Modeling the Effect of Resolution Inhomogeneity in LES

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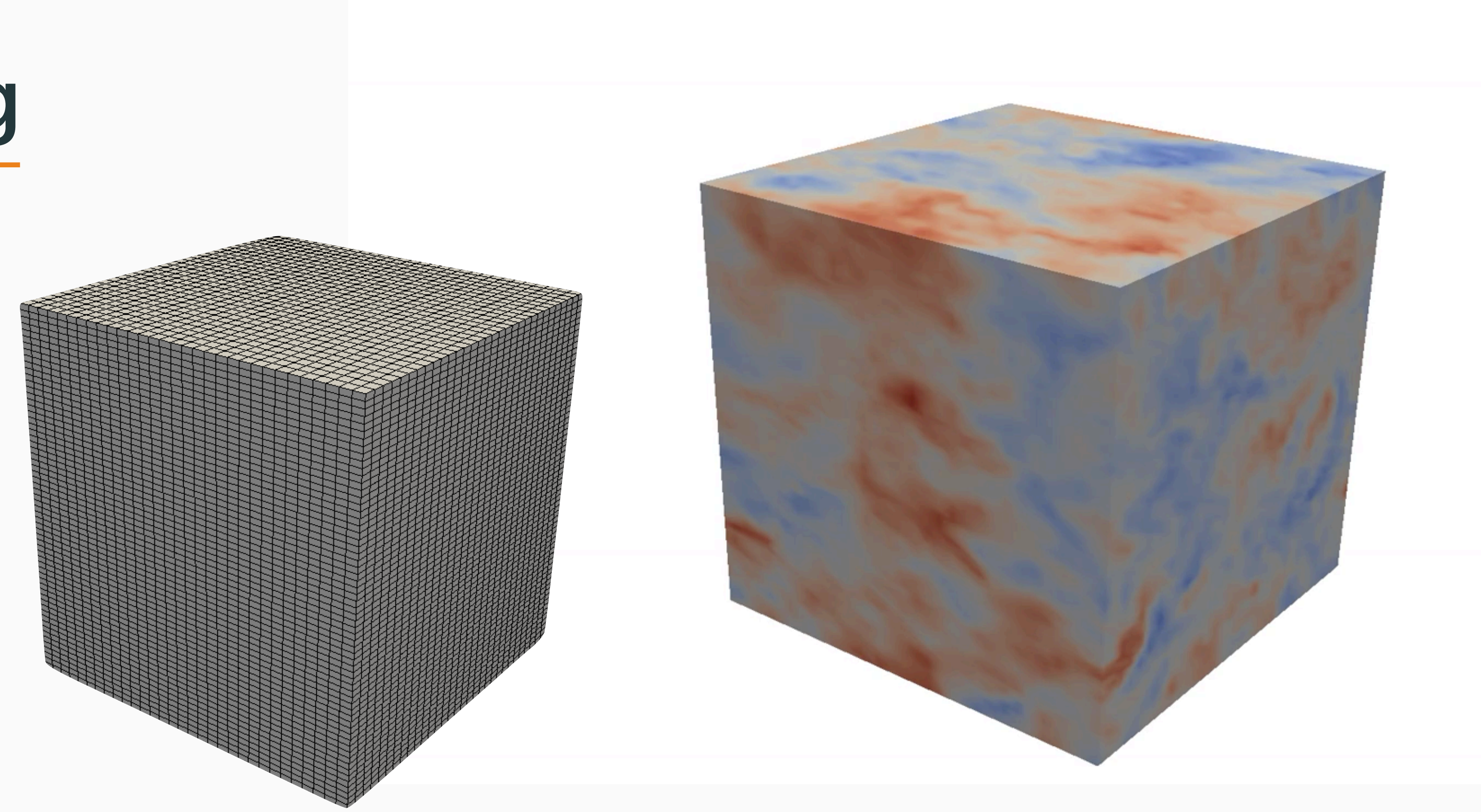
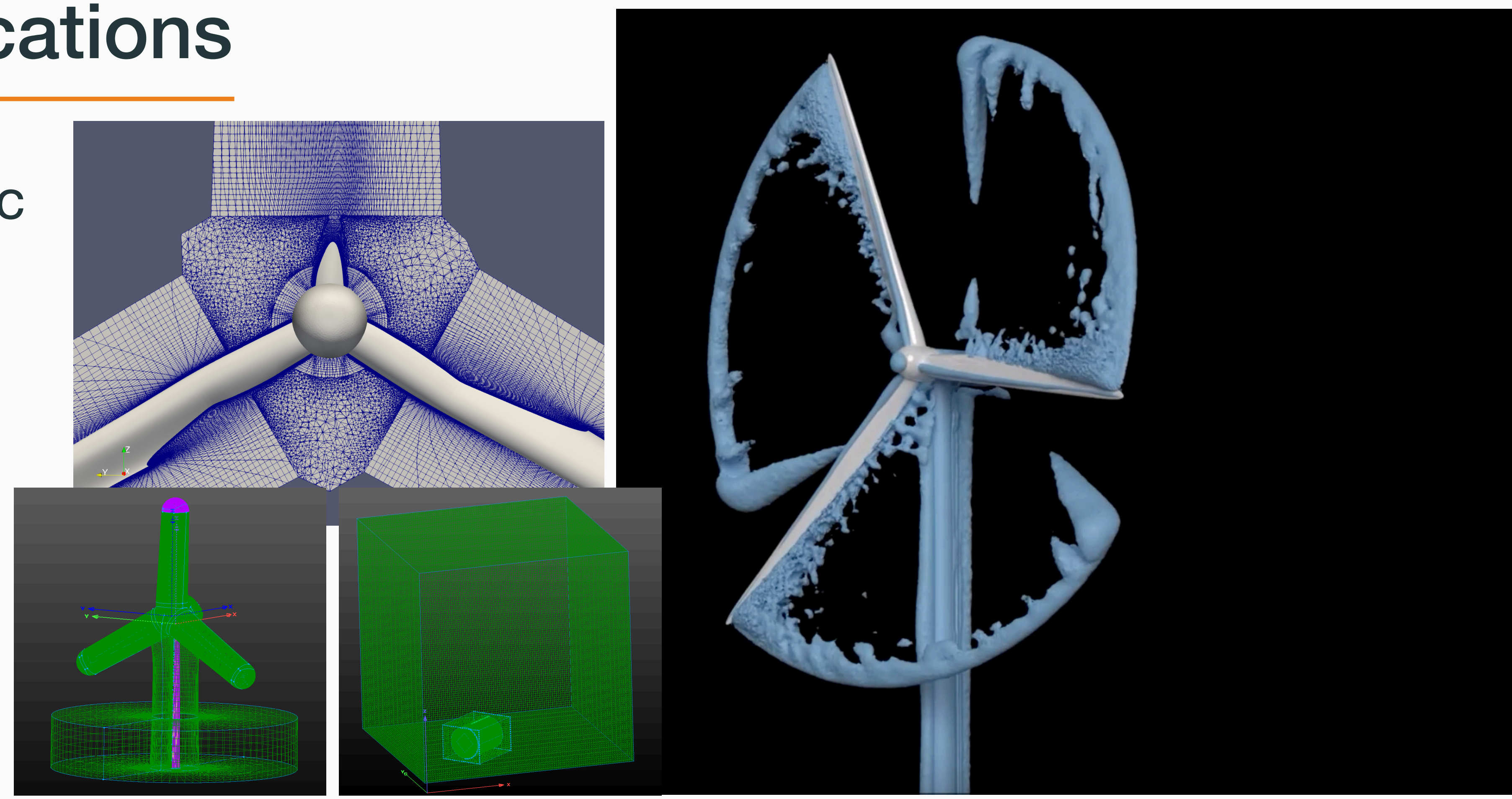
Traditional LES Modeling

- Homogeneous, Isotropic Filtering / Resolution
- Isotropic Unresolved Turbulence
- Spectral representation by the underlying numerics



Practical LES Applications

- Inhomogeneous, Anisotropic Filtering / Resolution
- Anisotropic Unresolved Turbulence
- Lower order numerical representation



1D Advection Example

$$\overline{\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}} = 0$$

Commutation Error[1]:

$$\mathscr{C}(u) = \left(\overline{\frac{du}{dx}} - \frac{d\bar{u}}{dx}\right)$$

$$\approx -C\Delta\frac{d\Delta}{dx}\frac{d^2\bar{u}}{dx^2} + \text{Higher Order Terms}$$

Without Commutation Terms

$$\frac{\partial \bar{u}}{\partial t} + a\frac{\partial \bar{u}}{\partial x} = 0$$

[1] S. Ghosal and P. Moin "The basic equations for the large eddy simulation of turbulent flows in complex geometry," *Journal of Computational Physics*, vol. 118, no. 1, pp. 24-37, 1995

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Without Commutation Terms

$$\frac{\partial \bar{u}}{\partial t} + a\frac{\partial \bar{u}}{\partial x} = 0$$

Analysis was incomplete due to a dimensionally inconsistent assumption ($\Delta \ll \kappa\Delta$) that leads to the absence of several terms of the commutation error.

Complete Commutation Error

$$\mathscr{C}(u) = \left(\overline{\frac{du}{dx}} - \frac{d\bar{u}}{dx}\right) = \sum_{n=1}^{N/2}\sum_{m=1}^{2n} C_{mn}\Delta^{2n-m}\frac{d^{2n-m+1}\bar{u}}{dx^{2n-m+1}}\frac{d^m\Delta^m}{dx^m} + \mathcal{O}\left(\delta^{N+2}\right)$$

$$N = 2,4,6,\ldots$$

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1D Advection Example

$$\overline{\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x}} = 0$$

With Commutation Terms

$$\frac{\partial \bar{u}}{\partial t} + a^*(x)\frac{\partial \bar{u}}{\partial x} = \nu(x)\frac{\partial^2 \bar{u}}{\partial x^2}$$

$$a^*(x) = a\left[1 - \frac{C}{2}\left(\left(\frac{d\Delta}{dx}\right)^2 + \Delta\frac{d^2\Delta}{dx}\right)\right]$$

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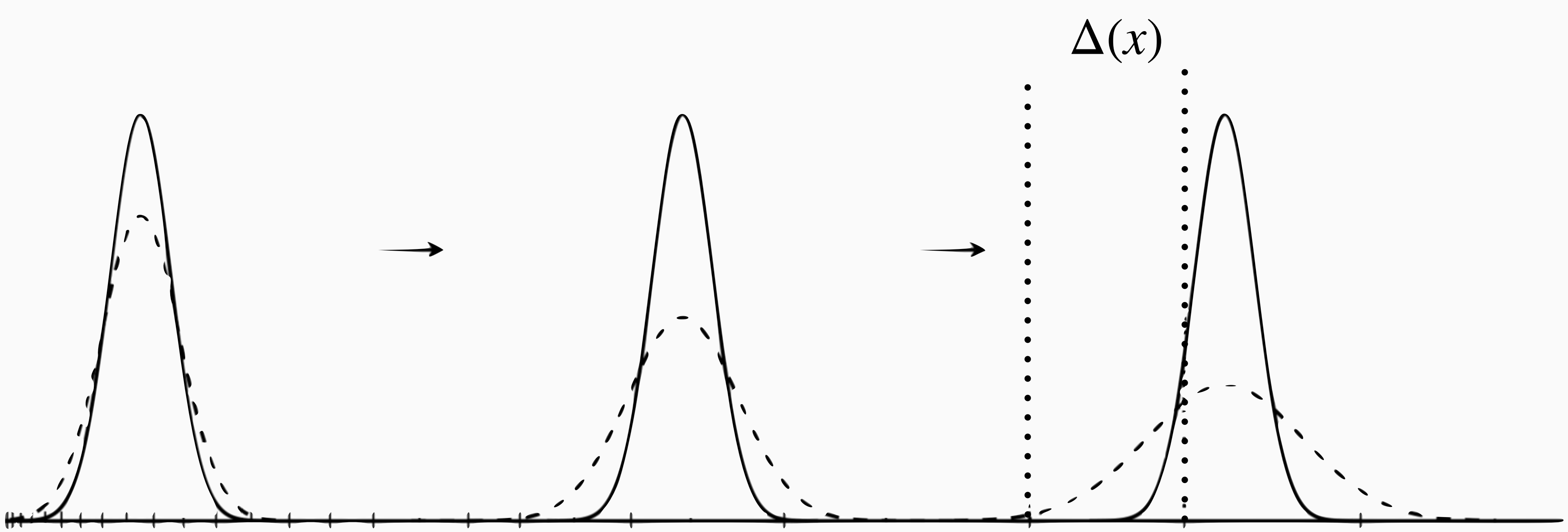

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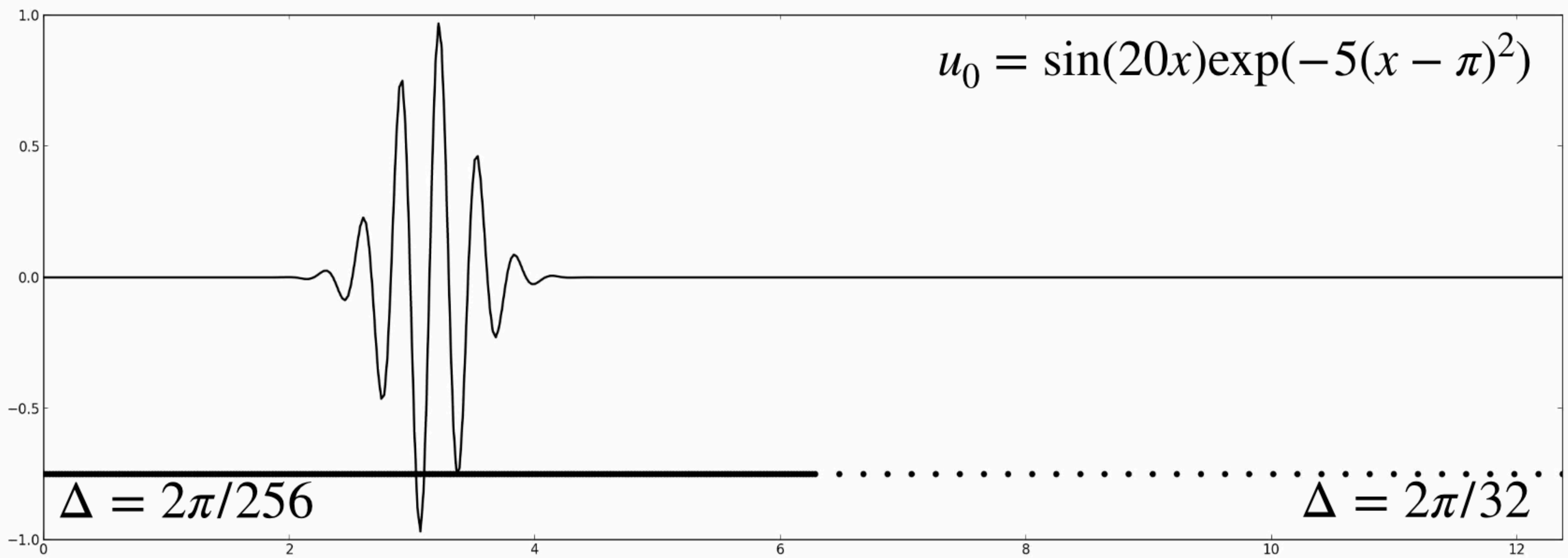

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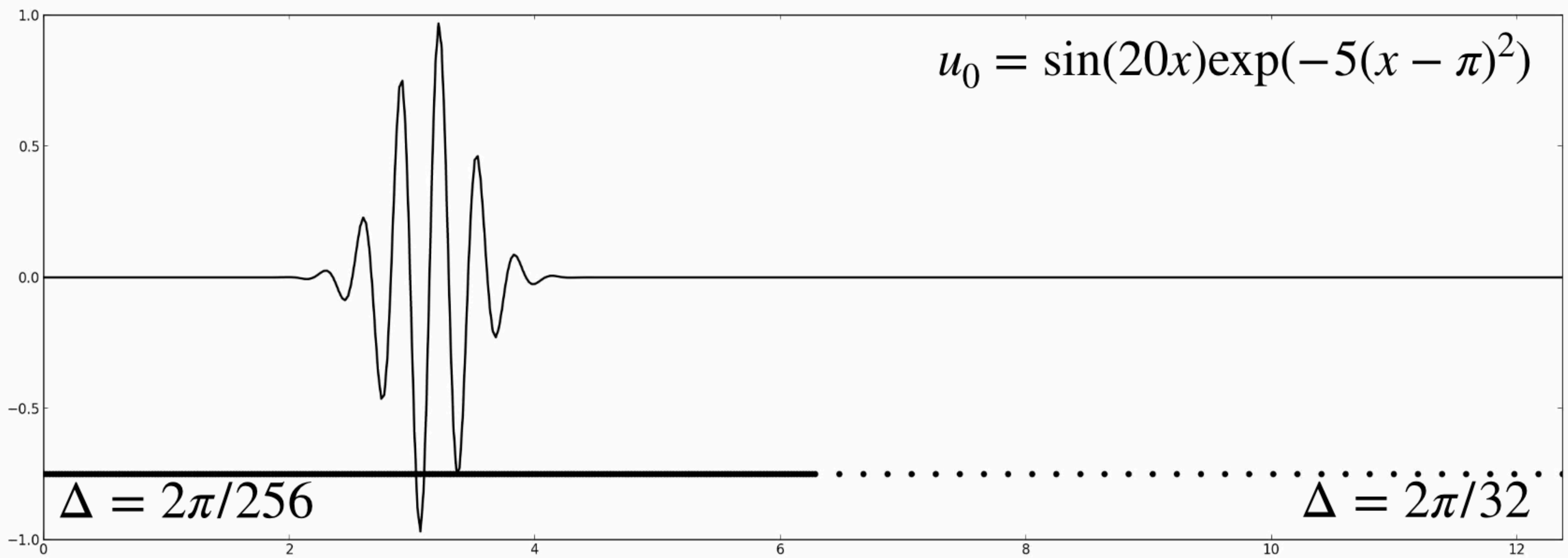
Without Commutation Terms

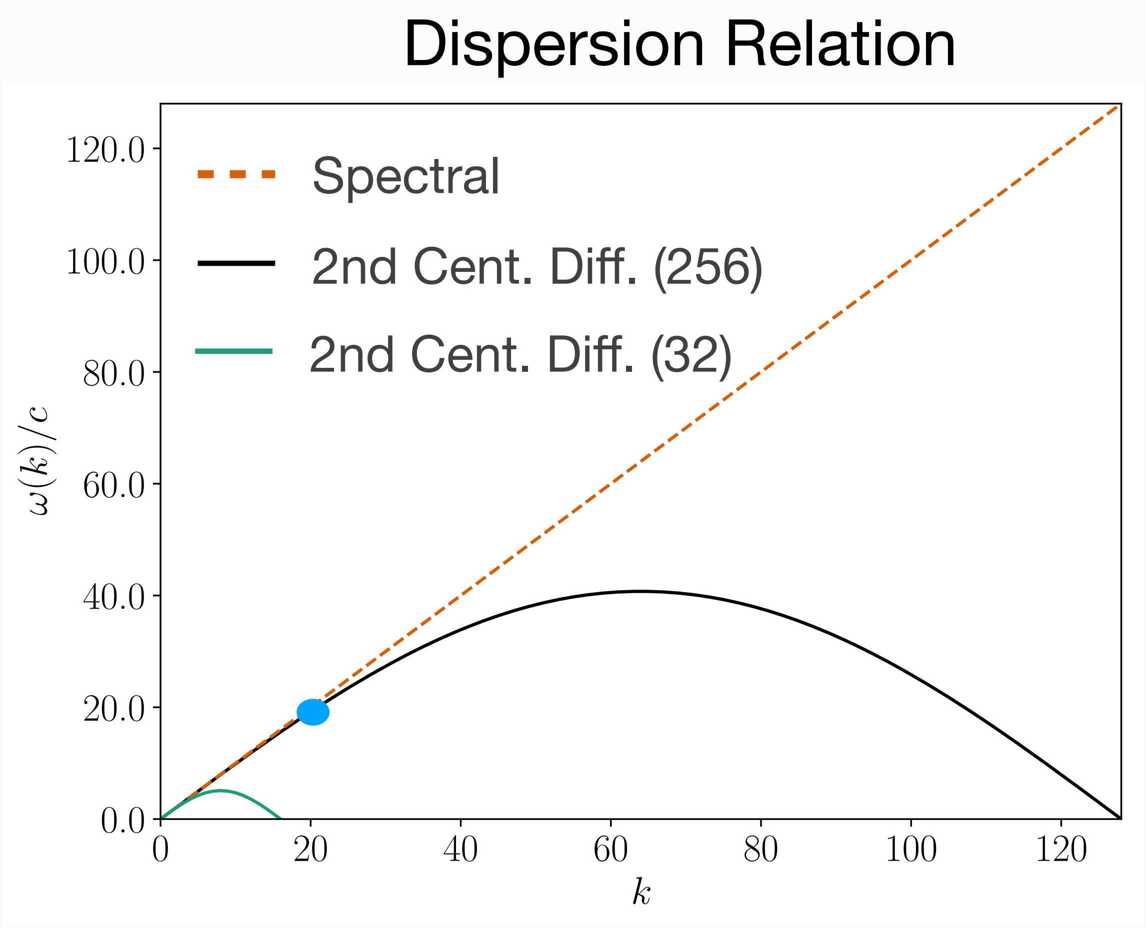
$$\frac{\partial \bar{u}}{\partial t} + a\frac{\partial \bar{u}}{\partial x} = 0$$

$$\downarrow$$

$$\frac{\partial u_j}{\partial t} + aB_1(u_j) = 0$$



1D Advection Example



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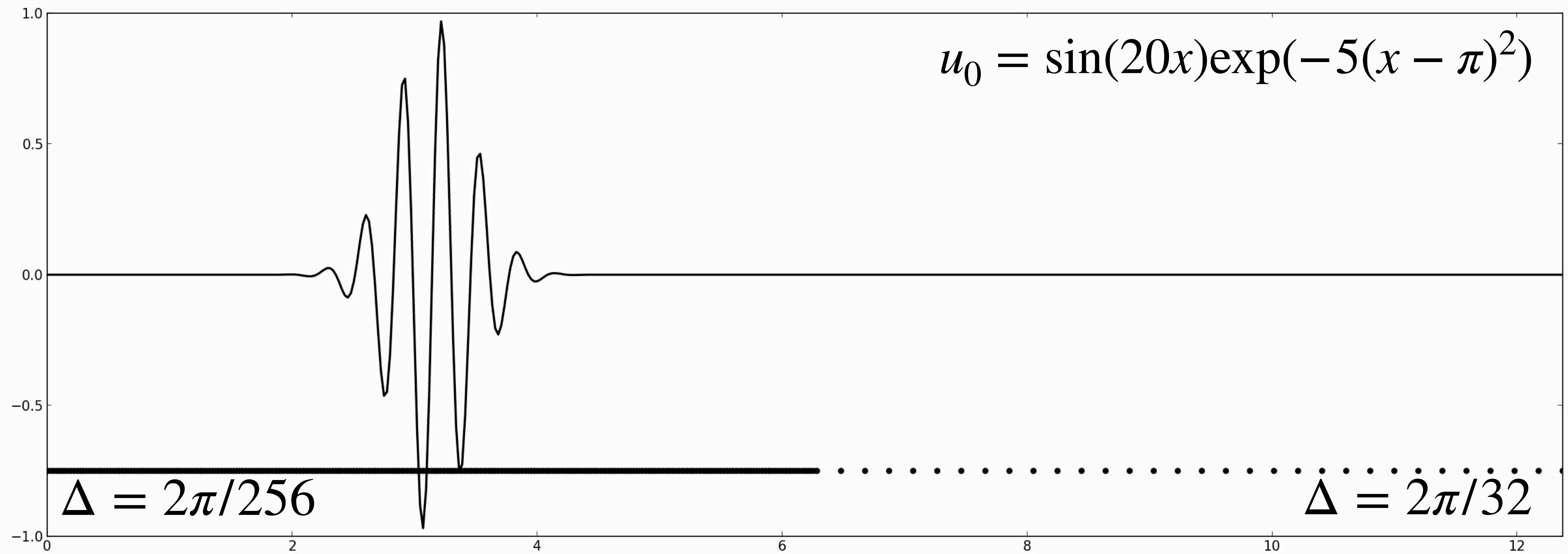
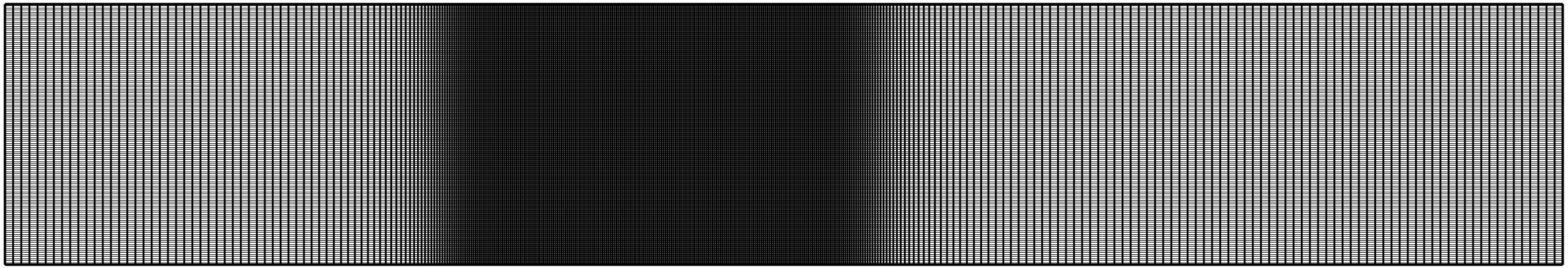
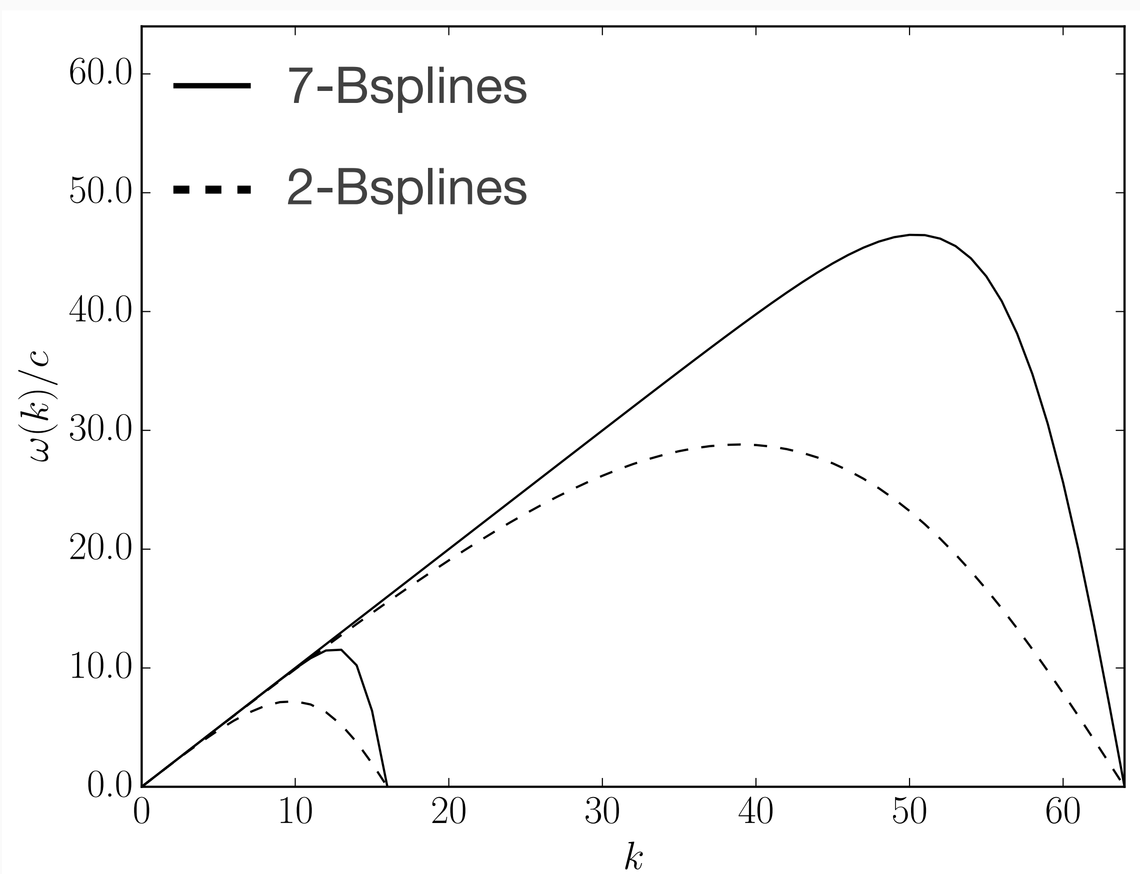


Impact on LES



$$\Delta_f = \frac{2\pi}{128} \qquad \Delta_c = \frac{2\pi}{32}$$



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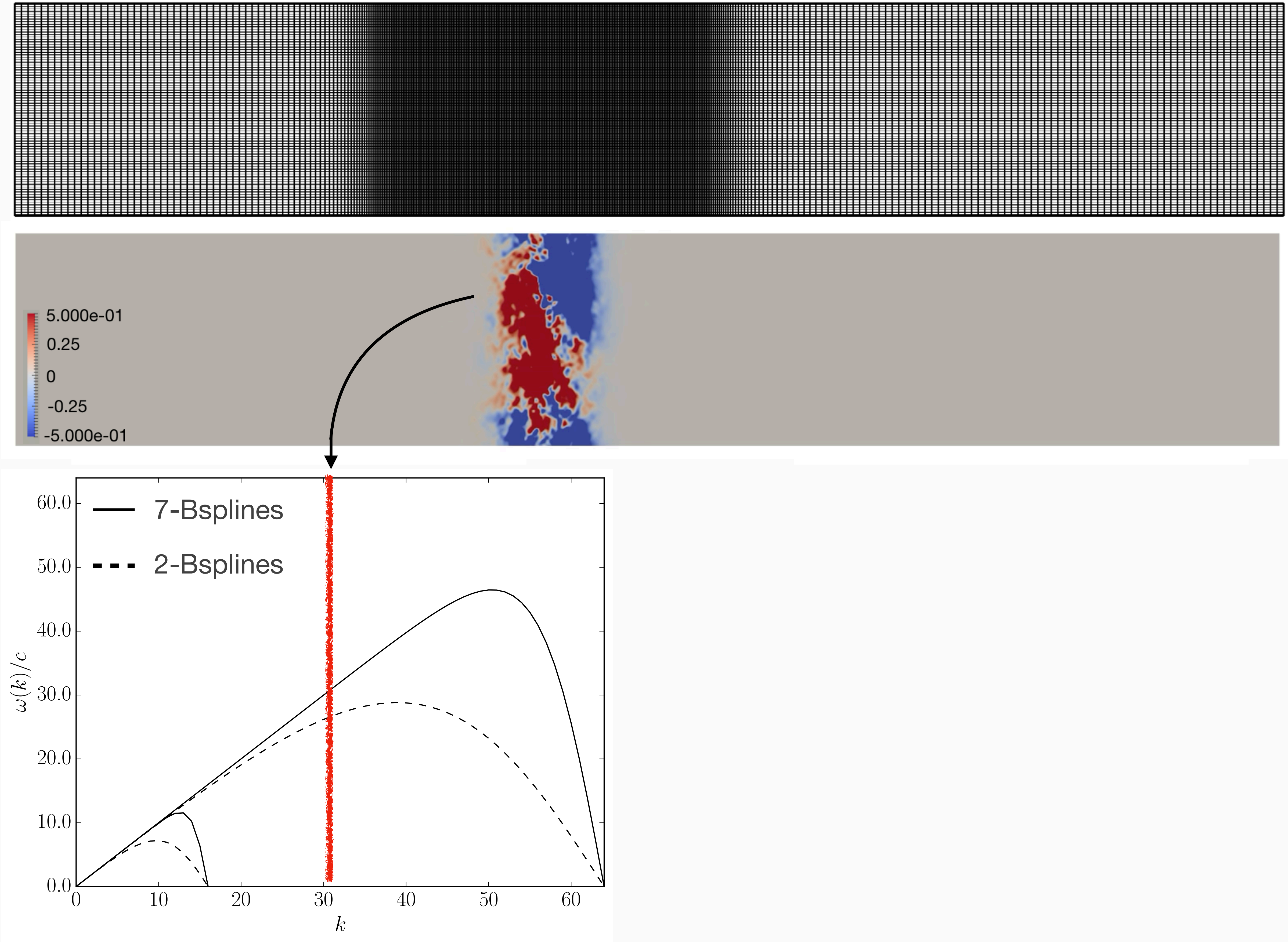

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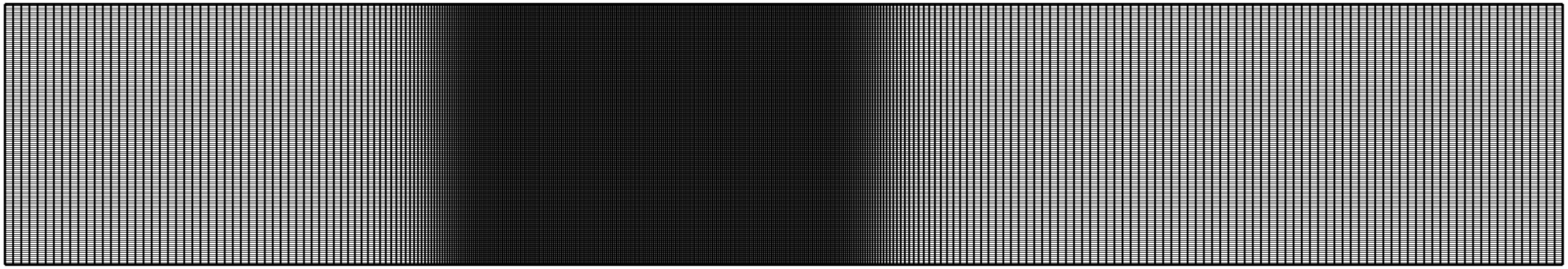

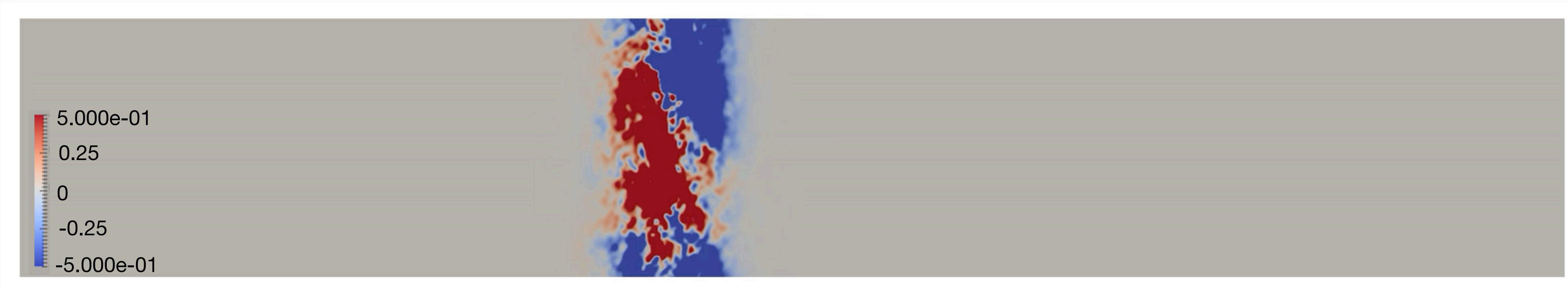

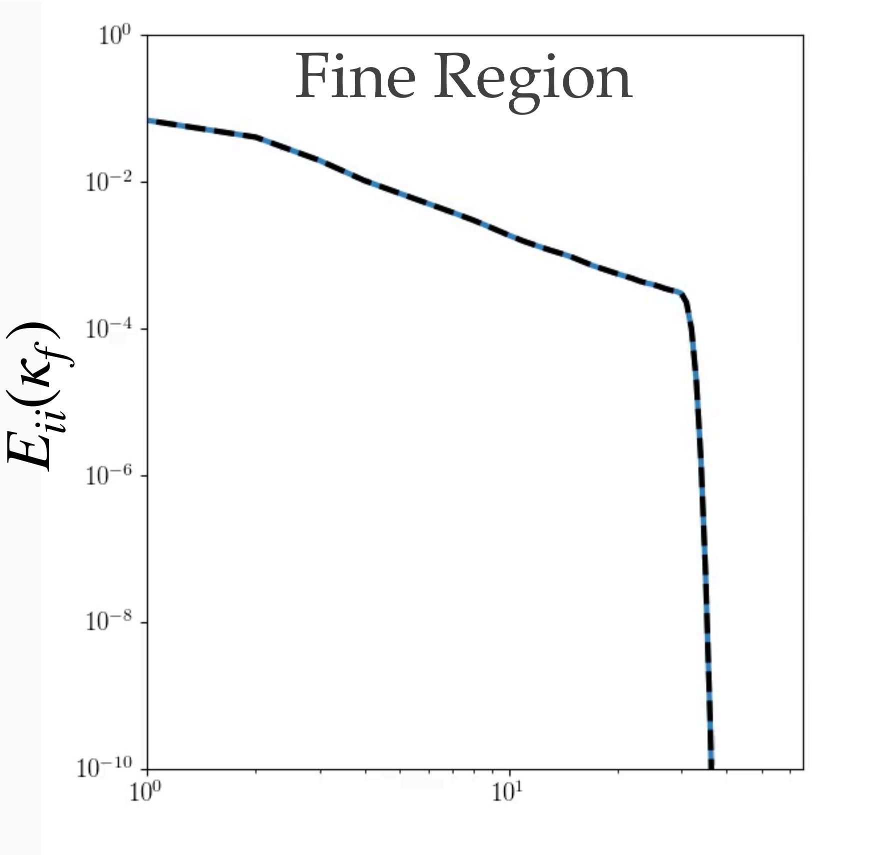

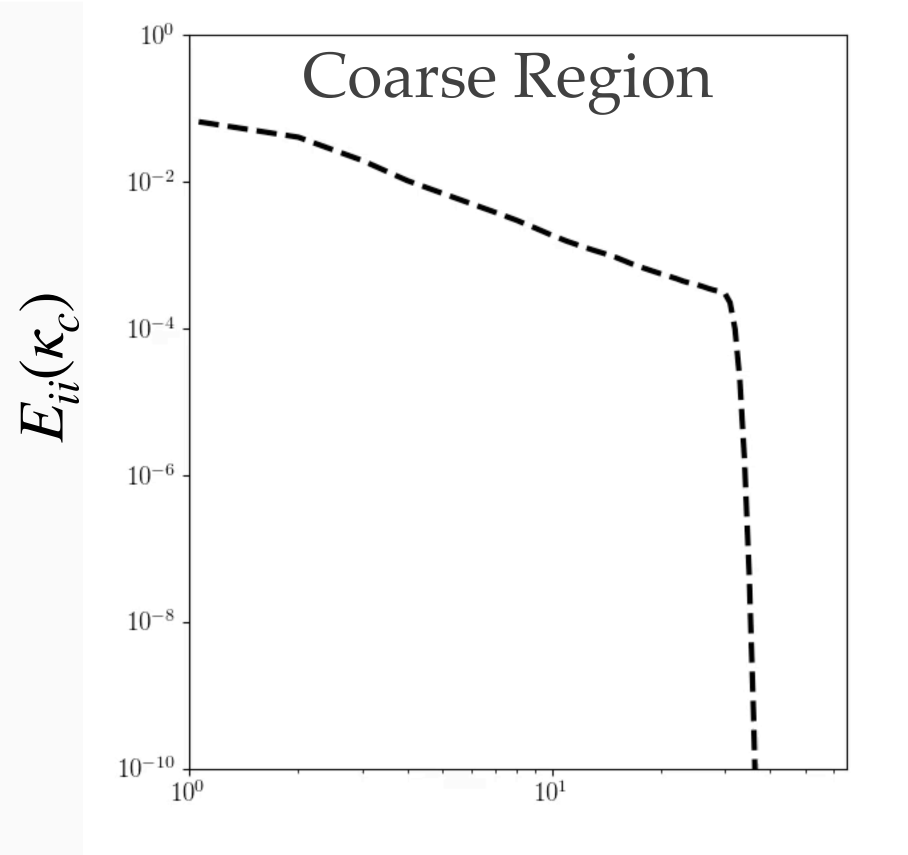

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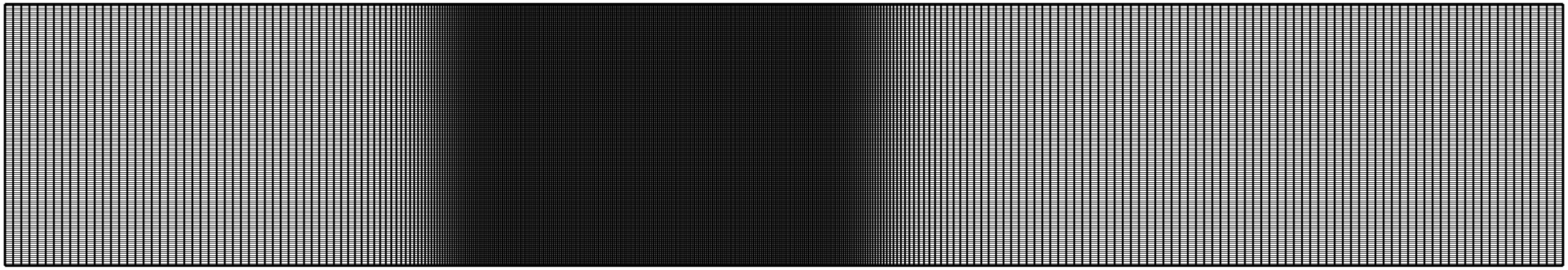
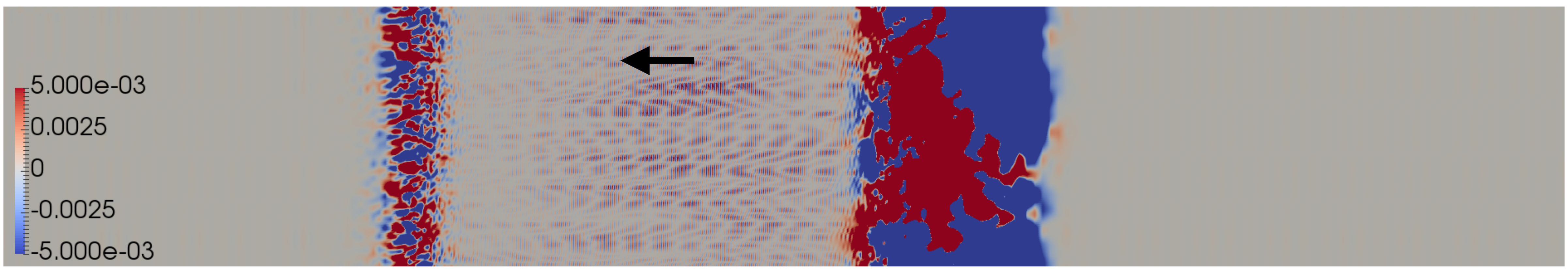
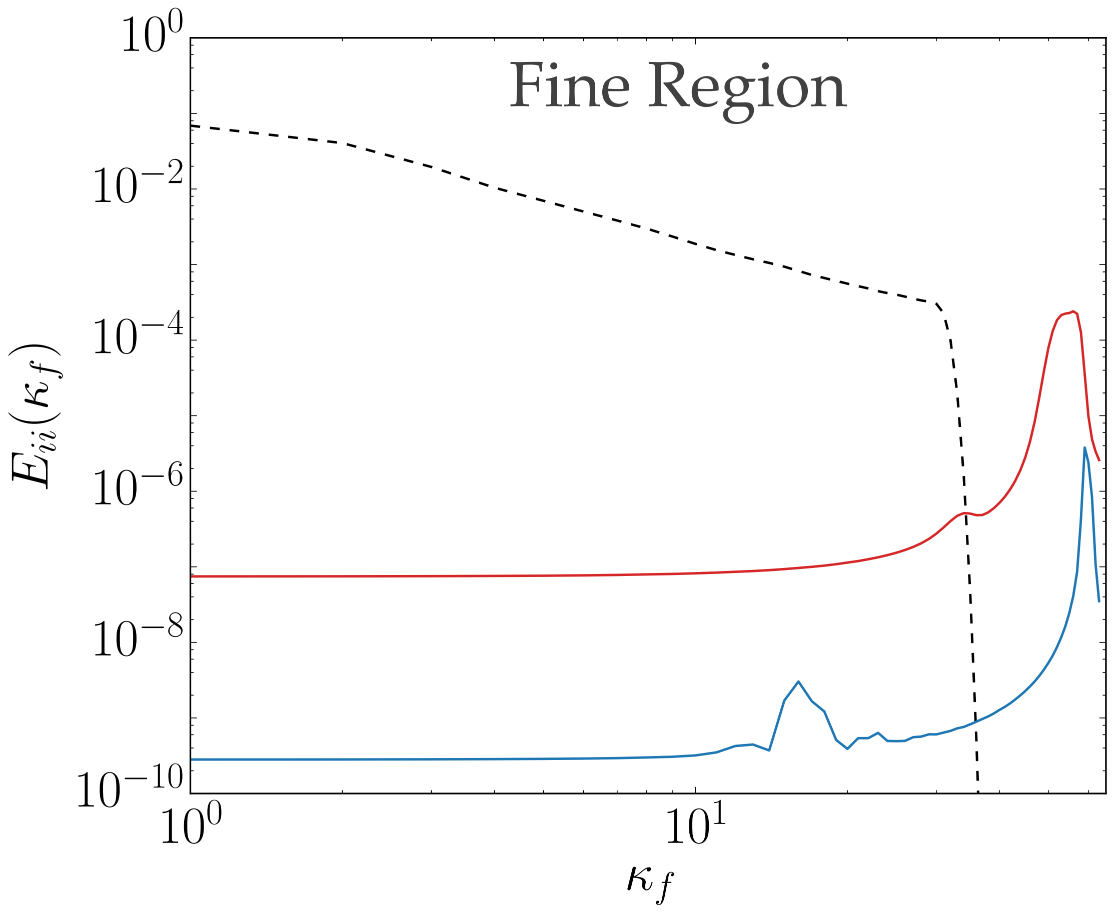






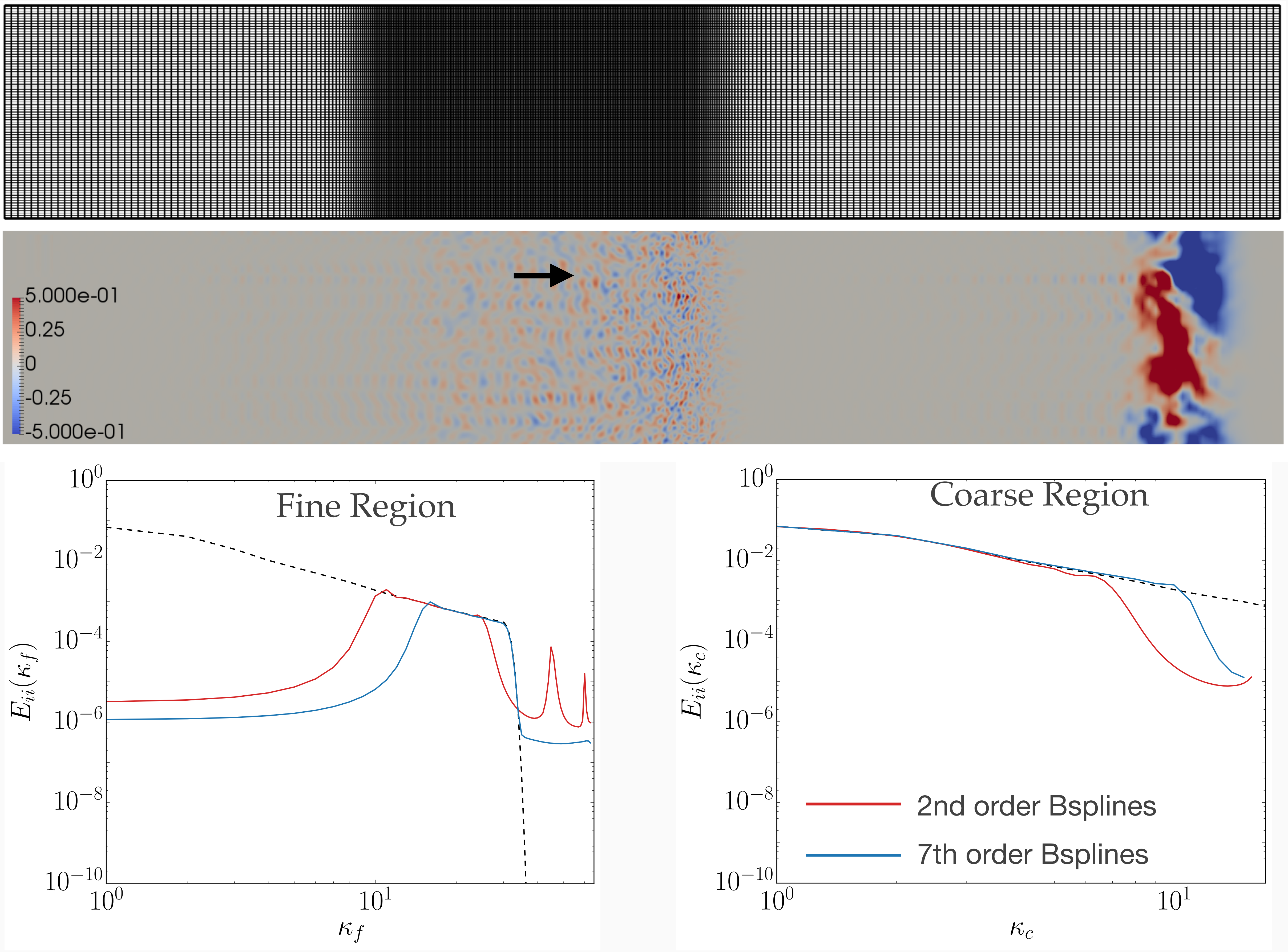
2nd order Bsplines

7th order Bsplines

Initial Reflections:

Unresolvable scales reflected at high wavenumber into the fine region

Impact on LES



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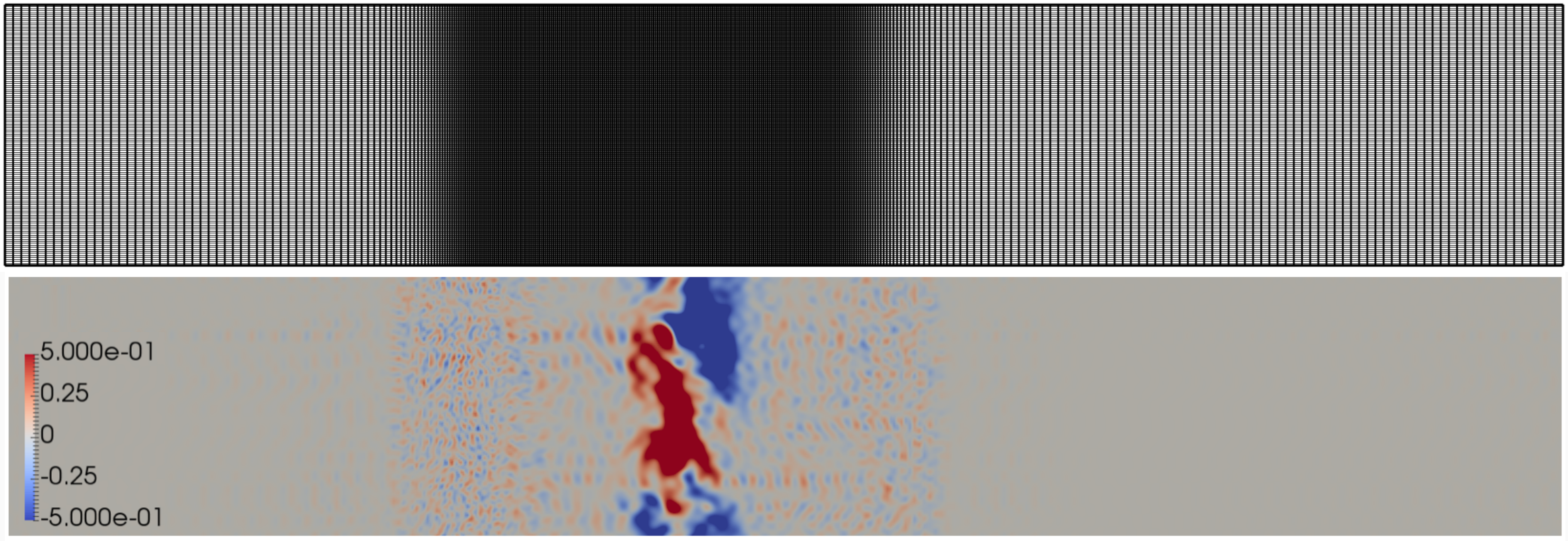

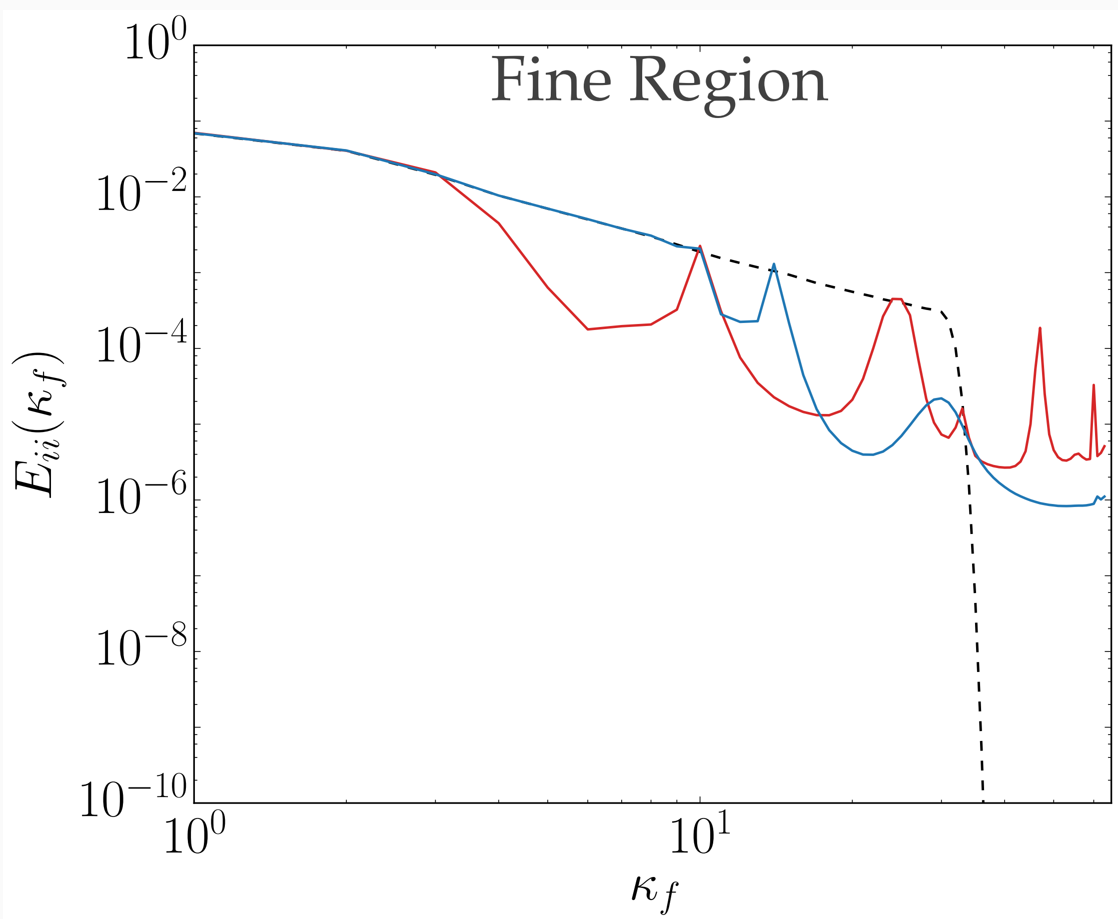

2nd order Bsplines

7th order Bsplines

One flow through:

Incoming turbulence interacts with spurious scales.

Model Formulation

For large κ, the most significant commutation term is:

$$\mathscr{C}(\psi) \approx C \frac{d\Delta}{dx}\left(\Delta^{N-1}\frac{d^N\bar{\psi}}{dx^N}\right) = C\Delta\frac{d\Delta}{dx}\left(\Delta^{N-2}F_N(\bar{\psi})\right)$$

for N (even) as large as possible, and $F_N(\bar{\psi}) = \dfrac{d^N\bar{\psi}}{dx^N}$.

- N limited by the underlying numerics based on number of available derivatives of filtered field.

- Analytical constant needs adjusting to compensate for omitted commutation terms and numerical behavior.

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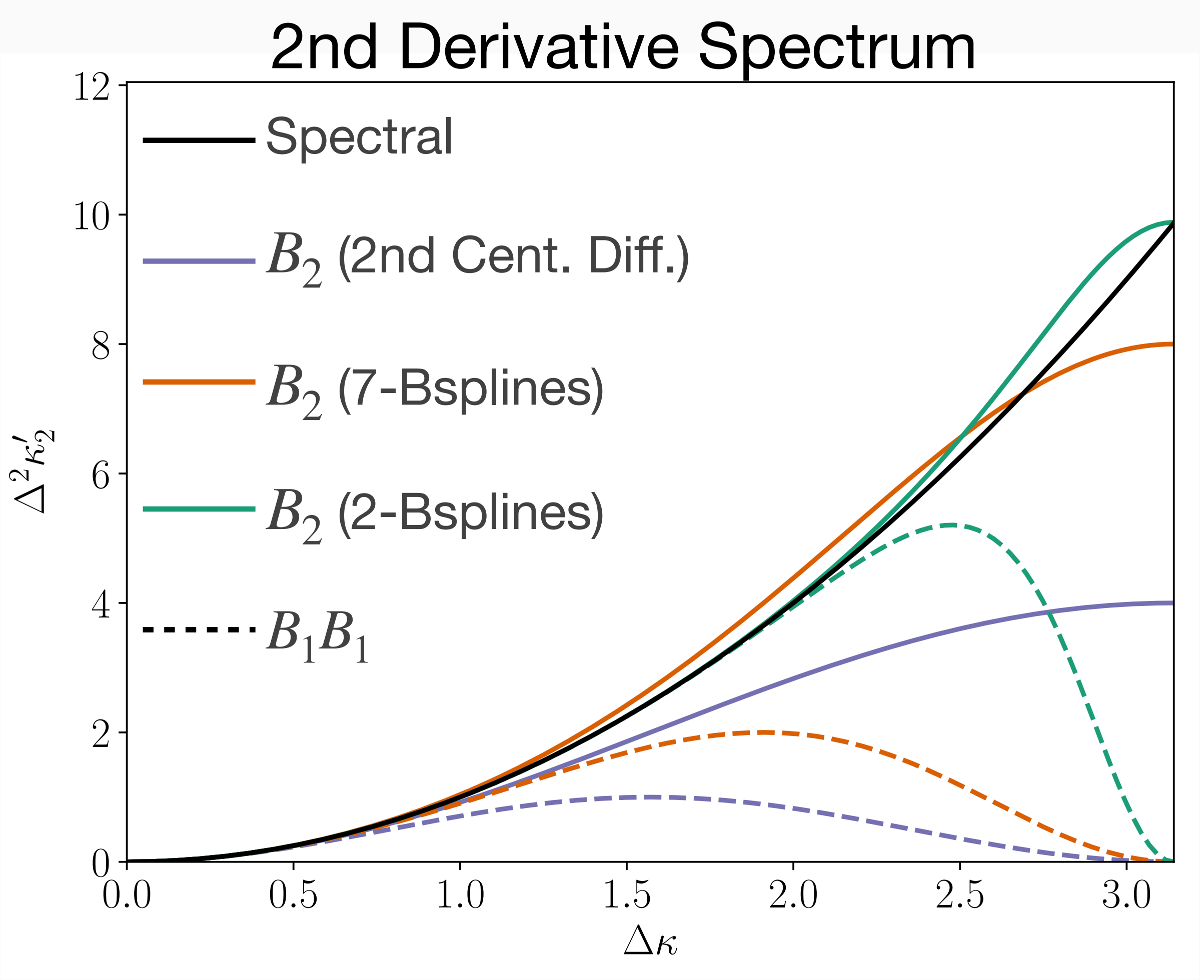
We can create higher order filters from lower order differential operators. Consider the $B_2 - B_1B_1$ operator:



Taylor Expansion:

<u>2nd Centered Difference</u>

$$(B_2 - B_1B_1)\psi = \Delta x^2 F_4(\psi) + \mathcal{O}(\Delta x^6)$$

<u>7-Bsplines</u>

$$(B_2^7 - B_1^7B_1^7)\psi \sim \Delta x^8 F_{10}(\psi) + \mathcal{O}(\Delta x^{12})$$

Model Formulation

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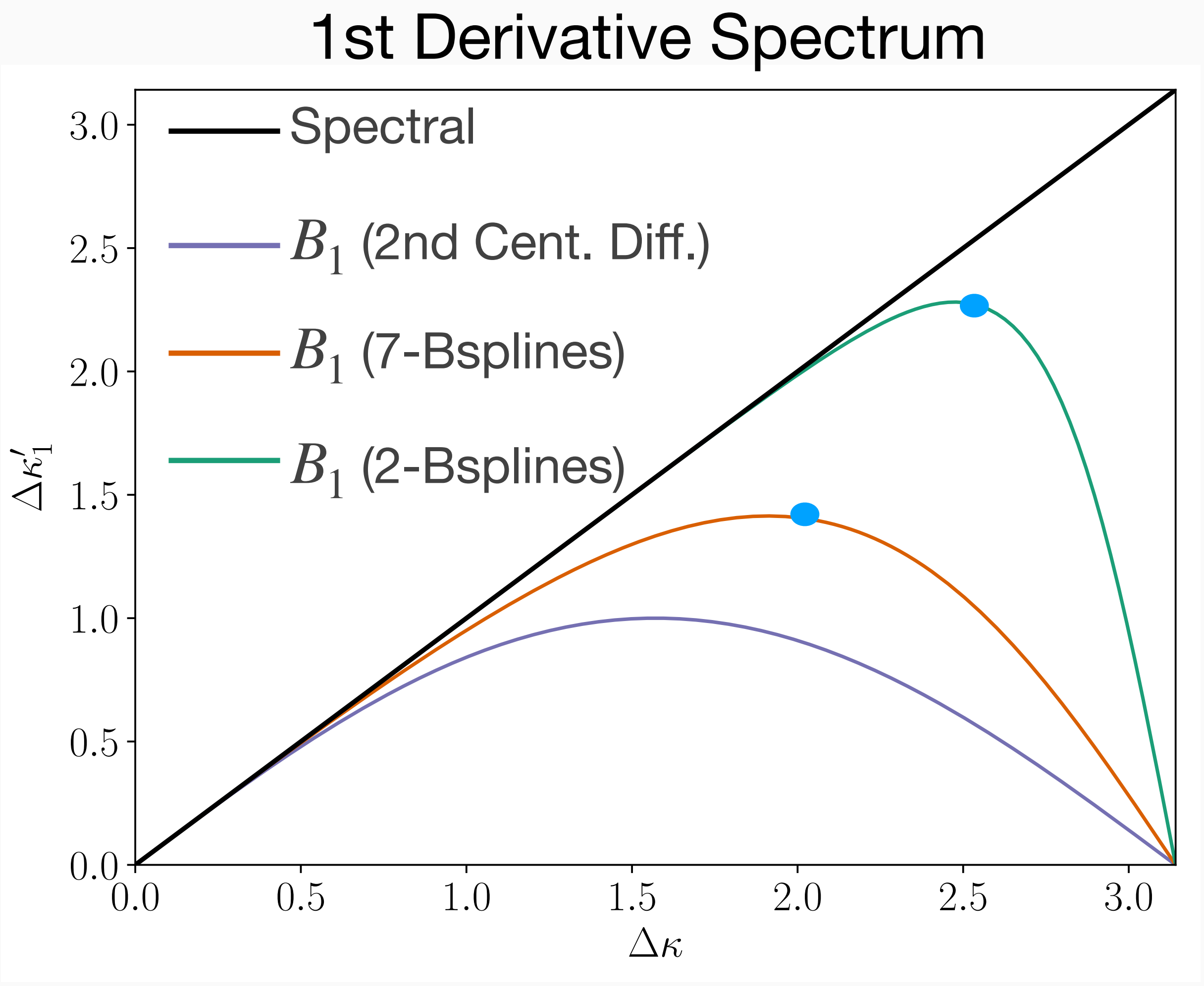
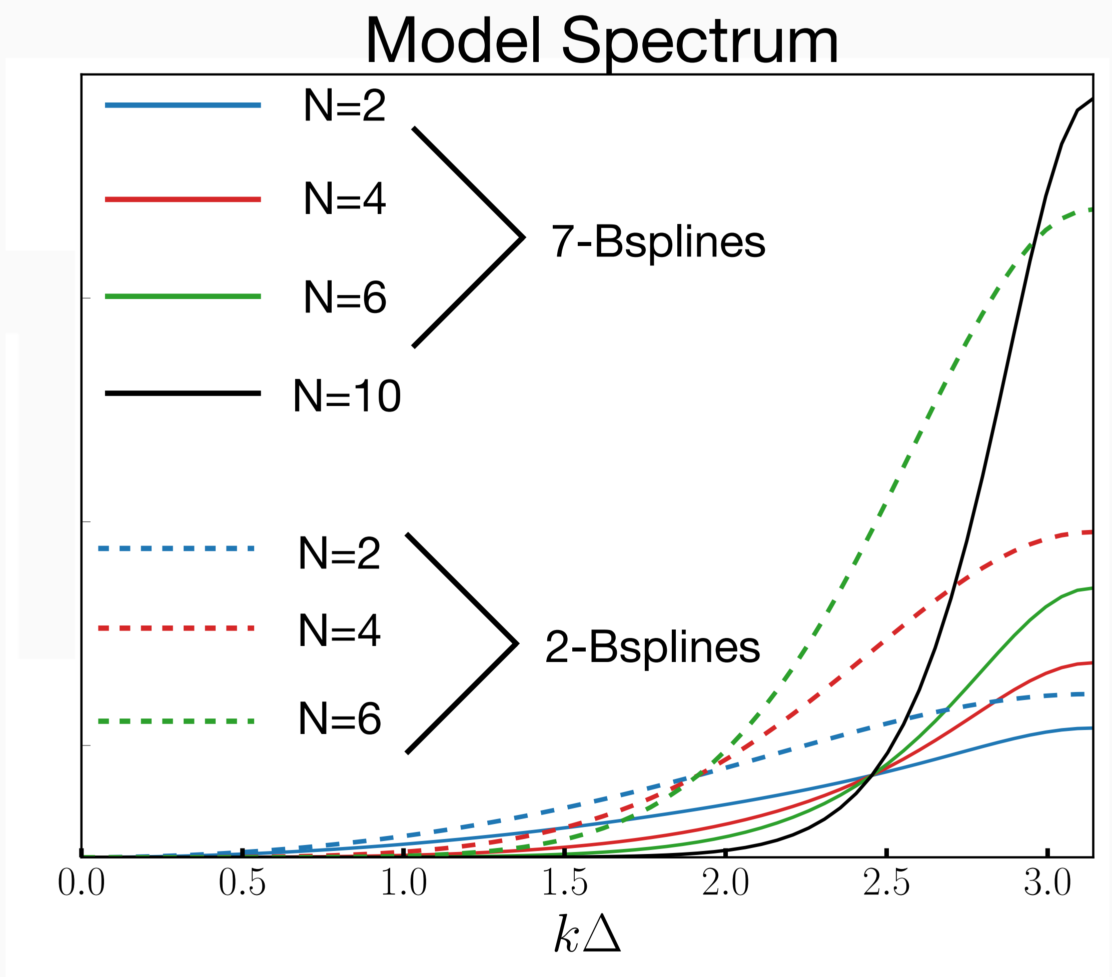
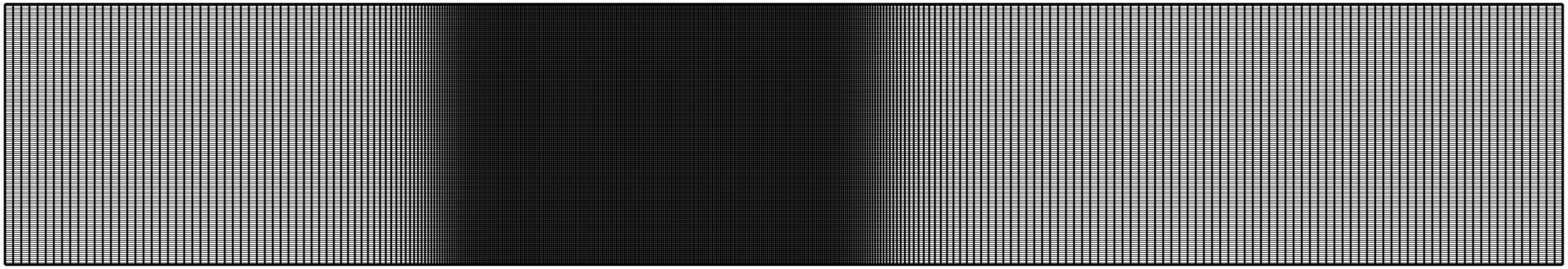
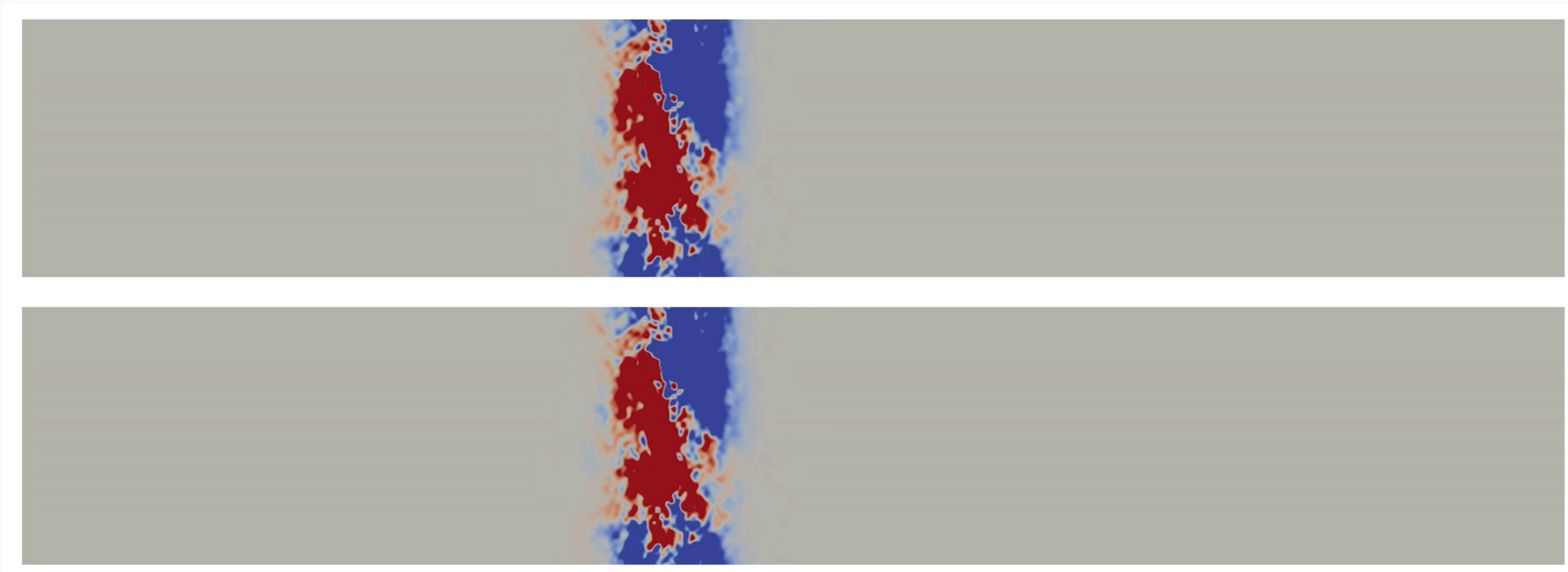
for N as large as possible, and $F_N(\bar{\psi}) = \frac{d^N\bar{\psi}}{dx^N}$.





Model Results

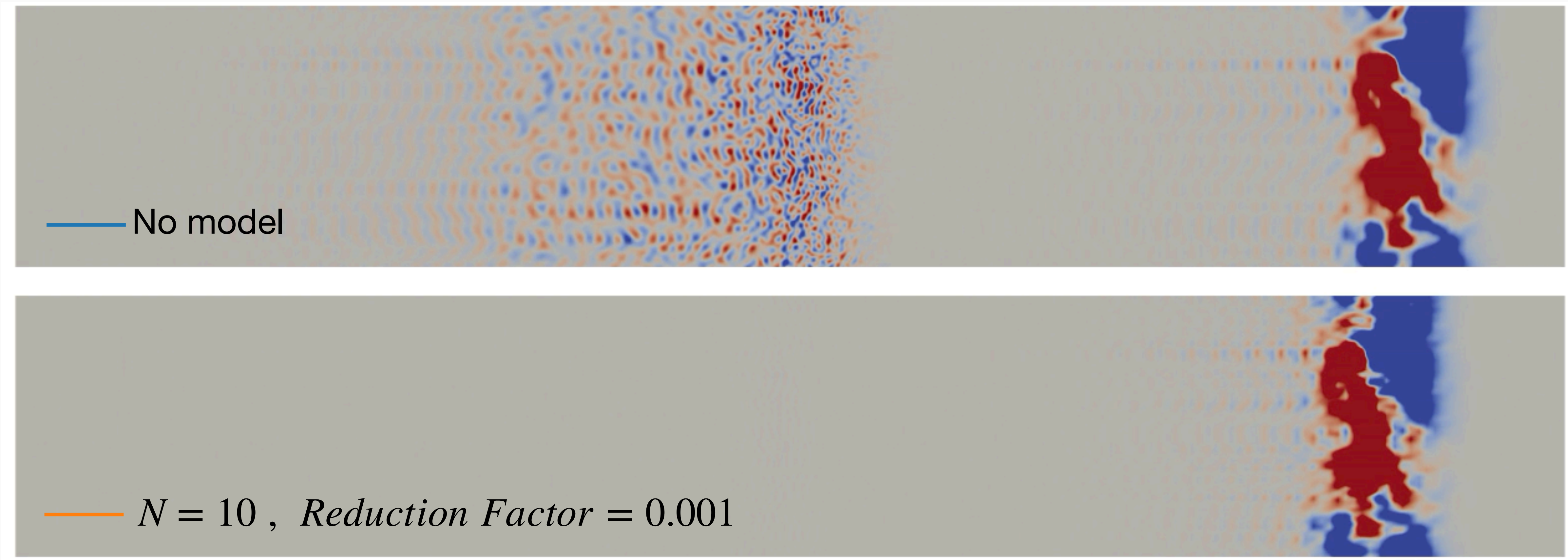


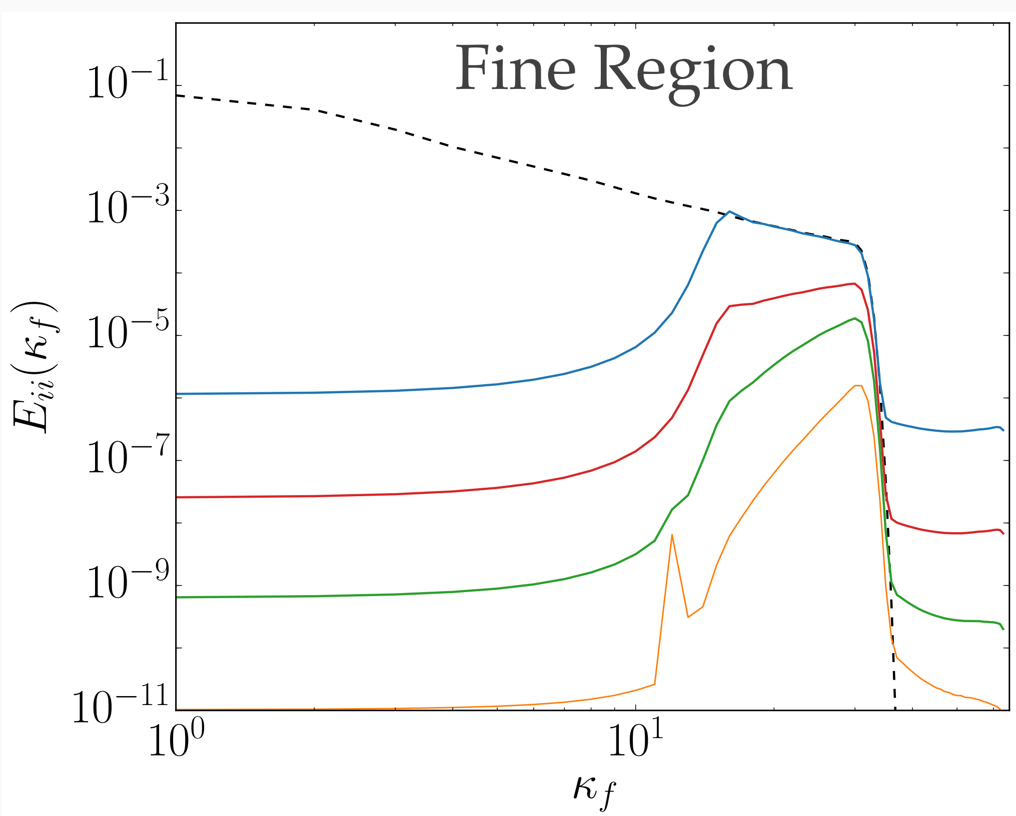


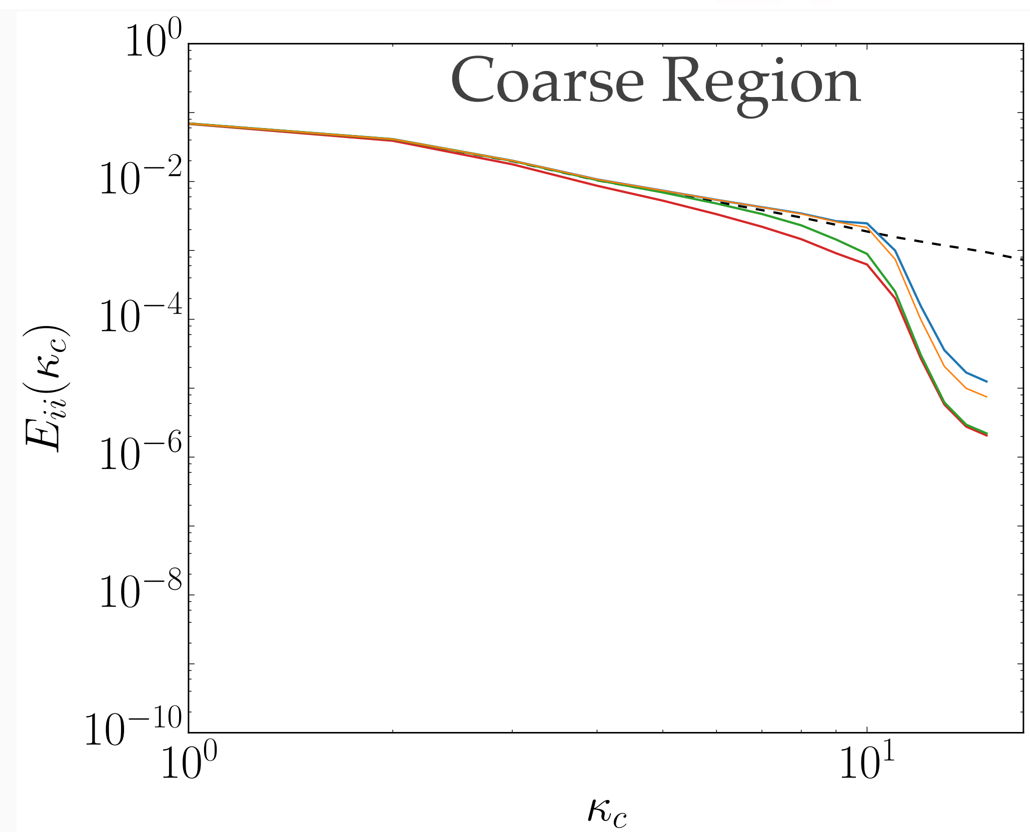
Top: $\frac{\partial \bar{\mathbf{u}}}{\partial t} + a_x \frac{\partial \bar{\mathbf{u}}}{\partial x} = 0$

Bottom: $\frac{\partial \bar{\mathbf{u}}}{\partial t} + a_x \frac{\partial \bar{\mathbf{u}}}{\partial x} = C a_x \frac{d\Delta}{dx} \Delta \left(\Delta^{N-2} \frac{d^N \bar{\mathbf{u}}}{dx^N} \right) \left[(B_2 - B_1^7 B_1^7)(x) \right] \bar{\mathbf{u}}$ $(N = 10)$

Model Results: 7-Bsplines

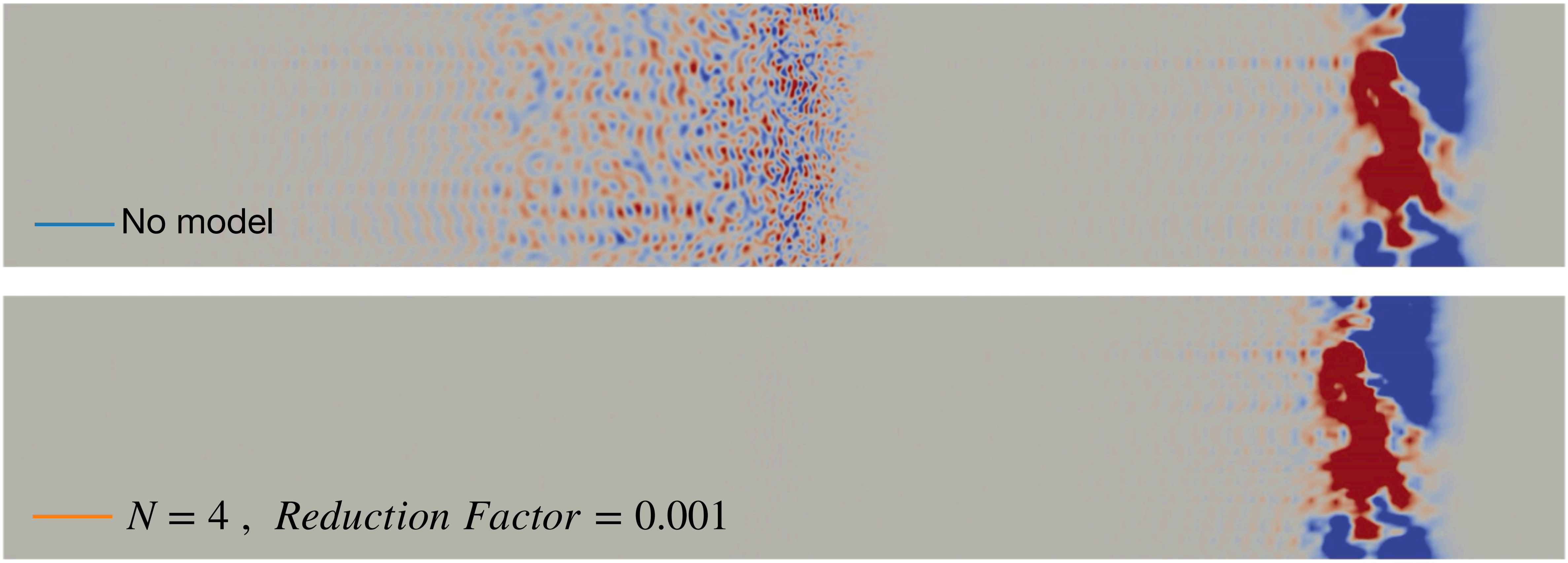


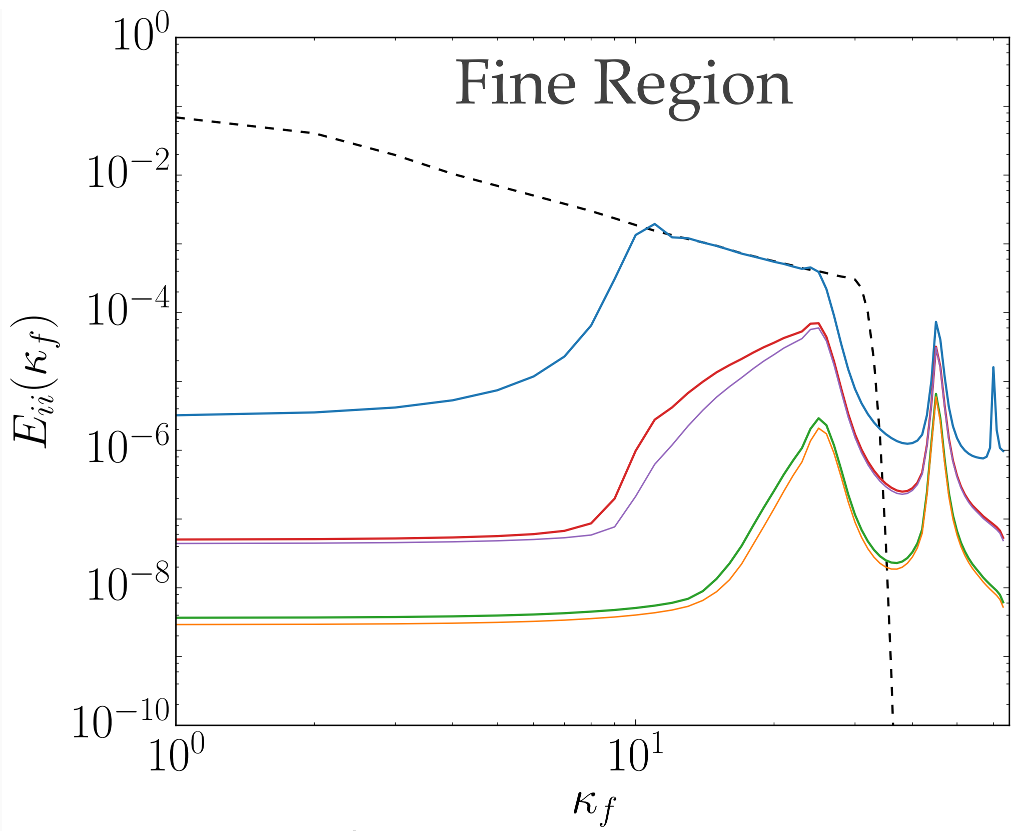


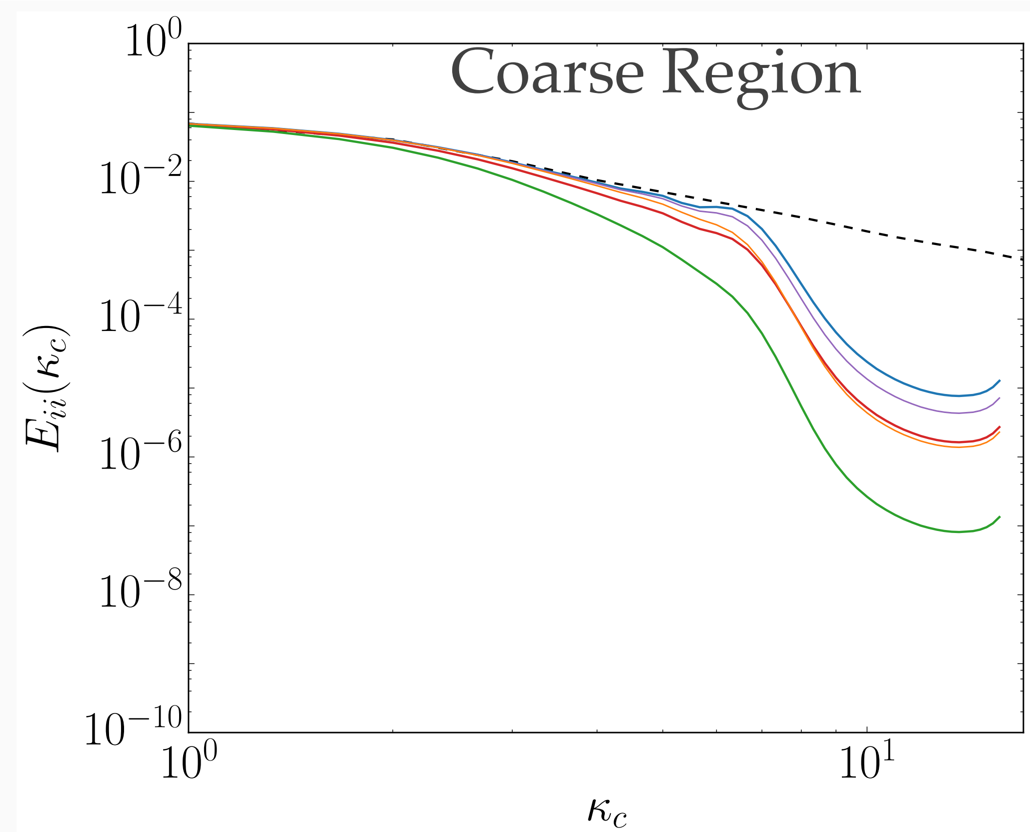


$N = 2\ ,\ \ Reduction\ Factor = 0.1$ $N = 4\ ,\ \ Reduction\ Factor = 0.01$

Model Results: 2-Bsplines







$N = 4$, $N = 2$ } $Reduction\ Factor = 0.1$

$N = 2$, $Reduction\ Factor = 0.001$

Conclusion

- Much more is required of LES models in practical applications than in the settings where they are typically developed.
  - Models need to capture *more than just the dissipation rate*!
- **Probed the issue of resolution inhomogeneity** to uncover what is required of LES models in this setting.
  - Need to account for commutation error and corresponding numerical behavior.
- Developed a model based on the extended commutation error analysis and numerical analysis and demonstrated impact on turbulence statistics.

References

S. Ghosal and P. Moin "The basic equations for the large eddy simulation of turbulent flows in complex geometry," *Journal of Computational Physics,* vol. 118, no. 1, pp. 24-37, 1995

L. N. Trefethen, "Group velocity in finite difference schemes," *SIAM review*, vol. 24, no. 2, pp. 113–136, 1982.

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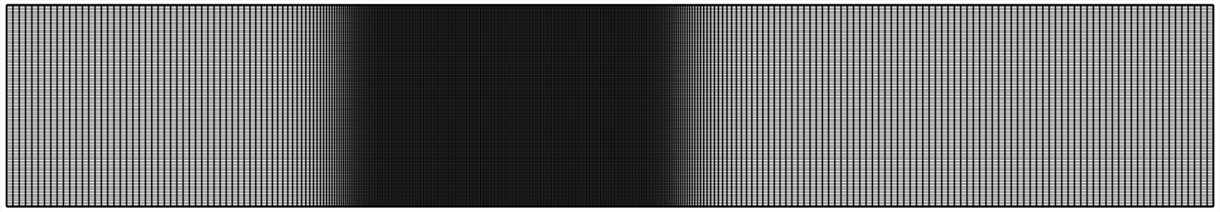
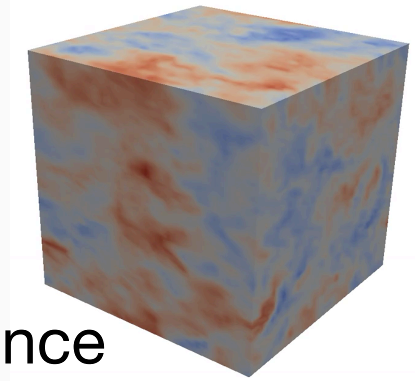
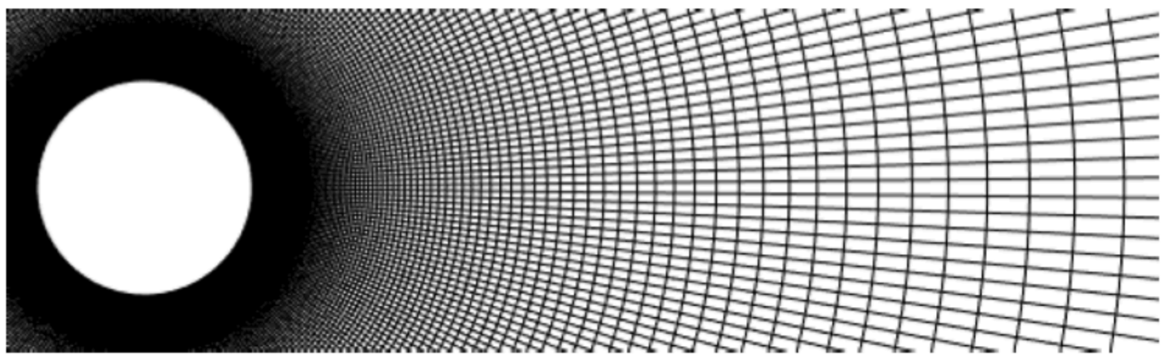
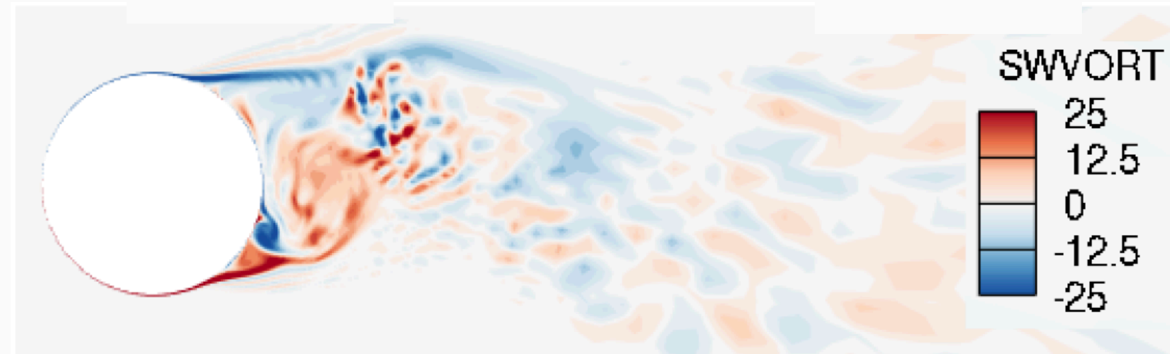


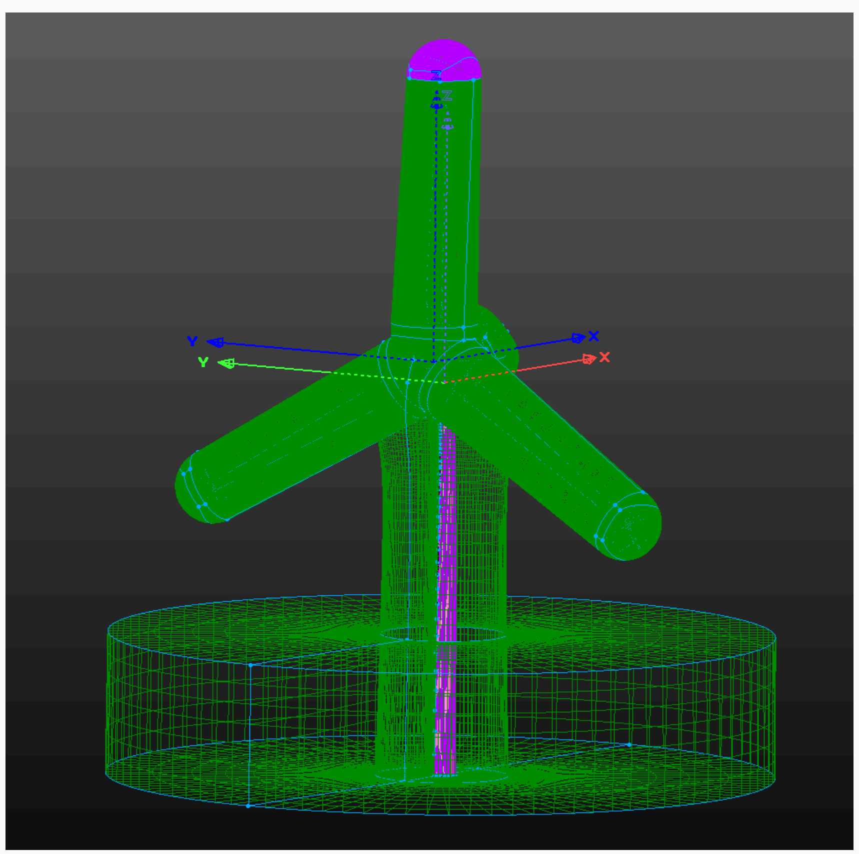


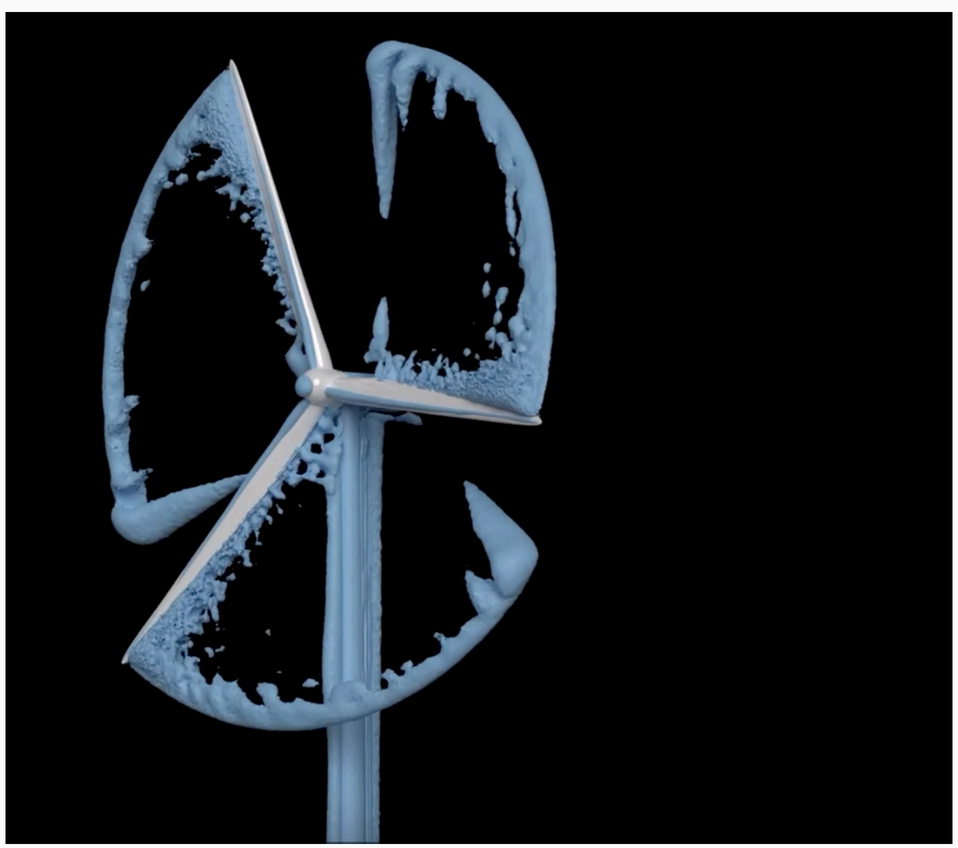
Homogeneous, Isotropic Turbulence





Flow over a cylinder





Flow through a wind turbine

Recommended References

S. Ghosal and P. Moin "The basic equations for the large eddy simulation of turbulent flows in complex geometry," *Journal of Computational Physics,* vol. 118, no. 1, pp. 24-37, 1995

L. N. Trefethen, "Group velocity in finite difference schemes," *SIAM review*, vol. 24, no. 2, pp. 113–136, 1982.

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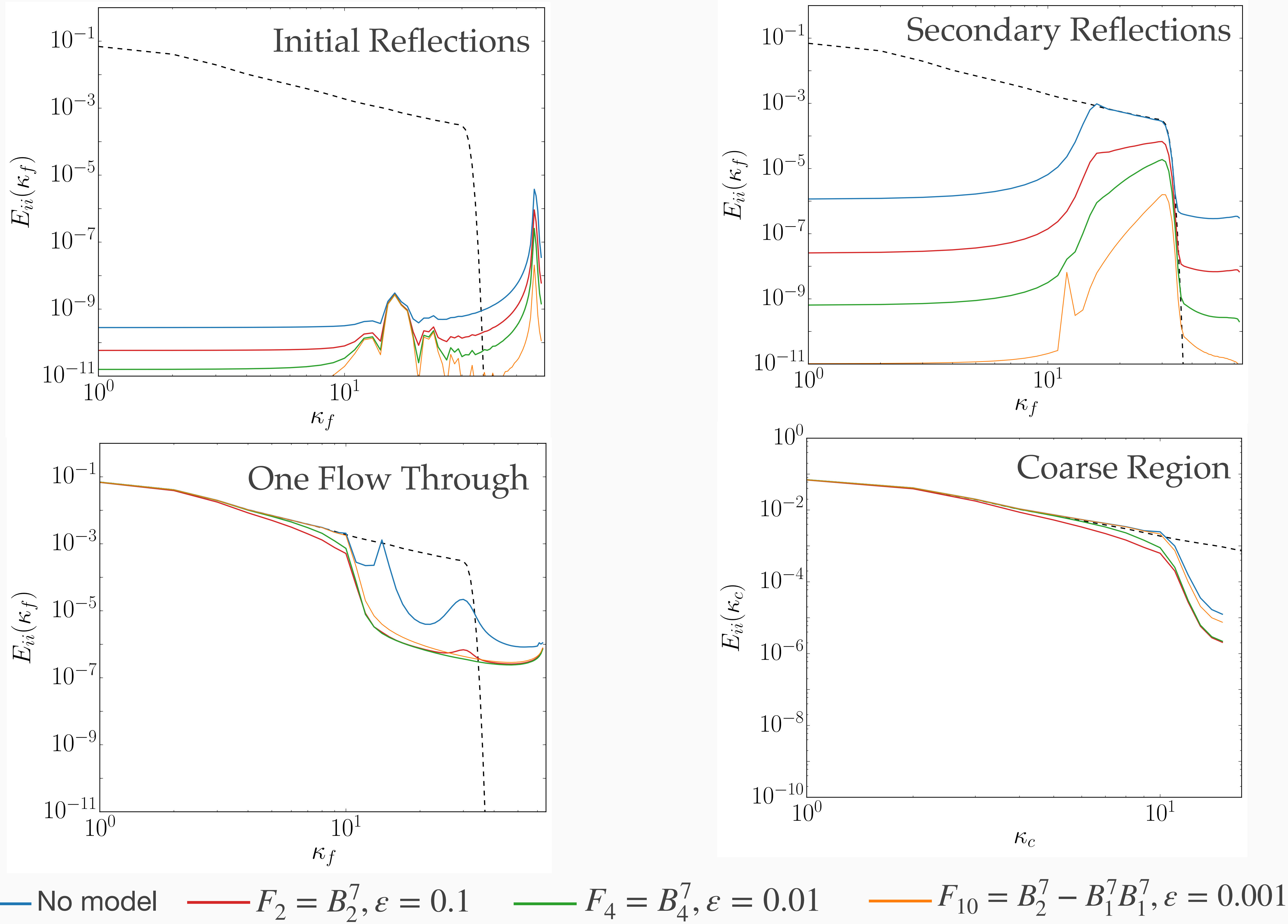
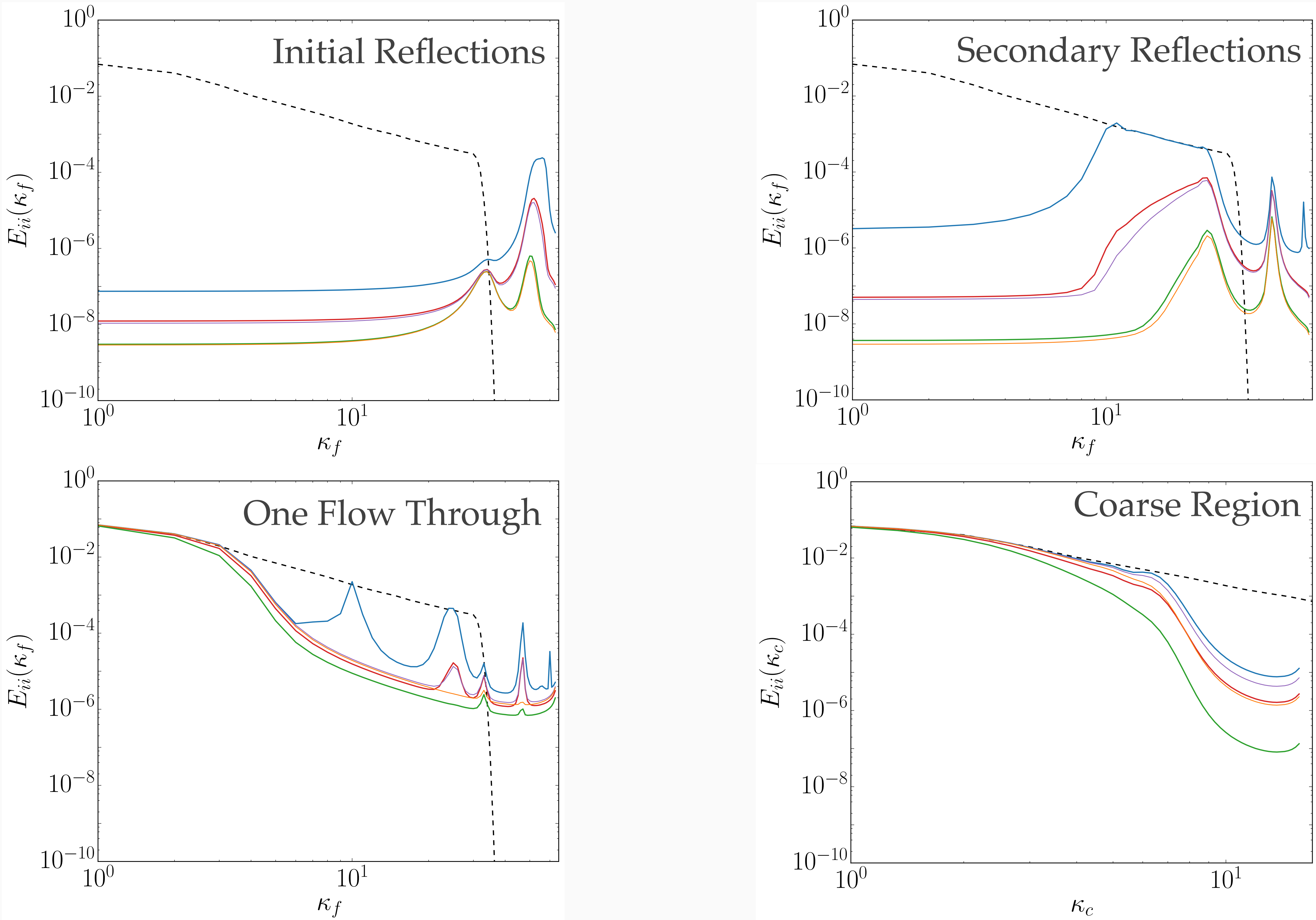
J. Frank and S. Reich, "On spurious reflections, nonuniform grids and finite difference discretizations of wave equations. cwi report mas-e0406," *Center for Mathematics and Computer Science*, 2004.

Model Results: 7-Bsplines



Model Results: 2-Bsplines



No model

$F_2 = B_2^2, \varepsilon = 0.1$

$F_2 = B_2^2, \varepsilon = 0.001$

$F_4 = B_2^2 - B_1^2 B_1^2, \varepsilon = 0.1$

$F_4 = B_2^2 - B_1^2 B_1^2, \varepsilon = 0.001$