1. Define the following notions and provide a non-trivial example (2+2 points each):
   - (topological) transpose of a continuous operator,
   - (topological) transpose of a closed operator,
   - orthogonal complement of a subspace in a Hilbert space,
   - orthonormal basis in a Hilbert space,
   - Riesz operator.

See the book.

2. State and prove three out of the four theorems (10 points each):
   - Properties of the transpose of a continuous operator (Prop. 5.16.1)
   - Characterization of injective operators with closed range (Thm. 5.17.1)
   - Completeness of quotient Banach space (Lemma 5.17.1)
   - The Orthogonal Decomposition Theorem (Thm. 6.2.1)

See the book.

3. Let $X$ be a Banach space, and $P : X \to X$ be a continuous linear projection, i.e., $P^2 = P$. Prove that the range of $P$ is closed. (10 points)

   Let $u_n \in \mathcal{R}(P), u_n \to u$. We need to show that $u \in \mathcal{R}(P)$ as well. Let $v_n \in X$ be such that $u_n = P v_n$. Then $P u_n = P^2 v_n = P v_n = u_n \to P u$. By the uniqueness of the limit, it must be $u = P u$. Consequently, $u$ is the image of itself and must be in the range of the projection.

4. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal family in a Hilbert space $V$. Prove that the following conditions are equivalent to each other.

   (i) $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis, i.e., it is maximal.

   (ii) $u = \sum_{n=1}^{\infty} (u, e_n) e_n \quad \forall u \in V$.

   (iii) $(u, v) = \sum_{n=1}^{\infty} (u, e_n) \overline{(v, e_n)}$. 
(iv) \[ \| u \|^2 = \sum_{n=1}^{\infty} |(u, e_n)|^2. \]

(15 points)

(i)⇒(ii). Let \[ u_N := \sum_{j=1}^{N} u_j e_j, \quad u_N \to u \]

Multiply both sides of the equality above by \( e_i \), and use orthonormality of \( e_j \) to learn that \[ u_i = (u_N, e_i) \to (u, e_i) \] as \( N \to \infty \)

(ii)⇒(iii). Use orthogonality of \( e_i \) to learn that \[ (u_N, v_N) = \sum_{i=1}^{N} u_i \overline{v_i} = \sum_{i=1}^{N} (u, e_i) (\overline{v}, e_i) \to \sum_{i=1}^{\infty} (u, e_i) (\overline{v}, e_i) \]

(iii)⇒(iv). Substitute \( v = u \).

(iv)⇒(i). Suppose, to the contrary, the \( \{e_1, e_2, \ldots\} \) can be extended with a vector \( u \neq 0 \) to a bigger orthonormal family. Then \( u \) is orthogonal with each \( e_i \) and, by property (iv), \[ \| u \| = 0. \] So \( u = 0 \), a contradiction.

5. Consider an elementary boundary-value problem:
\[ \begin{cases} u(0) = 0 \\ u' + u = f \end{cases} \]

Use elementary means (variation of a constant) to derive the explicit formula for the solution, \[ u(x) = \int_{0}^{x} e^{(s-x)} f(s) \, ds. \]

Use the formula then to demonstrate that the operator \[ A : L^2(0, 1) \supset D(A) \to L^2(0, 1) \]
where \[ D(A) = \{ u \in H^1(0, 1) : u(0) = 0 \}, \quad Au := u' + u \]
is bounded below. (10 points)
The first part is elementary. The second follows from Cauchy-Schwarz inequality:

\[
\int_0^1 |u(x)| \, dx = \int_0^1 \left| \int_0^x e^{(s-x)} f(s) \, ds \right|^2 \, dx \\
\leq \int_0^1 \left( \int_0^x e^{(s-x)} \, ds \right)^2 \int_0^1 |f(s)|^2 \, ds \, dx \\
\leq e^2 \int_0^1 |f(s)|^2 \, ds
\]

6. Analyze well posedness of the variational problem;

\[
\begin{align*}
\left\{
\begin{array}{l}
\quad u \in H^1(0,1), \, u(0) = 0 \\
\quad \int_0^1 (u' + u)v \, dx = \int_0^1 fv \, dx \quad \forall v \in L^2(0,1)
\end{array}
\right.
\]

*Hint:* Use the result from the previous problem. (15 points)

Check the assumptions of Babuška-Nečas Theorem. The only non-trivial condition is the inf-sup condition. But this follows from the previous problem. Indeed,

\[
\sup_v \frac{|\int_0^1 (u' + u)v|}{\|v\|_{L^2}} = \|u' + u\|_{L^2} \geq c \|u\|_{L^2}
\]

At the same time,

\[
\|u'\|_{L^2} \leq \|u' + u\|_{L^2} + \|u\|_{L^2} \leq (1 + c^{-1}) \|u' + u\|_{L^2}
\]

Combining the two inequalities, we prove the inf-sup condition.