1. Define the following notions and provide a non-trivial example (2+2 points each).
   - Lagrangian (in analytical mechanics),
   - Legendre transformation,
   - Green – St. Venant strain tensor
   - velocity gradient
   - second Piola-Kirchhoff stress tensor
   
   Please clearly distinguish between material and spatial coordinates.

2. Use Principles of Linear and Angular Momentum and formula for the velocities of points belonging to a rigid body (you need not derive them) to derive the rigid body equations of motion (15 points).

3. Derive the formulas for the velocity of a particle in a deformable body in terms of its displacement in both Lagrange and Euler coordinates. Illustrate the formulas with an example (15 points).

4. Derive equations of motion for a continuum in both Euler (Cauchy stress tensor) and Lagrange (Piola-Kirchhoff stress tensor) (15 points).

5. Consider the “three-quarter” homogeneous thin plate with radius $R$ and mass $m$ shown in Fig. 1. Compute the 3D inertia tensor at point $A$. Determine the direction through $A$ for which the corresponding moment of inertia is maximal and determine its value (20 points).

![Figure 1: A semicircular plate.](image-url)
We will use the standard polar coordinates to parametrize the domain

\[
\begin{align*}
  x &= r \cos \theta & 0 < r < R \\
  y &= r \sin \theta & \frac{\pi}{2} < \theta < 2\pi
\end{align*}
\]

Area:

\[
A = \int_{\pi/2}^{R} \int_{0}^{2\pi} r \, dr \, d\theta = \int_{0}^{R} r \, dr \int_{\pi/2}^{2\pi} d\theta = \frac{R^2}{2} \frac{3\pi}{2} = \frac{3}{4} \pi R^2
\]

density:

\[
\rho = \frac{m}{A} = \frac{4}{3 \pi R^2}
\]

Moment of inertia with respect to axis \( x \) for a homogeneous thin body (variation in \( z \) neglected):

\[
I_x = \int_A \rho(y^2 + z^2) \, dA = \int_A \rho y^2 \, dA = \rho \int_A y^2 \, dA
\]

and

\[
\int_A y^2 \, dA = \int_{\pi/2}^{R} \int_{0}^{2\pi} r^2 \sin^2 \theta r \, dr \, d\theta = \int_{0}^{R} r^3 \, dr \int_{\pi/2}^{2\pi} \sin^2 \theta \, d\theta = \frac{R^4}{4} \left( \frac{\theta}{2} + \frac{\sin 2\theta}{4} \right)_{\pi/2}^{2\pi} = \frac{R^4}{4} \frac{3\pi}{4} = \frac{3\pi R^4}{16}
\]
gives

\[
I_x = \frac{4m}{3\pi R^2} \frac{3\pi R^4}{16} = \frac{1}{4} mR^2
\]

(units OK). By symmetry,

\[
I_y = I_x = \frac{1}{4} mR^2
\]

For thin bodies,

\[
I_z = \int_A \rho(x^2 + y^2) \, dA = \int_A \rho x^2 \, dA + \int_A \rho y^2 \, dA = I_y + I_x = \frac{1}{mR^2}
\]

Product of inertia for a homogeneous body with respect to axes \( x \) and \( y \):

\[
I_{xy} = \rho \int_A xy \, dA
\]

and,

\[
\int_A xy \, dA = \int_{\pi/2}^{R} \int_{0}^{2\pi} r^2 \cos \theta \sin \theta r \, dr \, d\theta = \int_{0}^{R} r^3 \, dr \int_{\pi/2}^{2\pi} \sin 2\theta \, d\theta
\]
\[
= \frac{R^4}{4} \left( -\cos 2\theta \right)_{\pi/2}^{2\pi} = \frac{R^4}{4} \left( -\frac{1}{4} \right)(1 + 1) = -\frac{R^4}{8}
\]
gives

\[
I_{xy} = \frac{4m}{3\pi R^2} \left( -\frac{R^4}{8} \right) = -\frac{1}{6} mR^2
\]

As products of inertia \( I_{xz} = I_{yz} = 0 \) (\( z \approx 0 \)), the whole tensor of inertia at point \( A \) is:

\[
\mathbf{I}_A = \begin{pmatrix}
  \frac{1}{4} & \frac{1}{6\pi} & 0 \\
  \frac{1}{6\pi} & \frac{1}{4} & 0 \\
  0 & 0 & \frac{1}{2}
\end{pmatrix} mR^2
\]
The form of the tensor proves that $\frac{1}{2}$ equals one of the eigenvalues with the $z$ axis being the corresponding eigendirection. The eigenproblems reduces thus to two space dimensions ($x, y$) only.

Invariants:

$\begin{align*}
I_1 &= \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \\
I_2 &= \frac{1}{16} - \frac{1}{36\pi^2}
\end{align*}$

Characteristic equation:

$\begin{align*}
\lambda^2 - \frac{1}{2}\lambda + \frac{1}{16} - \frac{1}{36\pi^2} &= 0 \\
\Delta &= \frac{1}{4} - 4\left(\frac{1}{16} - \frac{1}{36\pi^2}\right) = \frac{1}{9\pi^2} \\
\sqrt{\Delta} &= \frac{1}{3\pi}
\end{align*}$

Eigenvalues

$\begin{align*}
\lambda_1 &= \frac{1}{4} - \frac{1}{6\pi}, \\
\lambda_2 &= \frac{1}{4} + \frac{1}{6\pi}
\end{align*}$

represent principal moments of inertia with respect to corresponding eigendirections. Both of them, however, are smaller then $I_z$. In other words, $I_z$ is the largest possible moment of inertia with respect to an axis passing through point $A$.

6. Reproduce the Coleman–Noll argument and discuss its consequences (15 points).