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Some Current Trends in Finite Element Research

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Introduction

In the 1960's the finite element method was under rapid development in the field of structural mechanics. The essential features of the methodology had been identified and it was clear to the research community what additional extensions were needed to achieve general capabilities for the bulk of structural mechanics problems which had traditionally faced analysts. The main thrusts of research at the time were directed towards the development of curved two- and three-dimensional higher-order continuum elements, and the development of effective shell elements.

Straight edged higher-order triangular and tetrahedral continuum elements were available early on, but were inconvenient in many situations, and did not permit accurate geometrical modeling of curved domains. The importance of elements of general shape, particularly curvilinear triangles, quadrilaterals, wedges and brick-shaped elements, was acknowledged by researchers as a key step in the development of the finite element method for it would enable convenient modeling of intricately shaped, real-world engineering designs. All classical analytical techniques, and even competing numerical techniques of the time, were severely limited due to the inability to handle complex geometries. One may view the "geometry problem" as one which has beleaguered the history of analysis. Essentially with the development of curved "isoparametric" elements [E1-E3, I1, I2, T1, Z2] the problem was solved once and for all for most practical purposes.

The ability to solve general shell configurations was of considerable interest to the developers of finite element methods in the 1960's due to its obvious importance throughout structural analysis, and particularly because most researchers were actively engaged in aerospace projects, a focal point of engineering endeavors at the time. By the latter part of the decade a wide variety of shell elements had been developed which were adequate for most linear analyses [A7, B24, C2, C7, C8, G17, P7, S6, Z8].

With the main problems of finite element/structural mechanics research (as perceived at the time) essentially solved, and the occurrence of a simultaneous decline in the aerospace industry, which had provided the major support for finite element developments, one would have expected a decrease in finite element research. In fact, to some extent this did occur. In aerospace, interest shifted to computer-program development employing, by and large, the existing methodology. Basic research activities dwindled. Evidence which indicates the reduced amount of finite element research may be garnered from the number of papers published per year. From 1960 onward, the number increased "exponentially" through 1969. For example, ten papers were published in 1961 and five hundred and thirty-one[1] were published in 1969. The first (and only) year that there was a decrease in the number of papers was in 1970 in which five hundred and ten appeared. Considering the usual delay between performing research and getting it published of about two years, it may be seen that the decline in number of papers may be correlated with the publication of major works on curved elements and shells, and the general decline in aerospace activities.

The predictable decline cited above was immediately reversed in 1971 in which eight hundred and forty-four papers on finite elements appeared. The subsequent year to year increases have been dramatic. In 1974, the last year for which complete data is available, thirteen hundred and seventy-seven papers were published.[2] What happened was that a counter trend of greater magnitude had begun and overwhelmed the decline associated with aerospace structural research: the finite element method had spread to other areas of engineering and analysis. The techniques that had proven so successful in structural analysis were seen to be more general and were being applied elsewhere. The seeds for this development had already been planted in the 1960's [W9, V1, Z4, Z6] and considerable momentum had been created by the early 1970's. This trend continues unabated today.

Within each field to which finite elements have been applied, success on simple problem classes has encouraged bolder applications, typically nonlinear and time-dependent.

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[1] All data quoted are from [N4].

[2] As of the writing of this manuscript, thirty-two texts on finite elements had been published. See [H14] for a compilation.

These have in turn created new problems of methodology and so research has continued to grow. Structural mechanics is a case in point. The emphasis in recent research has been oriented more to nonlinear dynamic phenomena involving, for example, finite deformations of inelastic materials, contact-impact, etc., and this has led to fundamentally new problems which continue to be grappled with.

In the remainder of this review I will try to identify some of the main contemporary trends in finite element research and their significance. The present work builds upon Zienkiewicz's earlier review in these pages [Z1], and complements several other feature articles dealing with other aspects of this subject, namely, Finlayson and Scriven's review of the method of weighted residuals [F4], Argyris and Patton's discourse on the role of the computer in research [A12] and, most recently, Oden and Bathe's [O3] commentary on the state of computational mechanics.[3]

The selection of topics discussed is necessarily a personal one shaped by my experience. The theme is essentially positive and optimistic as this is my overall impression of activity in the field. Due to the enormity of the literature on finite elements, no attempt has been made to mention all important recent works. Nevertheless, a large number of references are cited which should prove useful to readers wishing to delve further into particular areas.

Fluid Mechanics

The vast majority of problems in solid and structural mechanics involve symmetric, positive, spatial differential operators. The finite element method, as is most commonly practiced, is taken to be the *Galerkin* finite element method, a special member of the family of so called weighted residual methods [F3] in which both trial solutions and weighting functions are selected from the *same* class of functions. This canonical symmetry of the approximation method turns out to exploit the structure of symmetric operators. It is a simple exercise to show that a "best approximation" property is achieved in the natural energy norm defined by the operator [S4]. This optimality is the underlying reason for the high accuracy consistently attained by Galerkin/finite element methods in solid and structural mechanics. (Certain elements, incapable of convergence in the limit, have actually exhibited good accuracy in practical situations. See [B9].)

The basic problems of fluid mechanics involve nonsymmetric "convection" operators. In most physical problems of engineering interest convection is dominant. The Galerkin/finite element method loses the best approximation property under these circumstances and there are many situations in which the solution may be shown to be very poor indeed [H6, H32]. This issue was raised in [S4] and the potential usefulness of Galerkin/finite element methods in simulating flows was cast in doubt. It turns out that, despite the noted deficiencies, the basic Galerkin/finite element method is capable of high accuracy in *many* flow problems, even convection dominated ones, and a great deal of success and progress has been achieved in this mode on a wide range of fluid mechanics problems [G4-G8, G18, T2].

Nevertheless, it is still correct to point out that the basic Galerkin/finite element method exhibits spurious behavior for *some* convection dominated situations. These problems have been attributed to downstream essential, or "hard," boundary conditions [G9, Z3], but are still not fully understood.

This deficiency in the basic "recipe" has caused the realization that the degree of generality and reliability achieved by Galerkin/finite element methods in symmetric problem classes can never be achieved in fluid mechanics without a fundamental breakthrough in the analytical treatment of nonsymmetric operators. This necessarily will take us beyond the Galerkin/finite element method as classically used, but to what is the question.

The research problem is thus clearly set: To develop efficient finite element schemes with all the usual attributes vis-à-vis geometry, etc., which retain a best approximation property in an appropriate sense. Presently, there is considerable activity on this topic and it is being pursued in diverse areas of engineering and the physical sciences. A variety of schemes have been proposed and are under investigation [B31, G19, H6-H8, H18, H23, H24, H32].[4] A name which has been used to describe some of these techniques is "upwind finite element methods," since some of the initial efforts were very similar in intent to classical upwind finite difference methods [R2]. The name has stuck, which may be unfortunate, because some of the new finite element methods are very different from upwind finite difference techniques and are not subject to the well-known defects of the latter. An Applied Mechanics Division Symposium at the 1979 Winter Annual Meeting attempts to assess the state-of-the-art in this area [H19].

If a methodology emerges from these endeavors which attains the reliability of Galerkin/finite element methods on symmetric problems, then an era of development of "general purpose" computer programs in fluid mechanics may be anticipated. In the meantime we can at least expect many special purpose finite element programs with (hopefully) accompanying admonitions that in some situations the performance may be inadequate.

The whole effort to develop new finite element methods for fluid mechanics problems has been criticized by some investigators because of the current ability to accurately solve *particular* problems with Galerkin/finite element methods. This begs the central *numerical* problem of the general class of nonsymmetric operators which needs to be satisfactorily resolved to have significant effect on engineering practice.

Prior to any serious hope of attempting a numerical study of complicated physical phenomena such as turbulence, a much greater degree of reliability and *a priori* optimality must be achieved. The use of techniques which work well "sometimes" is simply not good enough.

The types of flow problems to which finite element methods have been applied are by now too numerous to describe in an article of this scope. The interested reader

[3] An excellent early review and history of matrix methods of structural analysis may be found in J. H. Argyris, "On the Analysis of Complex Elastic Structures," *Applied Mechanics Reviews*, Vol. 11, 331-338 (1958).

[4] Morton and colleagues at the University of Reading have made several recent important contributions. See K. W. Morton and J. W. Barrett, "Optimal Finite Element Methods for Diffusion–Convection Problems," pp. 134-148 in *Proceedings of the Conference on Boundary and Interior Layers–Computational and Asymptotic Methods*, J. J. H. Miller (Ed.), Boole Press, Dublin, 1980, and references therein.

may profitably consult [G4-G6, T2], and references cited therein, for exposure to the literature.

Fluid-Structure Interaction

Traditionally, fluid-structure interaction analysis has been an area in which special techniques have been developed to exploit the structure of specific problems. With advances being made in finite element methods for fluid mechanics, it becomes clear that a merging with the now well established discipline of finite element structural analysis will lead to greater analytical ability in this area of significant contemporary interest. Given basically sound numerical capabilities for fluids and structures, some way of "interfacing" the fluid and structural domains is, of course, required. This is complicated by the fact that different kinematical descriptions are generally favored for each. For example, it is usually convenient in modeling structures to adopt the classical Lagrangian description in which the finite element mesh moves with the material particles. On the other hand, the Eulerian description, in which the mesh is fixed in space, is generally favored for fluids due to typically large fluid motions. Clearly, some compromise in description must be adopted to model the fluid interface region between the purely Eulerian and Lagrangian subdomains. More flexible descriptions are also required to model free-surface flows. Several techniques have been developed with these ends in mind [B13, D3, D4, H33, W2]. The terminology generally applied to these techniques is "mixed Lagrangian-Eulerian finite element methods." They possess the ability to continuously interpolate between the classical descriptions, and/or directionally split descriptions at each point. Generalizations along these lines [H15] and further applications of techniques such as these promise to considerably extend problem solving capabilities in ensuring years. Areas other than fluid-structure interaction will no doubt also benefit from these developments (e.g., metal forming).

Transient Analysis

Until a few years ago, classical "off-the-rack" techniques were used to time-integrate the large systems of ordinary differential equations generated by finite element spatial discretizations. These schemes could be segregated into two distinct classes: *explicit* and *implicit*.

Explicit algorithms involve no matrix equations, consequently computer storage requirements and cost per time step are relatively small. The shortcoming is that numerical stability considerations dictate the use of small time steps, often smaller than necessary for accuracy. Explicit algorithms are generally felt to be cost effective for analysis of wave-propagation phenomena in continua.

Implicit algorithms, on the other hand, require solution of matrix equations during each time step which engenders a considerable increase in storage and number of operations per step over explicit algorithms. The attribute is that numerical stability is improved, and in many cases implicit algorithms can be shown to be unconditionally stable (i.e., no restriction, aside from accuracy, is imposed on the time step). In cases in which the response is dominated by the low modes, the larger time step permitted often makes implicit schemes more cost-effective.

It has been concluded that neither explicit nor implicit algorithms are superior for all problem classes. Some classes, such as structure-continuous media interaction, suggest that an optimal scheme might employ both implicit and explicit concepts within one algorithm. Finite element applications of this idea were first introduced in [B14, B15]. More recent works [H29, H30, H34, H35] have shown how to develop such schemes within the context of the standard finite element "assembly" algorithm. This procedure may be implemented in many existing implicit computer programs with only minor modifications. Furthermore, the basic structure of the schemes has suggested many other generalizations [F2, H20, H34, H35, P3]. It now appears that many different algorithmic concepts such as implicit-explicit schemes, alternating direction methods [T8], staggering schemes [P4], etc., may be deducible from a general coherent theory, and the implementation may be facilitated in similar fashion. The ideas are general in the sense that they pertain to both linear and nonlinear analysis in the context of any field theory. This represents a considerable generalization and consolidation of ideas and is a significant step toward optimally efficient transient analysis.

Various other improvements in transient algorithms have also been recently achieved. Examples which may be mentioned are refined damping characteristics which enable filtering of spurious higher modes, without adverse effect on the accuracy in the lower modes [H11-H13], and algorithms which rigorously achieve unconditional stability in nonlinear elastodynamics [H5, H25].

Two areas which are now being given increased attention, for their obvious importance in lowering computational cost, are automatic time stepping strategies based upon accuracy considerations [H10, P5, U3] and subcycling techniques in which subdomains of the mesh are integrated at different time steps [B18, W10]. Although progress has been made, neither of these areas seems to have reached full fruition yet, and thus it may be anticipated that considerable further activity will be seen.

Synthesis of Theoretical Concepts

At one point in the development of finite element methods, a bewildering array of approaches confronted the potential finite element developer for even the simplest class of problems. The so called displacement, force, mixed and hybrid formulations may be mentioned as examples. In linear elastostatics, displacement and force models are based upon the variational theorems of minimum potential and complementary energy, respectively, whereas mixed and hybrid models are based upon particular forms of mixed-field variational theorems such as those of Hellinger-Reissner and Hu-Washizu. There was a period when much activity took place in developing elements based on the various formulations, and the relative strengths and weaknesses were strongly argued. This has for the most part subsided. The majority of work is presently being performed with the simplest formulation. (This is the displacement method in elastostatics.) The reason for this is that success can generally best be achieved with the simplest formulation.

The trend in recent years has been to establish the equivalences or similarities between the various approaches rather than emphasize their differences [B19, pp. 276-280 in G1, H17, H32, L2, M2-M4]. These efforts have to some extent lessened interest in the more exotic approaches since equivalent results have been achieved with much simpler

procedures. A case in point has been the increased use of reduced integration and allied techniques in constrained media applications (e.g., incompressible solids and fluids, and beam, plate and shell models based upon theories which account for transverse shear deformations; see [F6, G14, H21, H26-H28, H32, H36, H37, K1, K2, M1, N1-N3, P10, Z5, Z9] and references therein). In the same spirit, penalty-function methodology is increasingly being used as an alternative to Lagrange multiplier formulations [B19, H32, H37, M3].

Equivalences between some finite element and finite difference techniques have also been established recently [J2, K10]. One consequence is that dominant areas of finite-difference methodology, such as the so called "Lagrangian hydrocodes" [G13], are now being subsumed by competing finite element codes [B10, G15, H1-H4, K4] which are often faster and always much more versatile with respect to mesh topology.

Additional Topics

Inelastic Analysis. The numerical ability to solve large-scale nonlinear inelastic problems has advanced beyond the ability to characterize nonlinear materials. This fact has provided increased motivation for the development and study of new constitutive models which more faithfully model the response of materials, especially geological ones such as soils and rocks [C6, M11, P8, P9]. New models of this type often represent a considerable increase in complexity, and their practical usefulness depends upon efficient and reliable implementation in finite element computer programs. This has lead to the study of "stress point algorithms," the methodology concerned with the integration of constitutive equations. Considerable progress and understanding have been achieved in relatively simple settings [K7-K9, S1] and much additional work may be foreseen along these lines.

Forming Processes. Metal forming processes (e.g., extrusion, stamping, bending, etc.) have been based for the most part upon empiricism since the large-deformation, inelastic behavior characteristic of these phenomena was in the past outside the realm of analytical capabilities. The situation is now changing due to the increased ability to solve large-deformation problems by finite element methods. Considerable attention to forming processes has been given recently [A13, K5, Z7] and this may eventually have considerable beneficial consequences to the efficient design of forming tools and machinery.

Nonlinear Equation Solving. The solution of large-scale finite element problems is contingent upon solution of the matrix-equation systems developed. Although numerous studies have been undertaken to develop effective iterative solution procedures, in hope of lessening storage requirements and calculations associated with direct elimination schemes, very little significant progress has been made to date. Indeed, direct procedures are still almost universally relied upon for finite element equation systems, and are included in all major codes available to the general public. Presently, "compacted column" [B7, F1, M9, M10, T3, W7, W8] and "frontal" [I3, J1] techniques have replaced "band solvers" in most major codes. These newer procedures are virtually identical in speed and storage requirements, assuming each is optimally coded. The compacted column technique is often favored due to its conceptual simplicity and greater efficiency with respect to re-solutions [T5].

Many attempts are being made to improve the efficiency of direct nonlinear equation-solving algorithms [M5]. In these efforts, use is being made of direct, iterative and combined concepts. Of particular note lately is the adoption of conjugate gradient and quasi-Newton updates [B6, B30, C10, C11, G12, I4, M6, S5] which have been employed extensively in other fields, such as optimization. It is anticipated that in the future increased use will be made of transient analysis methodology in static and quasi-static situations [U1] due to the higher-level of understanding of transient algorithms and variety of approaches now available.

Contact-Impact. Many important engineering problems involve contact, impact and/or frictional sliding between two or more bodies (e.g., problems emanating from weapons technology, bearing design, vehicle safety and crash-worthiness, etc.). A completely general theoretical framework, suitable as a basis for the development of finite element methods, does not yet exist for the entire class of problems of this type, although variational inequalities may be used to characterize a subclass [K6]. Despite this, considerable recent progress has been made in the practical resolution of many difficult problems [B16, F5, H1, H10, H38]. It is hoped that the gap between theory and practice will close in the coming years.

Fracture Mechanics. The technological importance of fracture mechanics has motivated considerable finite element research. Several approaches have been proposed to model the singularities which are present in problems of this type [A2, A7, B3, B4, B22, H9, S3, T7]. Perhaps the most general approach thus far has been to include special singular functions, determined from analytical procedures, amongst finite element basis functions. This insures full rate-of-convergence of the method in the presence of the singularity. Considerable success has been achieved for two-dimensional cases and current attention is focused on three-dimensional cases, surface cracks, crack propagation, and nonlinear and dynamic situations [G3, P6].

Special Elements. As in the case of fracture mechanics, the special features of particular problem classes, such as singularities, may be embedded in a finite element formulation if sufficient analytical results are available. This marriage of classical and numerical concepts has proven quite successful in several areas of interest [A3]. Recent applications include boundary-layer elements [H39] for viscous flow calculations, and infinite elements [B21, G10] for modeling unbounded domains. A concise computer algorithm has been developed for producing finite element interpolation schemes from mixed classes of functions, necessary for achieving "special" element properties [H22]. Techniques are now available for developing elements with an arbitrary number of boundary surfaces, each of which takes on a prescribed analytical form [W1]. For most practical purposes, such generality is unnecessary, as isoparametric elements suffice. However, special circumstances will no doubt be encountered in the future when elements of this more general kind will prove useful.

Boundary Element Methods. Finite element discretizations of boundary integral formulations – termed "boundary element methods" [B12] – have gained increased attention in recent years. The popularity of this procedure stems from the fact that a reduction in dimensionality can often be

achieved which may result in a significant decrease in computational effort. For example, a problem of three-dimensional elastostatics may require only a discretization of its boundary, which is two-dimensional. The complex boundaries of three-dimensional engineering geometries are conveniently discretized using isoparametric finite elements [L1]. Although some nonlinear and time-dependent applications of the boundary element method have been given [G11, M7, U2], the major area of success so far has been linear static problems [B26, C12]. Boundary element procedures are also well suited for the modeling of infinite domains [U2], such as occur in the analysis of soil-structure interaction.

Electromagnetic Field Theory. One of the many areas that has recently been impacted upon by finite element methodology is electromagnetism [A5, A6, C3, Z4]. As is often the case, the geometric complexity of electromagnetic devices favors the use of finite element procedures over competing numerical techniques. Typically, problems in electromagnetic field theory involve infinite domains. A particularly interesting recursive procedure has been developed for this purpose which does not entail use of special elements or analytical results [S2].

Mesh Optimization. Some recent progress has been made in the development of mesh optimization algorithms and adaptive mesh refinement strategies [B1, C1]. Nevertheless, it is fair to say that procedures of this type are to date little used in practical problem solving. Based on current developments, one would anticipate that limited purpose computer codes, which employ optimization/refinement techniques, will be available in the near future for certain problem classes.

Recently, an alternative procedure to mesh refinement has been proposed in which the mesh is fixed once and for all, but additional functions are added to the finite element basis. It has been shown that implementation may be expeditiously carried out and that exceptional convergence characteristics are achieved in practice [S7]. The terminology "p-convergence" has been applied to these methods due to the fact that convergence rate has been shown to be proportional to the order of the polynomial, p, contained in the finite element basis.

Finite Deformation Shell Analysis. Much recent progress has been made in the large deformation analysis of three-dimensional shell structures by finite elements [A8-A11, B5, B11, B12, B17, B23, B29, G2, H31, I5, P1, P2, R1, W3]. Many of these works employ the "degeneration technique" [A1] in which shell element properties are derived directly from general three-dimensional nonlinear continuum concepts. This avoids some of the limitations and inconveniences inherent in the use of shell theories. Difficult nonlinear post-buckling phenomena exhibiting imperfection sensitivity have been shown to be computable in certain cases by high precision elements [B29, I5]. The primary drawback to these developments is the computer cost associated with the formulation of elements. Consequently, simpler, lower-order alternatives are under investigation. These necessarily involve more sophisticated concepts and methodology since the low-order functions employed are incapable in themselves of exhibiting good bending behavior (see [A11, B8, B20, H16, H31, H36, K1-K3, M1, R3, T4] for contributions along these lines). It is sometimes frustrating to the non-specialist that, at this stage of the development of finite element methods, new elements, such as those cited, are still being proposed. These efforts, however, are important as it is generally acknowledged that optimally accurate and efficient shell analysis is far from a reality.

Mathematical Theory. In the early years of development of the finite element method there was little concern for, or awareness of, its mathematical basis. The situation is now completely changed. Since the late 1960's, the mathematical theory of finite elements has steadily grown [A14, C5, D1, M8, O1, S4, W4, W5]. Many of the basic questions regarding stability, order-of-accuracy, and convergence were established in the early 1970's for simple classes of problems. To some extent the recent thrust is associated with developing mathematical theories of the governing classes of nonlinear differential equations themselves (e.g., the Navier-Stokes equations [T6], first order hyperbolics [B2], the equations of nonlinear elasticity [O2], etc.), since it is recognized that without a complete mathematical theory of the underlying problem, there is no rigorous basis for a mathematical theory of approximation via finite elements.

The development of mathematical theories has tended to lag behind the realm of feasible computations and thus the emphasis in computer model validation has been on comparisons with experimental results. For example, despite a paucity of mathematical results, a degree of confidence has been attained in elastic-plastic computations.

Computers

It may be deduced from the preceding remarks that considerable advancement has been made in recent years in the development and understanding of finite element methods for engineering analysis. If digital computer capabilities were stagnant, one could still look forward to continual progress in the years to come based upon the current rate of improvements in algorithms and methodology. As almost everyone is aware, however, digital computer capabilities are far from stagnant. We are living in an era which has been aptly described as the "integrated circuit revolution." Digital computer capabilities are increasing in speed and capacity, while simultaneously decreasing in cost. The rate at which these events have been taking place is staggering, and the limits are nowhere in sight. Orders-of-magnitude improvements are still technically feasible and, in fact, are anticipated in the ensuring years.

This is of great significance to finite element research and application since digital computers are the technological foundation upon which finite element analysis rests. Every increase in the performance/cost ratio enhances the ability of existing finite element methods to solve engineering problems effectively.

Improved computer graphics hardware/software also contributes greatly to the use of element procedures in that time and cost involved in preparing data and assimilating results are considerably decreased. Finally, new types of hardware (e.g., microcomputers) are becoming available which promise to enlarge the scope of finite element computer analysis in the coming years [W6].

Concluding Remarks

The finite element method represents one of the most significant developments in the history of engineering analysis. What was viewed as analytically inconceivable as little as twenty years ago is often commonplace today. The primary reasons for the success of the method are its generality,

ability to handle arbitrarily complex geometries and its consistent treatment of difficult differential-type boundary conditions. These strengths are often decisive when compared against competing techniques. In addition, finite element techniques may be brought to bear on any problem which can be stated in terms of differential, integral, or integro-differential equations. Once the basic procedures are mastered, the door is open for the practitioner to apply his skills to a wide range of physical problems. Successful application of these skills must, of course, be buttressed by a sound physical understanding of the problem area in which the analyst works. We are presently seeing the development of finite element sub-disciplines, within several fields of engineering and science [B27, B28, C4, C9, D2, G16, G20, L3, L4, O4]), that seek to thoroughly combine both the physical and computational aspects of the problem. This trend will no doubt continue since proper application of complex element methodology cannot be performed independently of physical insight.

Perhaps the most salient feature of current finite element research is the variety of areas in which developments are being undertaken. A sampling of the more prominent disciplines has been discussed in the body of this review. The most important area of endeavor appears to be fluid mechanics. Successful resolution of the treatment of nonsymmetric operators, which dominate much fluid mechanical phenomena, will at once enhance the numerical calculation of flows and have significant spin-off effects on other disciplines.

Financial support for finite element development should continue to be good due to the area's "track record," its continued potential, widespread use in the industrial sector, and an increasing demand for sophisticated and accurate analyses of evermore complicated engineering systems.

The continual improvements in methodology and computers augers an era in which the development and use of finite element methods should continue to rapidly grow.

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