CSE386C
METHODS OF APPLIED MATHEMATICS
Fall 2014, Final Exam, 9:00-noon, Tue, Dec 16, ACES 6.304

1. Let A be a closed operator defined on a dense subspace $D(A)$ of a Hilbert space U.

   (a) Define the adjoint A^* of operator A.

   (b) Define a self-adjoint operator.

   (c) Discuss shortly how these concepts generalize the corresponding definitions of adjoint and self-adjoint operators for a continuous operators. Point out differences.

   (d) Prove that for a self-adjoint, continuous operator A,

   $$\|A\| = \sup_{\|u\|\le 1} |(Au,u)| =: \|A\|_*$$

   Hint: You will have to put together two facts:

   i. The formula:

   $$\begin{aligned}(A\boldsymbol{u},\boldsymbol{v}) = \; & \frac{1}{4}[(A(\boldsymbol{u}+\boldsymbol{v}),\boldsymbol{u}+\boldsymbol{v}) - (A(\boldsymbol{u}-\boldsymbol{v}),\boldsymbol{u}-\boldsymbol{v}) \\ & +i(A(\boldsymbol{u}+i\boldsymbol{v}),u+i\boldsymbol{v}) - i(A(\boldsymbol{u}-i\boldsymbol{v}),\boldsymbol{u}-i\boldsymbol{v})]\end{aligned}$$

   ii. For a complex-valued functional $l(v)$,

   $$\Re l(v) \le C\|v\| \quad \Longrightarrow \quad |l(v)| \le C\|v\|$$

   (Recall proof of Bohnenblust-Sobczyk Theorem),

   and remember that A is self-adjoint.

(20 points)

See book for the definitions.

The inequality

$$\|A\|_* \le \|A\|$$

is straightforward. Indeed,

$$\sup_{\|u\|\le 1} |(Au,u)| \le \sup_{\|u\|\le 1} \|Au\|\,\|u\| \le \sup_{\|u\|\le 1} \|Au\| = \|A\|$$

It is easier to see how to prove the reverse inequality starting with the real case first. For a self-adjoint operator, the "polarization formula" reduces then to:

$$(Au, v) = \frac{1}{4}\left[(A(u+v), u+v) - (A(u-v), u-v)\right]$$

Notice that the formula *does not* hold for an arbitrary operator A, as in deriving it you need to use the fact that $(Av, u) = (v, Au) = (Au, v)$. We have now,

$$\|A\| = \sup_{\|u\|\leq 1} \|Au\| = \sup_{\|u\|\leq 1} \sup_{\|v\|\leq 1} |(Au, v)|$$

But the polarization formula implies that the double supremum is bounded by

$$\begin{aligned} \tfrac{1}{4}\left[\|A\|_* \|u+v\|^2 + \|A\|_* \|u-v\|^2\right] &= \tfrac{1}{4}\|A\|_* \left[(u+v, u+v) + (u-v, u-v)\right] \\ &= \tfrac{1}{4}\|A\|_* \, 2(\|u\|^2 + \|v|^2) \\ &\leq \|A\|_* \end{aligned}$$

In the complex case, you need to use the second hint. This time,

$$\Re(Au, v) = \frac{1}{4}\left[(A(u+v), u+v) - (A(u-v), u-v)\right]$$

as the remaining two terms are purely imaginary. Notice that we have used the fact that, for self-adjoint A, expression (Au, u) is always real. Consequently, for unit vector u, v, we have

$$|\Re(Au, v)| \leq \|A\|_*$$

But Euler representation for a complex number,

$$(Au, v) = |(Au, v)| e^{i\theta}$$

implies that the modulus $|(Au, v)|$ can always be realized by "rotating"(changing phase of) function v[1],

$$|(Au, v)| = (Au, e^{i\theta} v) = \Re(Au, e^{i\theta} v) \leq \|A\|_*$$

Taking supremum with respect u, v, we get the final result.

An alternate, simpler argument used by Toshiwal: When we replace $\|Au\|$ with $\sup_{\|v\|\leq 1}(Au, v)$, we need not to consider the supremum over the *whole* unit ball as we actually *do know* that the supremum is attained for $v = Au/\|Au\|$. So, it is sufficient to restrict yourself to this uniqe v only. But then we also know that $(Au, v) = \|Au\|$ is actually real ! Consequently, we can proceed with the real part of the polarization formula only *without* invoking the more complex argument for the complex-valued functionals. In any event, perhaps it was useful to refresh this argument anyway...

[1] Rotation of v does not change its norm.

2. (a) Define discrete, residual and continuous spectrum for an operator $A : U \supset D(A) \to U$ where U is a Hilbert space.

   (b) Determine spectrum of operator A where

$$U = L^2(\mathbb{R}) \quad D(A) = H^1(\mathbb{R}) \quad Au = i\frac{du}{dx}$$

   Hint: Use Fourier transform.

(30 points)

This is a slight modification of the example discussed in the book for $Au = u'$. Multiplication by i makes the operator self-adjoint. Consequently, there is no residual spectrum. Direct computations using Fourier transform reveal that there is no point spectrum either. The continuous spectrum consists of the whole real line.

3. Consider a first order operator A in $L^2(0,1)$,

$$D(A) = \{u \in H^1(0,1) \,:\, u(0) = u(1) = 0\} \qquad Au = u' + u$$

where the derivative is understood in the sense of distributions.

(a) Define a closed operator and prove that operator A is closed. You may use the fact that pointwise value $u(x), x \in [0,1]$ of $u \in H^1(0,1)$ represents a continuous functional.

(b) Determine the adjoint operator A^* and its null space.

(c) Recall the explicit formula for the solution of a similar problem from your last exam,

$$u(0) = 0, \quad u' + u = f$$

and use it to prove it that operator A is bounded below in $L^2(0,1)$.

(d) Discuss the well posedness of the problem:

$$u \in D(A), \qquad Au = f$$

with an appropriate right-hand side f.

(25 points)

See book for definitions.

Answers:

(a) Let $u_n \in D(A)$ and $(u_n, Au_n) \to (u,v)$, i.e. $u_n \to u$, $Au_n \to v$, all convergence understood in the L^2-sense. Consequently, $u_n' \to v - u$. By definition,

$$\int u_n'\phi = -\int u_n\phi' \quad \forall \phi \in C_0^\infty(0,1)$$

Passing to the limit on both sides, we get

$$\int (v-u)\phi = -\int u\phi' \quad \forall \phi \in C_0^\infty(0,1)$$

which proves that $v - u = u'$ in the sense of distributions.
Thus $u' = v - u \in L^2(0,1)$ and, therefore, $u_n \to u$ also in $H^1(0,1)$ which in turn implies that $u(0) = u(1) = 0$. Consequently, $u \in D(A)$, and $v = u + u' = Au$ as required.

(b) Integration by part argument gives:

$$\begin{aligned} D(A^*) &= H^1(0,1) \quad A^*v = -v' + v \\ \mathcal{N}(A^*) &= \mathbb{R}e^x := \{ce^x \,:\, c \in \mathbb{R}\} \end{aligned}$$

(c) See the solution to the third exam problem. The fact that we have an extra boundary conditions does not matter. The formula is still valid.

(d) For every right hand side $f \in L^2(0,1)$ that satisfies the compatibility condition:

$$\int f e^x \, dx = 0 \,,$$

the problem has a unique solution that depends continuously upon the data

$$\|u\| \leq \alpha^{-1} \|f\|$$

where α is the boundedness below constant.

4. Consider the "ultraweak" variational formulation of the previous problem,

$$\left\{ \begin{array}{l} u \in U := L^2(0,1) \\ \underbrace{\int_0^1 u\, A^* v\, dx}_{b(u,v)} = \underbrace{\int_0^1 f v\, dx}_{l(v)} \quad \forall v \in V := H^1(0,1) \end{array} \right.$$

where A^* denotes the formal adjoint of A, $A^* v = -v' + v$.

(a) Define operator $B : U \to V'$ and its conjugate corresponding to the bilinear form $b(u,v)$.

(b) Use Babuška-Nečas Theorem and results from the previous problem to investigate the well-posedeness of the problem.

Hint: Can you relate the inf-sup constant for this problem with the boundedness below (Friedrichs) constant of operator A from the previous problem ? (25 points)

There are two operators associated with the bilinear form:

$$B : L^2(0,1) \to (H^1(0,1))'$$

$$B' : H^1(0,1) \to L^2(0,1) \sim (L^2(0,1))'$$

Due to reflexivity of Hilbert space, operator B' can be identified with the transpose of B. The whole point of this exercise is to realize that transpose B' *coincides* with the adjoint A^* discussed in the previous problem. Direct application of Cauchy-Schwartz inequality shows that both forms: $b(u,v)$ and $l(v)$ are continuous. Finally, the Closed Range Theorem for continuous operators implies that

$$\gamma = \inf_{u \in L^2} \sup_{v \in H^1} \frac{|\int_0^1 u(-v' + v)|}{\|u\|_{L^2}\, \|v\|_{H^1}} = \inf_{[v] \in H^1/\mathcal{N}(B')} \sup_{u \in L^2} \frac{|\int_0^1 u(-v' + v)|}{\|u\|_{L^2}\, \|[v]\|_{H^1/\mathcal{N}(B')}} .$$

Since $B' = A^*$, the right-hand side coincides with the boundedness below (Friedrichs) constant for the quotient operator corresponding to A^*. But, by the Closed Range Theorem for closed operators, this constant is equal exactly to constant α discussed in the previous problem. Consequently, there is no need to prove anything new. Application of Babuška-Nečas Theorem implies that, for any right-hand side f satisfying the compatibility condition, we have a unique solution that depends continuously upon the data. The subtle difference between the strong and (ultraweak) variational formulations is the regularity. The ultraweak formulation may accommodate "distributional loads":

$$l \in \left(H^1(0,1)\right)'$$

with the continuous dependence upon data modified accordingly:

$$\|u\| \leq \gamma^{-1} \|l\|_{(H^1(0,1))'} .$$