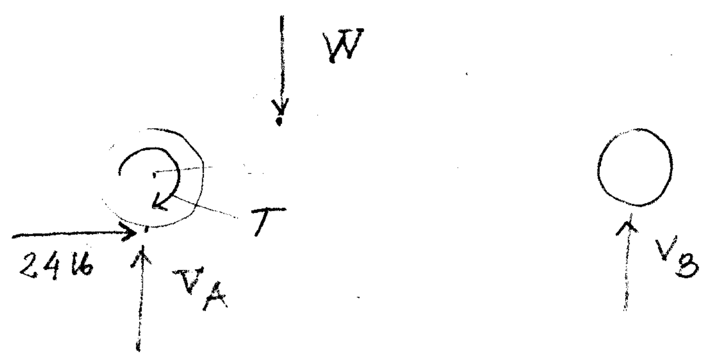
Since moments of inertia for wheels have not been specified, we are going to neglect them !

m_1 – mass of rear wheels (unknown)

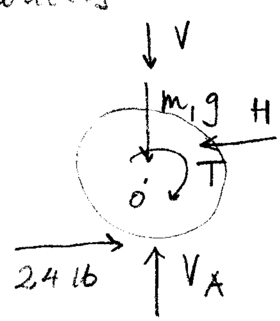
Free-body diagrams

- cart and wheels combined

T – torque (moment) exerted by the engine on the rear axle



- rear wheels



$$\cancel{I_o \alpha}^{\,0} = 24 \cdot \frac{6}{12} - T$$

$$\therefore \quad T = 12 \; [\text{lb} \cdot \text{ft}]$$

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- Eq of translational motion for the whole cart

$$\frac{240}{32.2}\, a = 24 \quad \Rightarrow \quad a = 3.22 \left[\frac{ft}{s^2}\right]$$

- $\sum F_y = 0 \qquad V_A + V_B - 240 = 0$

(10)

- $I_c \cancel{\alpha}^{\,0} = M_c$

$$24\,\frac{15}{12} - 12 - V_A\,\frac{16}{12} + V_B \cdot \frac{44}{12} = 0$$

Final system of equations:

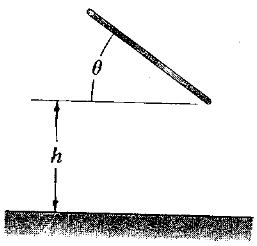
$$\begin{pmatrix} 1 & 1 \\ -1.333 & 3.667 \end{pmatrix}\begin{pmatrix} V_A \\ V_B \end{pmatrix} = \begin{pmatrix} 240 \\ -18 \end{pmatrix}$$

$$V_A = \frac{\begin{vmatrix} 240 & 1 \\ -18 & 3.667 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -1.333 & 3.667 \end{vmatrix}} = \frac{898}{5} = 179.6\ [lb]$$

$$V_B = \frac{\begin{vmatrix} 1 & 240 \\ -1.333 & -18 \end{vmatrix}}{5} = \frac{302}{5} = 60.4\ [lb]$$

(5)

10. The slender bar is released from rest with $\theta = 45^o$ and falls a distance $h = 2$ m onto the smooth floor. The length of the bar is 1m and its mass is 2 kg. If the coefficient of restitution of the impact is $e = 0.4$, what is the angular velocity of the bar just after it hits the floor ? (25 points)

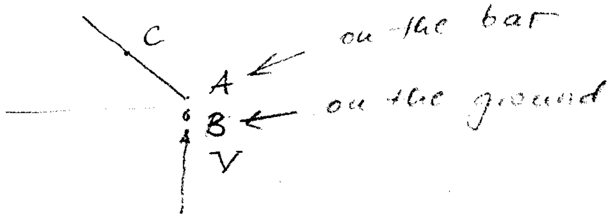


Step 1 : Free fall

$$\frac{m v^2}{2} = m g h$$

$$v = \sqrt{2gh} = 6.264 \left[\frac{m}{s}\right] \downarrow$$ (5)

Step 2 : Impact



- Conservation of linear momentum in x ⇒

$$V_{c_x}^{after} = V_{c_x}^{before} = 0$$

- Coefficient of restitution

$$V_{A_y}^{after} = -0.4\, V_{A_y}^{before} = 2.51 \left[\frac{m}{s}\right] \uparrow$$ (5)

- Kinematics

$$\underset{\sim}{V}_c^{after} = \underset{\sim}{V}_A^{after} + \omega^{after} \times \underset{\sim}{AC}$$ (5)

$\omega^{after}(0, 0, \omega^{after})$

$$= (V_{A_x}^{after}, 2.51, 0) + \times \underset{\sim}{AC}\,(-0.5\cos 45^\circ, 0.5\sin 45^\circ, 0)$$

$(-0.354\omega^a, -0.354\omega^a, 0)$

$$V_{c_y}^{after} = 2.51 - 0.354\,\omega^{after}$$

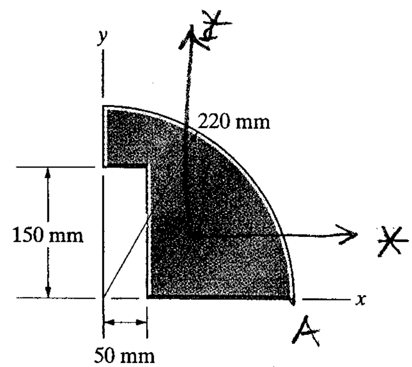
- Conservation of angular momentum at point B (= A)

$$m \cdot 6.264 \cdot \underbrace{0.5\cos 45^o}_{0.354} = -m(2.51 - 0.354\,\omega^a)\,0.354 + \underbrace{\frac{1}{12} m l^2}_{I_c} \omega^a$$

$$2.217 = -0.889 + 0.209\,\omega^{after}$$

$$\omega^{after} = 14.86 \frac{rad}{s}$$ (10)

11. The inertia matrix of the 2.4 kg plate in terms of the given coordinate system is shown. At $t = 0$, the plate is stationary and it is subjected to a force $\boldsymbol{F} = (0, 0, -10)$ N at point A(220,0,0) mm. No other forces nor couples act on the plate. Determine the acceleration of point A. (25 points)



$$\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} = \begin{bmatrix} 0.0318 & -0.0219 & 0 \\ -0.0219 & 0.0357 & 0 \\ 0 & 0 & 0.0674 \end{bmatrix} \text{ kg-m}^2.$$

Centroid

$A = \frac{1}{4}\pi 220^2 - 150 \cdot 50 = 30513 \; [mm^2]$

$S_y = \frac{4 \cdot 220}{3\pi} \cdot \frac{1}{4}\pi 220^2 - 25 \cdot 150 \cdot 50$

$= 3361833 \; [mm^3]$

$x_c = \frac{S_y}{A} = 110 \; [mm]$

$S_x = \frac{4 \cdot 220}{3\pi} \cdot \frac{1}{4}\pi 220^2 - 75 \cdot 150 \cdot 50$

$= 2986833 \; [mm^3]$

$y_c = \frac{S_x}{A} = 98 \; [mm]$

Inertia tensor in system XYZ

$I_X = I_x - m y_c^2 = 0.0318 - 2.4 \; 0.098^2 = 0.0088 \; [kg m^2]$

$I_Y = I_y - m x_c^2 = 0.0357 - 2.4 \; 0.11^2 = 0.0067 \; [kg m^2]$

$I_Z = I_z - m(x_c^2 + y_c^2) = 0.0674 - 2.4(0.098^2 + 0.11^2) = 0.0154 \; [kg m^2]$

$(= I_X + I_Y)$

$I_{XY} = I_{xy} - m x_c y_c = 0.0219 - 2.4(0.098 \cdot 0.11)$

$= -0.0040 \; [kg m^2]$

$$\underset{\sim}{I}_c = \begin{pmatrix} 0.0088 & 0.004 & 0 \\ 0.004 & 0.0067 & 0 \\ 0 & 0 & 0.0154 \end{pmatrix} [kg m^2]$$

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Translational motion

$$2.4\,a_{cx} = 0 \Rightarrow a_{cx} = 0$$

$$2.4\,a_{cy} = 0 \Rightarrow a_{cy} = 0 \qquad (7)$$

$$2.4\,a_{cz} = -10 \Rightarrow a_{cz} = -4.17\ [\tfrac{m}{s^2}]$$

Rotational motion

$$\underset{\sim}{I}_c \underset{\sim}{\alpha} + \cancel{\underset{\sim}{\omega} \times (\underset{\sim}{I}_c \underset{\sim}{\omega})}^{\,\underset{\sim}{0}} = \underset{\sim}{M}_c \qquad (8)$$

$$\underset{\sim}{M}_c = \begin{array}{r} \underset{\sim}{CA}\ (0.11, -0.098, 0) \\ \times\ \underset{\sim}{F}\ (0, \quad 0, -10) \\ \hline (0.98, \quad 1.1, \quad 0) \end{array}$$

$$\begin{pmatrix} 0.0088 & 0.004 \\ 0.004 & 0.0067 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \end{pmatrix} = \begin{pmatrix} 0.98 \\ 1.1 \end{pmatrix}$$

$$0.0154\,\alpha_z = 0 \Rightarrow \alpha_z = 0$$

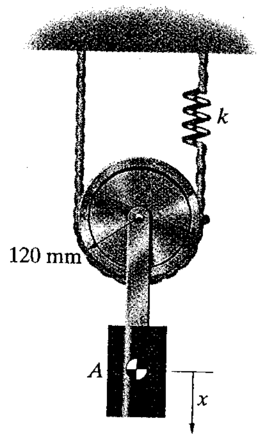
$$\alpha_x = \frac{\begin{vmatrix} 0.98 & 0.004 \\ 1.1 & 0.0067 \end{vmatrix}}{\begin{vmatrix} 0.0088 & 0.004 \\ 0.004 & 0.0067 \end{vmatrix}} = \frac{0.00217}{0.00004296} = 50.42\ [\tfrac{rad}{s^2}]$$

$$\alpha_y = \frac{\begin{vmatrix} 0.0088 & 0.098 \\ 0.004 & 1.1 \end{vmatrix}}{0.00004296} = \frac{0.009288}{} = 216.2\ [\tfrac{rad}{s^2}]$$

$$\underset{\sim}{a}_A = \underset{\sim}{a}_c + \underset{\sim}{\alpha} \times \underset{\sim}{CA} = (0, 0, -4.17) + \begin{array}{r} (50.42, 216.2, \quad 0) \\ \times\ (0.11, -0.098, 0) \\ \hline (0, \quad 0, -18.84) \end{array} \qquad (5)$$

$$= (0, 0, -23)\ [m/s^2]$$

12. The mass of the suspended object A is 4 kg. The mass of the pulley is 2 kg, and its moment of inertia is 0.018 N-m^2. The spring constant is $k = 150$ N/m. The spring is unstretched when $x = 0$. At $t = 0$, the system is released from rest with $x = 0$. Determine x as a funcion of time. (25 points)



Kinematics

$x\downarrow$ $\quad -\theta$

$$x = -R\theta \qquad \ddot{x} = -R\alpha$$

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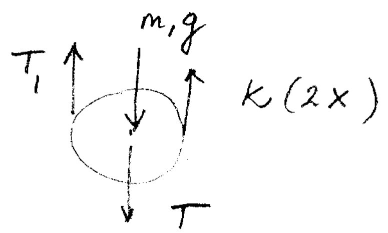
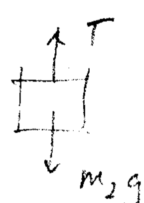
But : stretch $= 2x$!



Pulley :

$$I\alpha = (2kx - T_1)R \qquad (1)$$

$$m_1\ddot{x} = T - T_1 - 2kx + m_1 g \qquad (2)$$



Block :

$$m_2\ddot{x} = m_2 g - T \qquad (3)$$

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$$(1) \Rightarrow 2kx - T_1 = \frac{I}{R}\alpha = -\frac{I}{R^2}\ddot{x} \Rightarrow T_1 = \frac{I}{R^2}\ddot{x} + 2kx$$

$$(2)+(3) \Rightarrow (m_1 + m_2)\ddot{x} = (m_1 + m_2)g - 2kx - T_1$$

$$= (m_1 + m_2)g - 2kx - \frac{I}{R^2}\ddot{x} - 2kx$$

$$\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} + 4kx = (m_1 + m_2)g$$

$$7.25\,\ddot{x} + 600\,x = 58.86$$

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$$\ddot{x} + 82.76\,x = 8.12$$

$\omega_0 = 9.09\ [\frac{\text{rad}}{\text{s}}]$

$x_p = 0.098$

$$x = A\cos 9.09t + B\sin 9.09t + 0.098$$

$$t=0 \quad x=0 \Rightarrow A = -0.098$$

$$\dot{x} = 0 \Rightarrow B = 0$$

$$\underline{x = 0.098(1 - \cos 9.09t)}$$