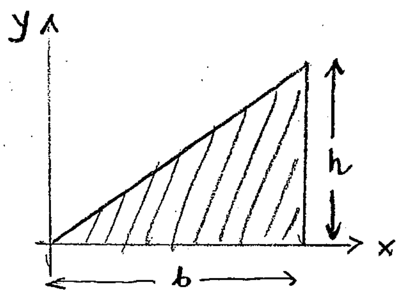
EM311M - Dynamics
Exam 3
Monday, Dec 1, 2008, 6:00-8:00 p.m., WEL 3.502

1. Use Cartesian coordinates to compute the product of inertia I_{xy}, for a homogeneous triangular plate of mass m shown below. (5 points)



$$\begin{cases} 0 < x < b \\ 0 < y < \frac{h}{b} x \end{cases}$$

$$m = \frac{1}{2} b h \rho \quad \Rightarrow \quad \rho = \frac{2m}{bh}$$

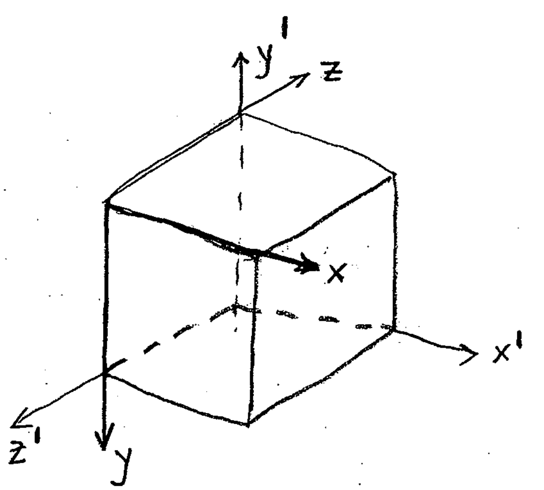
$$I_{xy} = \rho \int_0^b \int_0^{\frac{h}{b}x} xy \, dy \, dx = \rho \int_0^b x \left. \frac{y^2}{2} \right|_0^{\frac{h}{b}x} dx$$

$$= \frac{\rho}{2} \int_0^b x \frac{h^2}{b^2} x^2 \, dx = \frac{\rho h^2}{2b^2} \left. \frac{x^4}{4} \right|_0^b = \frac{\rho}{2} \frac{h^2}{\cancel{b^2}} \frac{b^{\cancel{4}2}}{4}$$

$$= \frac{\rho}{8} b^2 h^2 = \frac{2m}{\cancel{bh}\,8\,4} b^{\cancel{2}} h^{\cancel{2}} = \frac{1}{4} m b h \quad [kg\text{-}m^2]$$

5

2. Define the transformation matrix from system $Oxyz$ to system $O'x'y'z'$ and compute it for the systems shown below. (5 points)



$$\alpha_{ij} = \underset{\sim}{e}'_i \cdot \underset{\sim}{e}_j$$

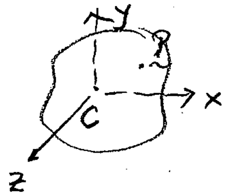
1

$$\begin{array}{cl} \underset{\sim}{\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{array}{l} e'_1 \text{ in } xyz \\ e'_2 \text{ in } xyz \\ e'_3 \text{ in } xyz \end{array} \end{array}$$

$\underset{\sim}{e}_1$ in $x'y'z'$ $\underset{\sim}{e}_2$ in $x'y'z'$ $\underset{\sim}{e}_3$ in $x'y'z'$

4

3. Derive the formula for the kinetic energy of a rigid body undergoing an arbitrary planar motion. (5 points)



$$\underset{\sim}{v}_P = \underset{\sim}{v}_c + \underset{\sim}{\omega} \times \underset{\sim}{CP} = (v_{cx}, v_{cy}, 0) + \underbrace{\underset{\sim}{\omega}(0,0,\omega) \times \underset{\sim}{CP}(x, y, 0)}_{(-\omega y,\ \omega x,\ 0)}$$

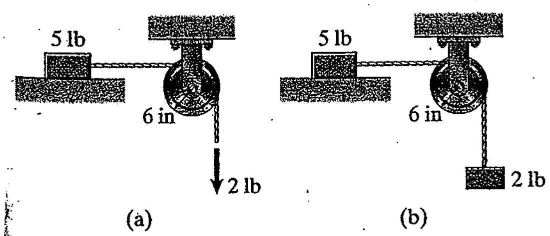
$$= (v_{cx} - \omega y,\ v_{cy} + \omega x,\ 0)$$

$$\int \rho\, (v_{cx} - \omega y)^2 + (v_{cy} - \omega y)^2 = (v_{cx}^2 + v_{cy}^2)\overbrace{\int\rho}^{m} \overbrace{- 2\omega v_{cx} \int \rho y}^{0} - \overbrace{2\omega v_{cy}\int \rho y}^{0}$$

$$\boxed{T = \frac{1}{2}\int \rho\, \underset{\sim}{v}_P^2 = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2} + \underbrace{\omega^2 \int \rho (y^2 + z^2)}_{I_c}$$

5

4. Review the two scenarios below. In which case do you expect the acceleration of 5 lb block be smaller? Explain. (5 points)



In case (b). Forces are the same, but the inertia is bigger..

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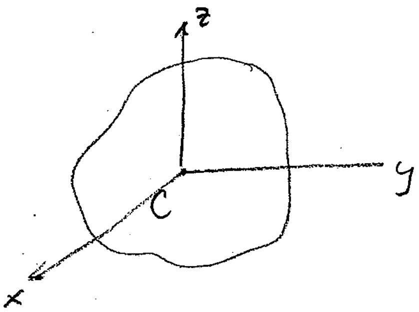
5. Derive the equation of angular motion for a rigid body undergoing an arbitrary motion, in the body-fitted system of coordinates (5 points)



xyz – a body-fitted system with center at $\underline{C}$.

Principle of angular impulse and momentum

$$(\underset{\approx}{I}_c \underset{\sim}{\omega})^{\circ} = \underset{\sim}{M}_c$$

In a rotating system: $\overset{\circ}{\underset{\sim}{f}} = \overset{\circ}{\underset{\sim}{f}}/_{rel} + \underset{\sim}{\Omega} \times \underset{\sim}{f}$

In a body-fitted system: $\underset{\sim}{\Omega} = \underset{\sim}{\omega}$

Hence $$\underbrace{(\underset{\approx}{I}_c \underset{\sim}{\omega})^{\circ}/_{rel}}_{\underset{\approx}{I}_c \dot{\underset{\sim}{\omega}}/_{rel}} + \underset{\sim}{\omega} \times (\underset{\approx}{I}_c \underset{\sim}{\omega}) = \underset{\sim}{M}_c$$

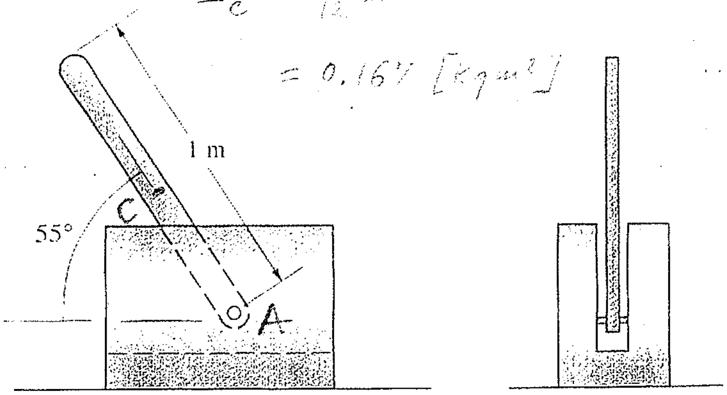
5

5

6. The 2-kg slender bar and 5-kg block are released from rest in the position shown. If friction is negligible, what is the block's acceleration at that instant? (25 points)

$I_c = \frac{1}{12} 2 \cdot 1^2$

$= 0.167 \ [kg\,m^2]$



$\underset{\sim}{a}_c = \underset{\sim}{a}_A + \underset{\sim}{\alpha} \times \underset{\sim}{AC} - \omega^2 AC$

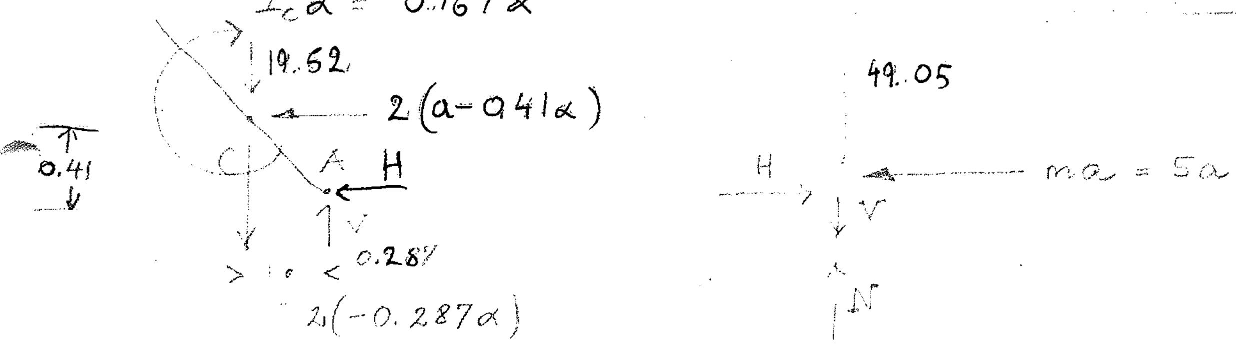
$\underset{\sim}{\alpha}\ (0, 0, \alpha)$

$\underset{\sim}{AC}\ (-0.287,\ 0.410,\ 0)$

$(-0.41\alpha, -0.287\alpha, 0)$

$= (a - 0.41\alpha\ ,\ -0.287\alpha,\ 0)$ (5)

$I_c\alpha = 0.167\alpha$



(5)

Bar : $\Sigma M_c = 0$

$19.62 \cdot 0.287 + 2(-0.287\alpha)\,0.287$
$+ 2(a - 0.41\alpha)\,0.41 - 0.167\alpha = 0$ (5)

$\Rightarrow \quad 0.82\,a = -5.631 + 0.165\alpha + 0.336\alpha + 0.167\alpha$
$= -5.631 + 0.668\alpha$

(1)

Bar + block $:\ \Sigma F_x$

(5)

$-2(a - 0.41\alpha) - 5a = 0$

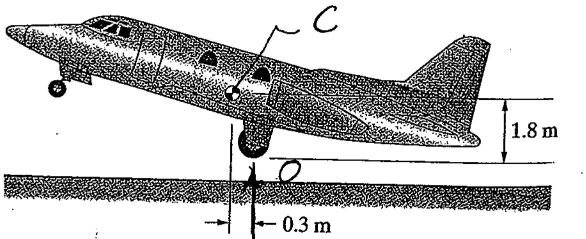
$7a = 0.82\alpha \Rightarrow a = 0.117\alpha$ (2)

$0.096\alpha = -5.631 + 0.668\alpha$

$\therefore \quad \alpha = 9.844$

$\Rightarrow \quad a = 1.151 \left[\frac{m}{s^2}\right]$ (5)

7. The horizontal velocity of the landing airplane ('s center of mass) is 50 m/s, its vertical velocity is 2 m/s, and its angular velocity is zero. The mass of the airplane is 12,000 kg, and the moment of inertia about its center of mass is 100,000 kg-m^2. The coefficient of restitution of the landing impact is $e = 0.4$. Determine the angular velocity of the airplane after the impact. (25 points)



impulsive reaction force

- Principle of angular impulse and momentum wrt point O on the ground:

Eq. 1 $$\underset{\sim}{OC} \times m\underset{\sim}{v}_C^{before} + (0, 0, I_C \cancel{\omega_C^{before}}) = \underset{\sim}{OC} \times m\underset{\sim}{v}_C^{after} + (0, 0, I_C\omega_C^{after})$$

$$\begin{array}{rl} & \underset{\sim}{OC}\ (-0.3,\ 1.8,\ 0) \\ \times & m\underset{\sim}{v}_C^{before}\ (-50m,\ -2m,\ 0) \\ \hline & (0,\ 0,\ 90.6m) \end{array}$$

- Principle of linear momentum in x-direction

$$m v_{Cx}^{before} = m v_{Cx}^{after} \Rightarrow v_{Cx}^{after} = v_{Cx}^{before} = -50$$ 5

- Coefficient of restitution $$v_{Oy}^{after} = -0.4\, \underset{-2}{\underset{\shortparallel}{v_{Oy}^{before}}} = 0.8$$ 5

- Kinematics $$\underset{\sim}{v}_C^{after} = \underset{\sim}{v}_O^{after} + \underset{\sim}{\omega}^{after} \times \underset{\sim}{OC}$$

$$= (v_{Ox}^{after},\ 0.8,\ 0) + \begin{array}{rl} & \underset{\sim}{\omega}^{after}\ (0,\ 0,\ \omega^{after}) \\ \times & \underset{\sim}{OC}\ (-0.3,\ 1.8,\ 0) \\ \hline & (-1.8\omega^{after},\ -0.3\omega^{after},\ 0) \end{array}$$

Hence $$v_{Cy}^{after} = 0.8 - 0.3\,\omega^{after}$$

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• angular momentum wrt point O after impact

$$\times \begin{array}{l} \underset{\sim}{OC}\,(-0.3, \quad 1.8, \quad 0) \\ m\,\underset{\sim}{v}_c^{after}\,(-50m, \ (0.8-0.3\omega^{after})m, \ 0) \end{array} \quad + \quad (0, 0, I_c\omega^{after})$$

$$(0, 0, \ -0.3(0.8-0.3\omega^{after})m + 90m \quad)$$

<u>Final equation for ω:</u> (from Eq. 1) 10

$$90.6\,m = -0.3(0.8-0.3\omega^{after}).m + 90m + I_c\,\omega^{after}$$

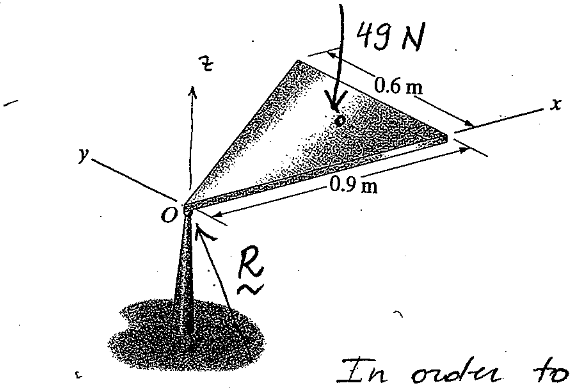
Divide by m

$$90.6 = -0.3(0.8-0.3\omega^{after}) + 90 + \underbrace{\frac{I_c}{m}}_{8.33}\,\omega^{after}$$

$$(8.33 + 0.09)\,\omega^{after} = 0.6 + 0.24 = 0.84$$

$$\underline{\omega^{after} \approx 0.1 \quad \left[\frac{rad}{s}\right]}$$

8. If the 5-kg plate is released from rest in the horizontal position, what force is exerted on it by the ball-and-socket support at that instant ? (25 points)



<u>Eqn's of translational motion :</u>

$$5a_{cx} = R_x$$
$$5a_{cy} = R_y \qquad (1) \qquad 5$$
$$5a_{cz} = R_z - 49$$

In order to determine $\underset{\sim}{R}$, we need to compute $\underset{\sim}{a}_c$

We use <u>the body-fitted system xyz</u>

$$\underset{\approx}{I}_o \underbrace{\dot{\underset{\sim}{\omega}}}_{\underset{\sim}{\alpha}} + \cancel{\overset{0}{\underset{\sim}{\omega}} \times \underset{\approx}{I}_o \underset{\sim}{\omega}} = \underset{\sim}{M}_o = \frac{\underset{\sim}{OC}(0.6,\ 0.2,\ 0) \times \underset{\sim}{W}(0,\ 0,\ -49)}{(-9.81,\ 29.4,\ 0)} \qquad 5$$

<u>Inertia :</u> Use parametrization from Problem #1

$$I_x = \rho \int_0^b \int_0^{\frac{h}{b}x} y^2\,dy\,dx = \rho \int_0^b \frac{y^3}{3}\Big|_0^{\frac{h}{b}x} dx = \frac{\rho}{3}\frac{h^3}{b^3}\int_0^b x^3\,dx$$

$$= \frac{\rho}{3}\frac{h^3}{\cancel{b^3}}\frac{b^4}{4} = \frac{\rho}{12} b h^3 = \frac{2}{12}\underbrace{\rho\frac{bh}{2}}_{m} h^2 = \frac{1}{6} m h^2 = 0.3\ kgm^2$$

$$I_y = \rho \int_0^b \int_0^{\frac{h}{b}x} x^2\,dy\,dx = \rho \int_0^b x^2\, y\Big|_0^{\frac{h}{b}x} dx = \rho \frac{h}{b}\frac{x^4}{4}\Big|_0^b$$

$$= \frac{\rho}{4} h b^3 = \frac{1}{2} m b^2 = 2.025\ kgm^2$$

$$I_{xy} = \frac{1}{4} m b h = 0.675\ kg\,m^2$$

$$\begin{pmatrix} 0.3 & -0.675 & 0 \\ -0.675 & 2.025 & 0 \\ 0 & 0 & 2.325 \end{pmatrix} \begin{pmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{pmatrix} = \begin{pmatrix} -9.81 \\ 29.4 \\ 0 \end{pmatrix} \qquad 10$$

$$\underline{\alpha_z = 0} \qquad \alpha_x = \frac{\begin{vmatrix} -9.81 & -0.675 \\ 29.4 & 2.025 \end{vmatrix}}{\begin{vmatrix} 0.3 & -0.675 \\ -0.675 & 2.025 \end{vmatrix}} = -\frac{0.02}{0.15} \approx -0.125\ \left[\frac{rad}{s^2}\right]$$

$$\alpha_y = \frac{\begin{vmatrix} 0.3 & -9.81 \\ -0.675 & 29.4 \end{vmatrix}}{0.15} \approx 14.5\ \left[\frac{rad}{s^2}\right] \qquad 5$$

Kinematics :

$$\underset{\sim}{a}_G = \cancel{\underset{\sim}{a}_0}^{\,0} + \underset{\sim}{\alpha} \times \underset{\sim}{OG} + \cancel{\underset{\sim}{\omega}}^{\,\underset{\sim}{0}} \times (\underset{\sim}{\omega} \times \underset{\sim}{OG})$$

$$= \begin{array}{c} (-0.125,\ 14.52,\ 0) \\ \times\ (0.6,\ 0.2,\ 0) \\ \hline (0,\ 0,\ -8.737) \end{array}$$

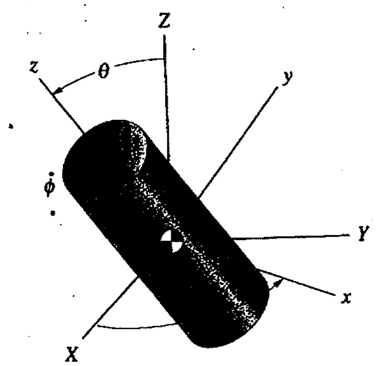
Consequently, from (1) (5)

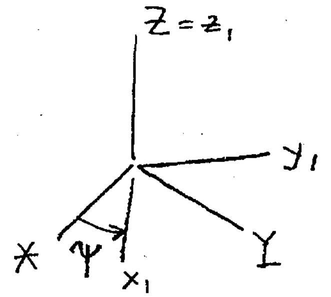
$$R_x = 0$$

$$R_y = 0$$

$$R_z = 49 + 5(-8.734) \simeq \underline{5.33}\ [N]$$

17. (bonus) A satellite can be modeled as an 1000-kg cylinder 4m in length and 2m in diameter. If the nutation angle is $\theta = 20°$, and the spin rate $\dot{\phi}$ is one revolution per second, what is the satellite's precession rate $\dot{\psi}$ in revolutions per second? (25 points)

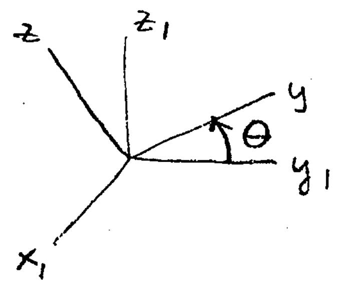




ang. velocity of $x_1 y_1 z_1$ wrt XYZ

$$\underset{\sim}{\omega}_1 = (0, 0, \dot{\psi})$$

(in both XYZ, $x_1 y_1 z_1$)



ang velocity of xyz wrt $x_1 y_1 z_1$

$$\underset{\sim}{\omega}_2 = \underset{\sim}{0} \quad (\theta = \text{const})$$

Transformation matrix from $x_1 y_1 z_1$ to xyz

$$\underset{\sim}{\alpha} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$\underset{\sim}{\omega}_1$ in system xyz

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\theta\, \dot{\psi} \\ \cos\theta\, \dot{\psi} \end{pmatrix}$$

Calculations done in system xyz

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$$\underset{\sim}{\Omega} = \underset{\sim}{\omega}_1 = (0, \sin\theta\, \dot{\psi}, \cos\theta\, \dot{\psi})$$

ang. velocity of the satellite wrt system xyz

$$\underset{\sim}{\omega}_3 = (0, 0, \dot{\varphi}) \quad (\text{in } xyz)$$

ang velocity of the satellite wrt XYZ (in xyz !)

$$\underset{\sim}{\omega} = \underset{\sim}{\Omega} + \underset{\sim}{\omega}_3 = (0, \sin\theta\, \dot{\psi}, \cos\theta\, \dot{\psi} + \dot{\varphi})$$

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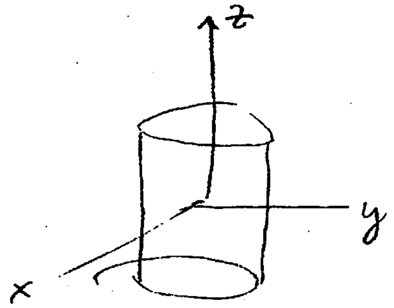

$$0 < r < R \qquad 0 < \theta < 2\pi \qquad -\frac{h}{2} < z < \frac{h}{2}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$I_x = \rho \int_0^R \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \underbrace{\left(r^2 \sin^2\theta + z^2\right)}_{y^2+z^2} \, dz \, d\theta \, r\,dr$$

$$= \rho\left(\frac{R^4}{4} h \cdot \pi + \frac{R^2}{\cancel{2}} \cancel{2}\pi \frac{h^3}{12}\right) = \rho \pi R^2 h \left(\frac{R^2}{4} + \frac{h^2}{12}\right)$$

$$= \frac{1}{24} m \left(6R^2 + 2h^2\right)$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$1 - 2\sin^2\theta = \cos 2\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \pi$$

$$I_z = \rho \int_0^R \int_0^{2\pi} \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(r^2\sin^2\theta + r^2\cos^2\theta\right) dz \, d\theta \, r\,dr$$

$$= \rho \frac{R^4}{4} 2\pi h = \frac{1}{2} m R^2$$

$$I_z = \frac{1}{2} \cdot 800 \cdot 1^2 = 400 \ [\mathrm{kg\, m^2}]$$

$$I_x = \frac{1}{24} 800 \ (6 + 16) = 1264 \ [\mathrm{kg\, m^2}]$$

5

Equation of rotational motion

$$\dot{\underset{\sim}{H}}_c = \underset{\sim c}{M} = \underset{\sim}{0}$$

$$(I_c \underset{\sim}{\omega})^{\circ} = \underset{\sim}{0}$$

$$I_c \cancel{\dot{\underset{\sim}{\omega}}} + \underset{\sim}{\Omega} \times (I_c \underset{\sim}{\omega}) = \underset{\sim}{0}$$

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Tensor of inertia at C

$$\underset{\sim}{I_c} = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{pmatrix}$$

$$\times \quad \underset{\sim}{\Omega} \quad (0, \; \sin\theta\,\dot{\psi}, \; \cos\theta\,\dot{\psi})$$

$$\underset{\sim c}{I}\,\underset{\sim}{\omega} \quad (0, \; I_x \sin\theta\,\dot{\psi}, \; I_z(\cos\theta\,\dot{\psi} + \dot{\varphi}))$$

$$(I_z(\cos\theta\dot{\psi} + \dot{\varphi})\sin\theta\,\dot{\psi} - I_x \cos\theta\sin\theta\,\dot{\psi}^2, \; 0, \; 0)$$

$$(I_z - I_x)\cos\theta\,\cancel{\sin\theta}\;\dot{\psi}^{\cancel{2}} + I_z\,\dot{\varphi}\,\cancel{\sin\theta\,\dot{\psi}} = 0$$

$$\dot{\psi} = \frac{I_z}{(I_x - I_z)\cos\theta}\,\dot{\varphi}$$

$$\dot{\psi} = \frac{400}{864\,\cos 20^{\circ}} \cdot 1 = 0.49 \; \left[\frac{rev}{s}\right]$$

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