SHALLOW WATER MODELING USING DISCONTINUOUS AND COUPLED FINITE ELEMENT METHODS

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Abstract. We consider the approximation of the depth-averaged two dimensional shallow water equations by two approaches. In both approaches, a discontinuous Galerkin (DG) method is used to approximate the continuity equation. In the first approach, a continuous Galerkin method is used for the momentum equations. In the second approach a particular DG method, the nonsymmetric interior penalty Galerkin method (NIPG), is used to approximate momentum.

Key words. shallow water equations, Galerkin finite element method, discontinuous Galerkin finite element method, coupled methods

1. Introduction. The shallow water equations (SWE) model flow in domains whose characteristic wave length in the horizontal is much larger than the water depth. The SWE consist of a first order hyperbolic continuity equation for the water elevation, coupled to momentum equations for the horizontal depth-averaged velocities [10]. These equations are often solved on domains with fairly irregular (land) boundaries.

Various finite element approaches have been developed for the SWE over the past two decades; see, for example, [7, 6, 5, 9, 11]. Much of this effort has been directed at deriving a finite element method which is stable under highly varying flow regimes, including advection dominant flows. Most of these finite element methods are based on continuous approximating spaces. In recent years, finite element methods which use discontinuous approximating spaces have been studied [2, 3, 1]. This

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discontinuous Galerkin (DG) approach has several appealing features; in particular, the ability to incorporate upwinding and stability post-processing into the solution to model highly advective flows, the ability to use different polynomial orders of approximation in different parts of the domain (and for different variables, if so desired), and the ability to easily use nonconforming meshes (e.g., with hanging nodes). Moreover, the DG method is "locally conservative," that is, the continuity equation relating the change in water elevation to water flux, is satisfied in a weak sense element by element.

In this paper, we briefly discuss two approaches for shallow water flow modeling based on discontinuous and continuous approximating spaces, additional details can be found in [4]. In the first approach, we discretize the primitive continuity equation using a DG method, coupled to a continuous finite element approximation of the momentum equations. This approach is useful when local conservation is important, and uses discontinuous approximations for the hyperbolic continuity equation, while allowing for the momentum equation to be approximated using more traditional continuous functions. In the second approach, we discretize both equations using DG methods. For the momentum equation, we use a particular DG method called the nonsymmetric interior penalty Galerkin (NIPG) method, developed in [8]. This approach allows for the flexibility of the DG method to be applied to both continuity and momentum, if so desired.

2. Problem definition. The SWE consist of the primitive continuity equation and momentum equations. Unknown variables are depth-averaged elevation $\zeta = \zeta(\mathbf{x}, t)$ and velocity $\boldsymbol{u} = \boldsymbol{u}(\mathbf{x}, t)$.

Let Ω be a bounded domain in $I\!\!R^2$, with Lipschitz boundary $\partial\Omega$, where $\boldsymbol{n}$ is the fixed unit outward normal to $\partial\Omega$. For the continuity equation, we decompose the boundary of the domain into an inflow portion $\partial\Omega_{\mathrm{I}}$ and an outflow portion $\partial\Omega_{\mathrm{O}}$ such that $\partial\Omega = \partial\Omega_{\mathrm{I}} \cup \partial\Omega_{\mathrm{O}}$, where $\partial\Omega_{\mathrm{I}} = \{x \in \partial\Omega : \boldsymbol{u} \cdot \mathbf{n} < 0\}$ and $\partial\Omega_{\mathrm{O}} = \{x \in \partial\Omega : \boldsymbol{u} \cdot \mathbf{n} \geq 0\}$. For the momentum equation, we assume for simplicity Dirichlet boundary conditions on $\boldsymbol{u}$ are specified everywhere on $\partial\Omega$.

Consider the following simplified form of the SWE: find ζ and $\boldsymbol{u}$ such that

$$\partial_t \zeta + \nabla \cdot (\boldsymbol{u}\zeta) = 0 \qquad (\mathbf{x}, t) \in \Omega,\ t > 0, \tag{1}$$

$$\partial_t \boldsymbol{u} + g\nabla\zeta - \nu\Delta\boldsymbol{u} = \boldsymbol{f} \qquad (\mathbf{x}, t) \in \Omega,\ t > 0. \tag{2}$$

This simplified model contains the primary coupling between the two equations. We will consider boundary and initial conditions of the form

$$\zeta = \hat{\zeta} \qquad \text{on } \partial\Omega_{\mathrm{I}}, \tag{3}$$

$$\zeta(\mathbf{x}, 0) = \zeta_0 \qquad \text{on } \Omega, \tag{4}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \qquad \text{on } \partial\Omega, \tag{5}$$

$$\boldsymbol{u}(\mathbf{x}, 0) = \boldsymbol{u}_0 \qquad \text{on } \Omega. \tag{6}$$

Here ν is the vertically averaged turbulent viscosity and g is acceleration due to gravity; $\boldsymbol{f}$ represents body forces.

2.1. Notation and Function Space Properties. Let $\{\mathcal{T}_h\}_{h>0}$ denote a family of finite element partitions of Ω such that no element Ω_e crosses $\partial\Omega$. We assume each element Ω_e has a element diameter h_e, with h being the maximal element diameter. Let $\mathcal{P}^k(\Omega_e)$ denote the space of complete polynomials of degree $k \geq 1$, defined on Ω_e.

We denote the trace of a function $v \in H^1(\Omega_e)$ on interior edges γ_i by $v^{\pm}$:

$$v^-(\mathbf{x}) = \lim_{s \to 0-} v(\mathbf{x} + s\boldsymbol{n}_i), \qquad v^+(\mathbf{x}) = \lim_{s \to 0+} v(\mathbf{x} + s\boldsymbol{n}_i),$$

then define

$$\overline{v} = \frac{1}{2}(v^+ + v^-), \qquad [v] = v^- - v^+,$$

where $\mathbf{x} \in \gamma_i$ and $\boldsymbol{n}_i$ denotes a fixed unit vector normal to γ_i. Let $\sum_i$ denote summation over all interior element edges γ_i.

We will use the $L^2(R)$ inner product notation $(\cdot, \cdot)_R$ for domains $R \in I\!R^2$, and the notation $\langle u, v \rangle_R$ to denote integration over one-dimensional surfaces.

3. The discontinuous and continuous Galerkin formulation. Multiply equation (1) by arbitrary, smooth test functions $v \in H^1(\Omega_e)$ and integrate by parts over each element Ω_e to obtain

$$(\partial_t \zeta, v)_{\Omega_e} - (\boldsymbol{u}\,\zeta, \nabla v)_{\Omega_e} + \langle \zeta\, \boldsymbol{u} \cdot \boldsymbol{n}_e, v\rangle_{\partial\Omega_e} = 0, \tag{7}$$

where $\boldsymbol{n}_e$ denotes a fixed unit normal to each edge $\partial\Omega_e$.

On each Ω_e, we approximate ζ in a space $\mathcal{S}^k(\Omega_e)$, where $\mathcal{P}^k(\Omega_e) \subset \mathcal{S}^k(\Omega_e)$, and such that if we define

$$V_h = \{v : \Omega \to I\!R : v|_{\Omega_e} \in \mathcal{S}^k(\Omega_e)\}, \tag{8}$$

then V_h^c, the space of continuous, piecewise polynomials of degree k is contained in V_h.

Multiply equation (2) by $\boldsymbol{w} \in (H_0^1(\Omega))^2$ and integrate by parts over the domain to obtain

$$(\partial_t \boldsymbol{u}, \boldsymbol{w})_\Omega + (g\nabla\zeta, \boldsymbol{w})_\Omega - \sum_i \langle g\,[\zeta], \boldsymbol{w}\cdot\boldsymbol{n}_i\rangle_{\gamma_i} + (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{w})_\Omega = (\boldsymbol{f}, \boldsymbol{w})_\Omega. \tag{9}$$

Note that the stabilization term $\sum_i \langle g[\zeta], \boldsymbol{w}\cdot\boldsymbol{n}_i\rangle_{\gamma_i}$ is actually zero since we are assuming our true solution sufficiently smooth to be continuous.

We approximate $\boldsymbol{u}$ in the finite-dimensional subspace $\boldsymbol{W}_h \subset (H^1(\Omega))^2 \cap \{\boldsymbol{u} : \boldsymbol{u} = \hat{\boldsymbol{u}} \text{ on } \partial\Omega\}$, consisting of continuous, piecewise polynomials of degree k. That is, each component of $\boldsymbol{u}$ is in V_h^c. Let $\boldsymbol{W}_{0,h}$ be the corresponding subspace of $(H_0^1(\Omega))^2$.

Approximate $\zeta(\cdot,t)$ by $Z(\cdot,t) \in V_h$, and $\boldsymbol{u}(\cdot,t)$ by $\boldsymbol{U}(\cdot,t) \in \boldsymbol{W}_h$. Sum (7) over all elements Ω_e, and let the value of ζ across inner element boundaries be approximated by the upwind value $Z^\uparrow$:

$$\zeta \approx Z^\uparrow = \begin{cases} Z^-, & \boldsymbol{U}\cdot\boldsymbol{n}_i > 0 \\ Z^+, & \boldsymbol{U}\cdot\boldsymbol{n}_i \le 0 \end{cases} \quad \text{on } \gamma_i.$$

The discrete weak formulation is: for each $t > 0$, find $(Z, \boldsymbol{U}) \in V_h \times \boldsymbol{W}_h$ satisfying $\forall\, v \in V_h$ and $\forall\, \boldsymbol{w} \in \boldsymbol{W}_{0,h}$:

$$\sum_e (\partial_t Z, v)_{\Omega_e} - \sum_e (\boldsymbol{U}\, Z, \nabla v)_{\Omega_e} + \sum_i \langle Z^{\uparrow} \boldsymbol{U} \cdot \boldsymbol{n}_i, [v] \rangle_{\gamma_i} + \langle Z\, \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{O}}} = -\langle \hat{\zeta}\, \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{I}}}, \quad (10)$$

$$(\partial_t \boldsymbol{U}, \boldsymbol{w})_{\Omega} + \sum_e (g \nabla Z, \boldsymbol{w})_{\Omega_e} - \sum_i \langle g\,[Z], \boldsymbol{w} \cdot \boldsymbol{n}_i \rangle_{\gamma_i} + (\nu \nabla \boldsymbol{U}, \nabla \boldsymbol{w})_{\Omega} = (\boldsymbol{f}, \boldsymbol{w})_{\Omega}. \quad (11)$$

The scheme above satisfies the following error estimate [4]:

THEOREM 3.1. *For $\boldsymbol{u}$,ζ sufficiently smooth, the scheme (10)-(11) satisfies the error estimate, for $e_\zeta = \zeta - Z$ and $\boldsymbol{e}_u = \boldsymbol{u} - \boldsymbol{U}$:*

$$||e_\zeta(\cdot, T)|| + ||\boldsymbol{e}_u(\cdot, T)|| \leq K_1 h^k, \quad (12)$$

where K_1 is a constant independent of h and k.

4. The discontinuous Galerkin and NIPG formulation. In this case, on each Ω_e, we will use the finite dimensional approximating spaces $V_h \subset H^1(\Omega_e)$ and $\boldsymbol{W}_h \subset (H^1(\Omega_e))^2$ defined by:

$$V_h = \{v : \Omega \to I\!R : v|_{\Omega_e} \in \mathcal{P}^{k_e^\zeta}(\Omega_e)\}$$

$$\boldsymbol{W}_h = \{\boldsymbol{w} : \Omega \to I\!R : \boldsymbol{w}|_{\Omega_e} \in (\mathcal{P}^{k_e^u}(\Omega_e))^2\},$$

where $k_e^\zeta, k_e^u \geq 1$. Approximate $\zeta(\cdot, t)$ by $Z(\cdot, t) \in V_h$, and $\boldsymbol{u}(\cdot, t)$ by $\boldsymbol{U}(\cdot, t) \in \boldsymbol{W}_h$.

The discrete weak formulation is: for each $t > 0$, find $(Z, \boldsymbol{U}) \in V_h \times \boldsymbol{W}_h$ satisfying

$$\sum_e (\partial_t Z, v)_{\Omega_e} - \sum_e (\boldsymbol{U}\, Z, \nabla v)_{\Omega_e} + \sum_i \langle Z^{\uparrow} \overline{\boldsymbol{U}} \cdot \boldsymbol{n}_i, [v] \rangle_{\gamma_i}$$
$$+ \langle Z\, \boldsymbol{U} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{O}}} + \langle \hat{\zeta}\, \boldsymbol{U} \cdot \boldsymbol{n}, v \rangle_{\partial\Omega_{\mathrm{I}}} = 0, \quad (13)$$

$$\sum_e (\partial_t \boldsymbol{U}, \boldsymbol{w})_{\Omega_e} + \sum_e (g\nabla Z, \boldsymbol{w})_{\Omega_e} - \sum_i \langle g\,[Z], \overline{\boldsymbol{w}} \cdot \boldsymbol{n}_i\rangle_{\gamma_i} + \sum_e (\nu \nabla \boldsymbol{U}, \nabla \boldsymbol{w})_{\Omega_e}$$

$$- \sum_i \langle \overline{\nu \nabla \boldsymbol{U}} \cdot \boldsymbol{n}_i, [\boldsymbol{w}]\rangle_{\gamma_i} + \sum_i \langle \overline{\nu \nabla \boldsymbol{w}} \cdot \boldsymbol{n}_i, [\boldsymbol{U}]\rangle_{\gamma_i} - \langle \nu \nabla \boldsymbol{U} \cdot \boldsymbol{n}, \boldsymbol{w}\rangle_{\partial\Omega}$$

$$+ \langle \nu \nabla \boldsymbol{w} \cdot \boldsymbol{n}, \boldsymbol{U} - \hat{\boldsymbol{u}}\rangle_{\partial\Omega} + \sum_i \langle \sigma[\boldsymbol{U}], [\boldsymbol{w}]\rangle_{\gamma_i}$$

$$+ \langle \sigma(\boldsymbol{U} - \hat{\boldsymbol{u}}), \boldsymbol{w}\rangle_{\partial\Omega} = \sum_e (\boldsymbol{f}, \boldsymbol{w})_{\Omega_e}, \quad (14)$$

where we have introduced the "interior penalty" term $\sum_i \langle \sigma[\boldsymbol{U}], [\boldsymbol{w}]\rangle_{\gamma_i}$ and the "boundary penalty" term $\langle \sigma(\boldsymbol{U} - \hat{\boldsymbol{u}}), \boldsymbol{w}\rangle_{\partial\Omega}$. Here $\sigma > 0$. We assume

$$\sigma|_{\gamma_i} = \mathcal{O}(h_{\gamma_i}^{-1}). \quad (15)$$

The scheme above satisfies the following estimate:

THEOREM 4.1. *For $\boldsymbol{u}$,ζ sufficiently smooth and positive penalty parameter σ satisfying (15), the scheme (13)-(14) satisfies the error estimate [4]*

$$||e_\zeta(\cdot, T)|| + ||\boldsymbol{e}_u(\cdot, T)|| \leq K_2 \left\{ \int_0^T \sum_e \left[h_e^{2k_e^\zeta} ||\zeta||^2_{H^{k_e^\zeta+1}(\Omega_e)} + h_e^{2k_e^u} ||\boldsymbol{u}||^2_{H^{k_e^u+1}(\Omega_e)} \right] dt \right\}^{1/2},$$

where K_2 is independent of h.

REFERENCES

[1] V. AIZINGER AND C. DAWSON, *Discontinuous Galerkin methods for two-dimensional flow and transport in shallow water*, Advances in Water Reources, 25 (2002), pp. 67–84.

[2] F. ALCRUDO AND P. GARCIA-NAVARRO, *A high-resolution godunov-type scheme in finite volumes for the 2d shallow-water equations*, Int. J. Num. Meth. Fluids, 16 (1993), pp. 489–505.

[3] S. CHIPPADA, C. N. DAWSON, M. L. MARTÍNEZ, AND M. F. WHEELER, *A godunov-type finite volume method for the system of shallow water equations*, Comput. Meth. Appl. Mech. Engrg., 151 (1998), pp. 105–129.

[4] C. Dawson and J. Proft, *Discontinuous and coupled discontinuous/continuous Galerkin methods for the shallow water equations*, Computer Methods in Applied Mechanics and Engineering. to appear.

[5] M. Kawahara, H. Hirano, K. Tsuhora, and K. Iwagaki, *Selective lumping finite element method for shallow water equations*, Int. J. Num. Meth. Eng., 2 (1982), pp. 99–112.

[6] I. King and W. R. Norton, *Recent application of RMA's finite element models for two-dimensional hydrodynamics and water quality*, in Finite Elements in Water Resources II, C. Brebbia, W.G.Gray, and G.F.Pinder, eds., Pentech Press, London, 1978.

[7] D. R. Lynch and W. R. Gray, *A wave equation model for finite element computations*, Computers and Fluids, 7 (1979), pp. 207–228.

[8] B. Rivière, M. Wheeler, and V. Girault, *Improved energy estimates for interior penalty, constrained and discontinuous Galerkin methods for elliptic problems. Part I*, Computational Geosciences, 3 (1999), pp. 337–360.

[9] R. Szymkiewicz, *Oscillation-free solution of shallow water equations for nonstaggered grid*, J. Hyd. Engg., 119 (1993), pp. 1118–1137.

[10] T. Weiyan, *Shallow Water Hydrodynamics*, vol. 55 of Elsevier Oceanography Series, Elsevier, Amsterdam, 1992.

[11] O. Zienkiewicz and P. Ortiz, *A split-characteristic based finite element model for the shallow water equations*, Int. J. Num. Meth. Fluids, 20 (1995), pp. 1061–1080.