Stochastic modeling and optimization methods in Investments

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Financial Mathematics

An interdisciplinary field on the crossroads of stochastic processes, stochastic analysis, optimization, partial differential equations, finance, financial economics, decision analysis, statistics and econometrics

It studies derivative securities and investments, and the management of financial risks

Started in the 1970s with the pricing of derivative securities

Derivative securities

Derivatives are financial contracts offering payoffs written on underlying (primary) assets

Their role is to eliminate, reduce or mitigate risk

Major breakthrough

Black-Scholes-Merton 1973

- The price of a derivative is the value of a portfolio that reproduces the derivative's payoff
- The components of this replicating portfolio yield the hedging strategies

This idea together with Itô's stochastic calculus started the field of Financial Mathematics

- Black, F. and M. Scholes (1973): *The pricing of options and corporate liabilities*, JPE
- Itô, K. (1944): *Stochastic integral*, Proc. Imp. Acad. Tokyo
  Döblin, W. (1940; 2000): *Sur l'équation de Kolmogoroff*, CRAS, Paris, 331

Louis Bachelier (1870-1946)

Father of Financial Mathematics

Thesis: *Théorie de la Spéculation*

Advisor: Henri Poincaré

Sciences
1899
25

8666.

N° D'ORDRE :
1018.

THÈSES

PRÉSENTÉES

A LA FACULTÉ DES SCIENCES DE PARIS

POUR OBTENIR

LE GRADE DE DOCTEUR ÈS SCIENCES MATHÉMATIQUES,

PAR M. L. BACHELIER.

1re THESE. — THÉORIE DE LA SPÉCULATION.
2e THÈSE. — PROPOSITIONS DONNÉES PAR LA FACULTÉ.

Soutenues le 29 mars 1900 devant la Commission d'Examen.

MM. APPELL, *Président.*
BOUSSINESQ, } *Examinateurs.*
H. POINCARÉ, }

BIBLIOTHÈQUE

PARIS,
GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE
DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE,
Quai des Grands-Augustins, 55.

1900

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Bachelier model (1900)

W_t a standard Brownian motion and μ, σ constants
increment evolves as

$$S_{t+\delta} - S_t = \mu\delta + \sigma\,(W_{t+\delta} - W_t)$$

Bachelier's work was neglected for decades
It was not recognized until Paul Samuelson introduced it to Economics

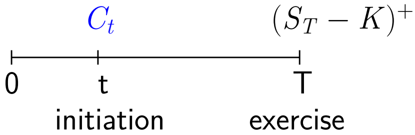
Samuelson model (1965)

log-normal stock prices

$$dS_s = \mu S_s ds + \sigma S_s dW_s$$

widely used in finance practice

European call



Replication (Black-Scholes-Merton)

- Market: stock S and a bond B $(dB_s = rB_s ds)$

- Find a pair of stochastic processes (α_s, β_s), $t \leq s \leq T$, such that a.s. at T,

$$\alpha_T S_T + \beta_T B_T = (S_T - K)^+$$

- The derivative price process C_s, $t \leq s \leq T$, is then given by

$$C_s = \alpha_s S_s + \beta_s B_s$$

- Price at initiation: $C_t = \alpha_t S_t + \beta_t B_t$

- Hedging strategies: (α_s, β_s), $t \leq s \leq T$

- **Replication** using self-financing strategies

$$C_s = C_t + \int_t^s \alpha_u dS_u + \int_t^s \beta_u dB_u$$

- Postulate a price representation $C_s = h(S_s, s)$ for a smooth function $h(x,t)$
- **Itô's calculus**

$$C_s = h(S_t, t)$$

$$+ \int_t^s \left(h_t(S_u, u) + \mu S_u h_x(S_u, u) + \frac{1}{2}\sigma^2 S_u^2 h_{xx}(S_u, u) \right) du$$

$$+ \int_t^s \sigma S_u h_x(S_u, u)\, dW_u^{\mathbb{P}}$$

Black and Scholes pde

The function $h(x,t)$ solves

$$rh = h_t + \frac{1}{2}\sigma^2 x^2 h_{xx} + rxh_x$$

with $h(0,t) = 0$ and $h(x,T) = (x-K)^+$

European call price process

$$C_s = h(S_s, s)$$

Hedging strategies

$$(\alpha_s, \beta_s) = \left(h_x(S_s, s), \frac{h(S_s, s) - h_x(S_s, s) S_s}{B_s} \right)$$

A universal pricing theory
for general price processes (semimartingales)

Arbitrage-free pricing of derivative securities

Harrison, Kreps, Pliska (1979,1981)

Arbitrage

A market admits arbitrage in $[t, T]$ if the outcome X_T of self-financing strategies satisfies $X_t = 0$, and

$$\mathbb{P}\left(X_T \geq 0\right) = 1 \quad \text{and} \quad \mathbb{P}\left(X_T > 0\right) > 0$$

In arbitrage-free markets, derivative prices are given by

$$C_t = E_{\mathbb{Q}}\left(\frac{B_\tau}{B_t} C_\tau \middle| \mathcal{F}_t\right)$$

$\mathbb{Q} \sim \mathbb{P}$ under which (discounted) assets are martingales

Model-independent pricing theory

$$\mathbb{P} \quad \rightarrow \quad \mathbb{Q} \quad \rightarrow \quad E_{\mathbb{Q}}\left(\cdot \middle| \mathcal{F}_t\right)$$

Linear pricing rule and change of measure

Mathematics and derivative securities

- Martingale theory and stochastic integration
  Derivative prices are martingales under $\mathbb{Q}$
  Hedging strategies are the integrands (martingale representation)

- Malliavin calculus for sensitivities ("greeks")

- Markovian models - (multi-dim) linear partial differential equations
  Early exercise claims - optimal stopping, free-boundary problems
  Exotics - linear pde with singular boundary conditions

- Credit derivatives - copulas, jump processes

- Bond pricing, interest rate derivatives, yield curve: linear stochastic PDE

Some problems of current interest

- Stochastic volatility
- Correlation, causality
- Systemic risk
- Counterparty risk
- Liquidity risk, funding risk
- Commodities and Energy
- Calibration
- Market data analysis

  ⋮

The other side of Finance: Investments

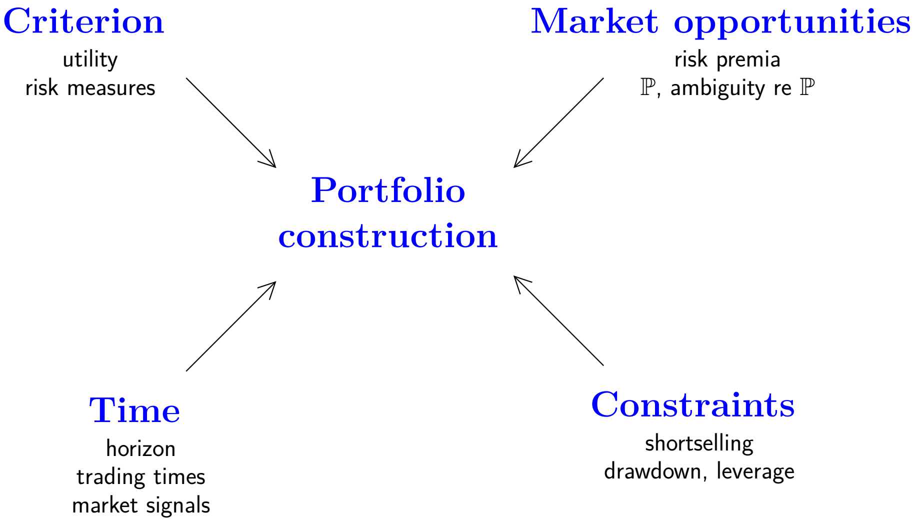
Derivative securities and investments

While in derivatives the aim is to eliminate the risk, the goal in investments is to profit from it

- Derivatives industry uses highly quantitative methods
- Academic research and finance practice have been working together, especially in the 80s and 90s

- Traditional investment industry is not yet very quantitative
- A unified "optimal investment" theory does not exist to date
- Ad hoc methods are predominantly used

Disconnection between academic research and investment practice

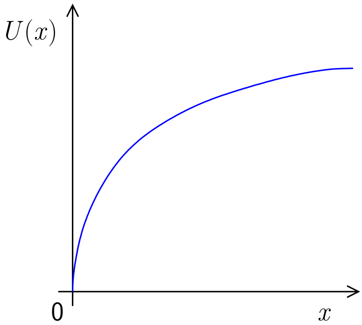

Criterion
utility
risk measures
Market opportunities
risk premia
ℙ, ambiguity re ℙ
Portfolio construction
Time
horizon
trading times
market signals
Constraints
shortselling
drawdown, leverage

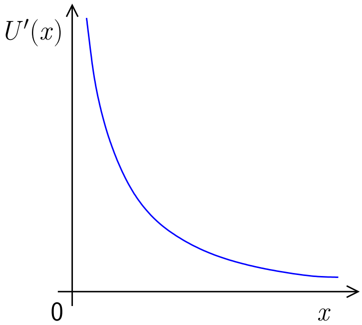
Academic research in investments

Modeling investor's behavior
Utility theory

- Created by **Daniel Bernoulli** (1738) responding to his cousin, **Nicholas Bernoulli** (1713) who proposed the famous St. Petersburg paradox, a game of "unreasonable" infinite value based on expected returns of outcomes
- D. Bernoulli suggested that utility or satisfaction has diminishing marginal returns, alluding to the **utility** being **concave** (see, also, Gabriel Cramer (1728))
- **Oskar Morgenstern and John von Neumann** (1944) published the highly influential work *"Theory of games and economic behavior"*. The major conceptual result is that the behavior of a rational agent coincides with the behavior of an agent who values uncertain payoffs using expected utility

Utility function and its marginal





Inada conditions: $\lim_{x \to 0} U'(x) = \infty$ and $\lim_{x \to \infty} U'(x) = 0$

Asymptotic elasticity: $\lim_{x \to \infty} \frac{xU'(x)}{U(x)} < 1$

Stochastic optimization
and optimal portfolio construction

Merton's portfolio selection continuous-time model

- start at $t \geq 0$ with endowment x,
  market $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, price processes

- follow investment strategies, $\pi_s \in \mathcal{F}_s$, $t < s \leq T$

- map their outcome $X_T^\pi \rightarrow E_{\mathbb{P}}\left(U\left(X_T^\pi\right) \middle| \mathcal{F}_t, X_t^\pi = x\right)$

- maximize terminal expected utility (value function)

$$V(x,t) = \sup_\pi E_{\mathbb{P}}\left(U\left(X_T^\pi\right) \middle| \mathcal{F}_t, X_t^\pi = x\right)$$

R. Merton (1969): *Lifetime portfolio selection under uncertainty: the continuous-time case*, RES

Stochastic optimization approaches

Markovian models

Dynamic Programming Principle
Bellman (1950)

Non-Markovian models

Duality approach
Bismut (1973)

Markovian models

Stock price: $dS_t = S_t \mu\left(Y_t, t\right) dt + S_t \sigma\left(Y_t, t\right) dW_t$

Stochastic factors: $dY_t = b\left(Y_t, t\right) dt + a\left(Y_t, t\right) d\tilde{W}_t;$ correlation ρ

Controlled diffusion (wealth): $dX_s^{\pi} = \pi_s \mu\left(Y_t, t\right) dt + \pi_s \sigma\left(Y_s, s\right) dW_s,\ X_t = x$

Dynamic Programming Principle (DPP)

$$V\left(X_s^{\pi}, Y_s, s\right) = \sup_{\pi} E_{\mathbb{P}}\left(V\left(X_{s'}^{\pi}, Y_{s'}, s'\right) \middle| \mathcal{F}_s\right)$$

Hamilton-Jacobi-Bellman equation

$$V_t + \max_{\pi \in \mathbb{R}}\left(\frac{1}{2}\pi^2 \sigma^2\left(y,t\right) V_{xx} + \pi\left(\mu\left(y,t\right) V_x + \sigma\left(y,t\right) a\left(y,t\right) V_{xy}\right)\right)$$

$$+\frac{1}{2} a^2\left(y,t\right) V_{yy} + b\left(y,t\right) V_y = 0,$$

with $V\left(x,y,T\right) = U\left(x\right)$

Optimal portfolios (feedback controls)

$$\pi^*(x,y,t) = -\frac{\mu(y,t)}{\sigma^2(y,t)}\frac{V_x}{V_{xx}} - \rho\frac{a(y,t)}{\sigma(y,t)}\frac{V_{xy}}{V_{yy}}$$

$$\pi_s^* = \pi^*(X_s^*, Y_s, s) \quad \text{and} \quad dX_s^* = \pi_s^*\mu(Y_t,t)\,dt + \pi_s^*\sigma(Y_s,s)\,dW_s$$

Difficulties

- set of controls non-compact, state-constraints
- degeneracies, lack of smoothness, validity of verification theorem
- value function as viscosity solution of HJB, smooth cases for special examples

- existence, smoothness and monotonicity properties of $\pi^*(x,y,t)$
- probabilistic properties of optimal processes π_s^*, X_s^* and their ratio

Karatzas, Shreve, Touzi, Bouchard, Pham, Z.,...

Duality approach
in optimal portfolio construction

Dual optimization problem - utility convex conjugate

$$\tilde{U}(y) = \max_{x>0} \left(U(x) - xy \right)$$

- Introduced in stochastic optimization by Bismut (1973)
- Introduced in optimal portfolio construction by Bismut (1975) and Foldes (1978)
- Further results by Karatzas et al (1987) and Cox and Huang (1989)
- Xu (1990) shows that the HJB linearizes for complete markets
- Kramkov and Schachermayer (1999) establish general results for semimartingale models

Semimartingale stock price models

$\mathcal{H}$: predictable processes integrable wrt the semimartingale S

$$\mathcal{X}(x) = \left\{ X : X_t = x + \int_0^t H_s \cdot dS_s,\ t \in [0, T],\ H \in \mathcal{H} \right\}$$

$$\mathcal{Y} = \{ Y \geq 0,\ Y_0 = 1,\ XY \text{ semimartingale for all } X \in \mathcal{X} \}$$

$$\mathcal{Y}(y) = y\mathcal{Y}, y > 0$$

Asymptotic elasticity condition: $\limsup_{x \to \infty} \left(\frac{xU'(x)}{U(x)} \right) < 1$

Primal problem

$$u(x) = \sup_{X \in \mathcal{X}(x)} E_{\mathbb{P}}(U(X_T))$$

Dual problem

$$\tilde{u}(y) = \inf_{Y \in \mathcal{Y}(y)} E_{\mathbb{P}}\left(\tilde{U}(Y_T)\right)$$

Duality results for semimartingale stock price models

If $u(x) < \infty$, for some x, and $\tilde{u}(y) < \infty$ for all $y > 0$, then:

- $\tilde{u}(y) = \sup_{x>0} (u(x) - xy)$
- $\tilde{u}(y) = \inf_{Q \in \mathcal{M}^e(S)} E_{\mathbb{P}} \left(\tilde{U} \left(y \frac{dQ}{d\mathbb{P}} \right) \right)$, $y > 0$, with $\mathcal{M}^e(S)$ the set of martingale measures $Q \sim \mathbb{P}$
- if Q^* optimal, the terminal optimal wealth (primal problem) $X_T^{x,*}$ is given by

$$U'\left(X_T^{x,*}\right) = u'(x) \frac{dQ^*}{d\mathbb{P}}$$

Kramkov and Schachermayer, Karatzas, Cvitanic, Zitkovic, Sirbu, ...

Some extensions

Coupled stochastic optimization problems

Systems of HJB equations

Delegated portfolio management

$$\text{investor} \xrightarrow{\text{fees, capital}} \text{manager}$$

$$\text{manager} \xrightarrow{\text{risk, return}} \text{investor}$$

$$V^{i}(x,t) = \sup_{\mathcal{A}^{i}} E_{\mathbb{P}}\left(U^{i}\left(X_T, I^{m}_{t\le s\le T}\right)\Big|\, \mathcal{F}_t, X_t = x\right)$$

$$V^{m}(y,t) = \sup_{\mathcal{A}^{m}} E_{\mathbb{P}}\left(U^{m}\left(Y_T, I^{i}_{t\le s\le T}\right)\Big|\, \mathcal{G}_t, Y_t = y\right)$$

I^m and I^i : inputs from the manager (performance, risk taken)
and the investor (investment targets)

Benchmarking and asset specialization among competing fund managers

$$V^{1}(x,t) = \sup_{\mathcal{A}^{1}} E_{\mathbb{P}_1}\left(U_1\left(X_T, Y_T^{*}\right)\right|\, \mathcal{F}_t, X_t = x)$$

$$V^{2}(y,t) = \sup_{\mathcal{A}^{2}} E_{\mathbb{P}_2}\left(U_2\left(Y_T, X_T^{*}\right)\right|\, \mathcal{G}_t, Y_t = y)$$

Proportional transaction costs

- N stocks, one riskless bond
- Pay proportionally α^i for selling and β^i for buying the i^{th} stock
  X_s : bond holding, $Y_s = \left(Y_s^1, ..., Y_s^N\right)$: stock holdings, $t \leq s \leq T$,

$$dX_s = rX_s ds + \Sigma_{i=1}^N \alpha^i dM_s^i - \Sigma_{i=1}^N \beta^i dL_s^i$$

$$dY_s^i = \mu^i Y_s^i ds + \sigma^i Y_s^i dW_s^i + dL_s^i - dM_s^i$$

Variational inequalities with gradient constraints

$$\min\Big(- V_t - \mathcal{L}^{(y_1, \dots y_N)} V - rxV_x, -a^1 V_x + V_{y_1}, \beta^1 V_x - V_{y_1},$$

$$\vdots$$

$$- a^N V_x + V_{y_N}, \beta^N V_x - V_{y_N}\Big) = 0,$$

with $V\left(x, y_1, \dots y_N, T\right) = U\left(x + \Sigma_{i=1}^N \alpha^i y_i 1_{\{y_i \geq 0\}} + \Sigma_{i=1}^N \beta^i y_i 1_{\{y_i \leq 0\}}\right)$

Liquidation of financial positions and price impact

big investor : delegates liquidation to "major agent"

major agent : liquidates in the presence of many small agents
small agents: **noise traders** and **high-frequency traders**

Optimal liquidation is an interplay between speed and volume

Too fast $\longrightarrow$ price impact
Too slow $\longrightarrow$ unfavorable price fluctuations

Mean-field games

- Aggregate impact from noise traders
- Aggregate impact from high-frequency traders

Model uncertainty
and optimal portfolio construction

Knightian uncertainty (model ambiguity)

Frank Knight (1921)

The historical measure $\mathbb{P}$ might not be a priori known

- Gilboa and Schmeidler (1989) built an axiomatic approach for preferences towards **both** risk and model ambiguity. They proposed the *robust utility* form

$$X_T^\pi \to \inf_{Q\in\mathcal{Q}} E_Q\left(U\left(X_T^\pi\right)\right),$$

where U is a classical utility function and $\mathcal{Q}$ a family of *subjective* probability measures

Standard criticism: the above criterion allows for very limited , if at all, differentiation of models with respect to their plausibility

Knightian uncertainty

- Maccheroni, Marinacci and Rustichini (2006) extended the above approach to

$$X_T^\pi \to \inf_{Q \in \mathcal{Q}} \left(E_Q \left(U \left(X_T^\pi \right) \right) + \gamma \left(Q \right) \right)$$

where the functional $\gamma(Q)$ serves as a *penalization* weight to each Q-market model

Entropic penalty and entropic robust utility

$$\gamma(Q) = H(Q|\mathbb{P}) \quad \text{with} \quad H(Q|\mathbb{P}) = \int \frac{dQ}{d\mathbb{P}} \ln\left(\frac{dQ}{d\mathbb{P}}\right) d\mathbb{P}$$

$$\inf_{Q \in \mathcal{Q}} \left(E_Q \left(U \left(X_T^\pi \right) \right) + \gamma \left(Q \right) \right) = \ln E_{\mathbb{P}} \left(e^{-U(X_T)} \right)$$

Hansen, Talay, Schied, Föllmer, Frittelli, Weber ...

Maxmin stochastic optimization problem

Stock dynamics: $S_t = \left(S_t^1, ..., S_t^d\right), \ t \in [0, T]$, semimartingales

Wealth dynamics: $X_t^\alpha = x + \int_0^t \alpha_s \cdot dS_s, \ X_t^\alpha \geq 0, \ t \geq 0$

Objective:

$$v(x) = \sup_{X \in \mathcal{X}(x)} \inf_{Q \in \mathcal{Q}} \left(E_Q\left(U\left(X_T\right)\right) + \gamma(Q)\right),$$

where $\mathcal{Q} = \{ Q \ll \mathbb{P} | \, \gamma(Q) < \infty\}$

Duality approach

$$u(y) = \inf_{Y \in \mathcal{Y}_Q(y)} \inf_{Q \in \mathcal{Q}} \left(E_Q\left(\tilde{U}\left(Y_T\right)\right) + \gamma(Q)\right)$$

where $\tilde{U}(y) = \sup_{x>0} \left(U(x) - xy\right)$

Investment practice

Portfolio selection criteria

Single-period criteria

- Mean-variance $\leftrightarrow$ maximize the mean return for fixed variance
- Black-Litterman $\leftrightarrow$ allows for subjective views of the investor

Long-term criterion

- Kelly criterion $\leftrightarrow$ maximize the long-term growth

While using these criteria allows for tractable solutions, they have major deficiencies and limitations which do not capture important features like the evolution of both the market and the investor's targets

Mean–variance optimization

Harry Markowitz (1952)

Performance of portfolio returns $\leftrightarrow$ mean, variance

- Single period: 0, T
- Allocation weights at $t = 0$: $a = (a^1, \cdots, a^n)$; $\sum_{i=1}^n a^i = 1$
- Asset returns: $R_T = (R_T^1, \cdots, R_T^n)$
- Return on allocation: $R_T^a = \sum_{i=1}^n a^i R_T^i$

Mean–variance optimization

For a fixed acceptable variance v maximize the mean,

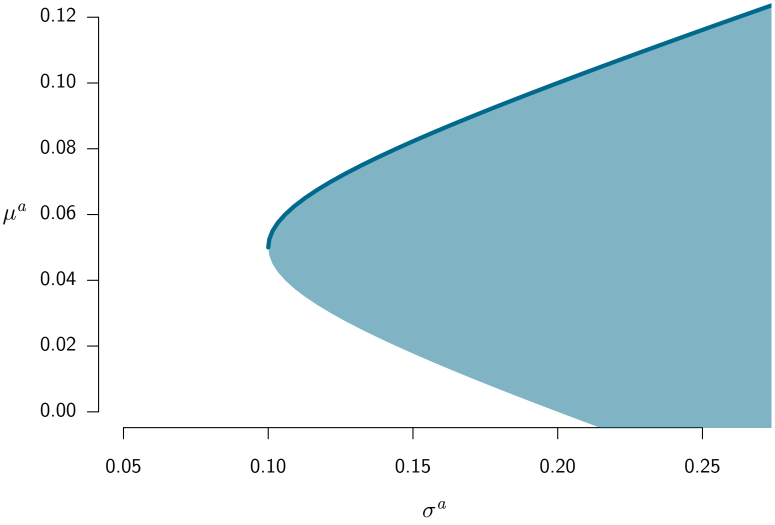
$$\max_{\substack{a:\sum_i a_i=1 \\ \mathrm{Var}\left(R_T^a\right)\leq v}} \mathrm{E}_{\mathbb{P}}\left(R_T^a\right)$$

or

For a fixed desired mean m minimize the variance,

$$\min_{\substack{a:\sum_i a_i=1 \\ \mathrm{E}_{\mathbb{P}}\left(R_T^a\right)\geq m}} \mathrm{Var}\left(R_T^a\right)$$

Efficient frontier



Despite its popularity and wide use,
MV has major deficiencies !

- Model error and unstable solutions
- Time–inconsistency of optimal portfolios
- Dynamic extensions not available to date

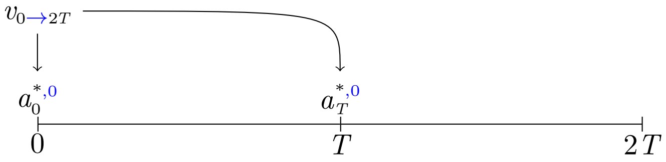
Model error and unstable solutions

- Quality and availability of market data not always good
- Estimation error very high
- Optimal allocation highly sensitive to this error
- Historical returns frequently used, not “forward–looking”
- Optimal portfolios are frequently “extreme”, unnatural, high short–selling
- Asset managers are familiar with only certain asset classes and sectors (“familiarity versus diversification” issue)

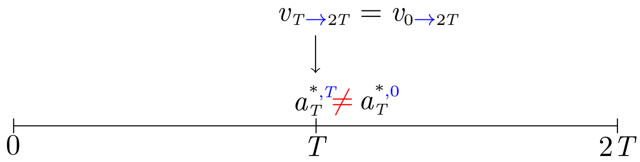
Some of these issues can be partially addressed by the Black–Litterman criterion which adjusts the equilibrium asset returns by the manager’s individual views on “familiar” assets (classes or sectors)

Time–inconsistency of MV problem

At time $t = 0$

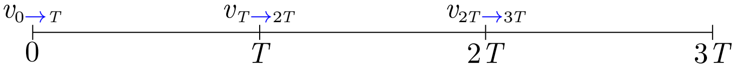


At time $t = T$



Game theoretic approach (Björk et al. (2014))

Dynamic (rolling investment times) MV optimization



- This problem is not the same as setting, at $t = 0$, the "three–period" variance target $v_{0\to 3T}$
- Is there a discrete process $v_{0\to 2T}, v_{T\to 2T}, \cdots, v_{nT\to (n+1)T}, \cdots$ modelling the targeted conditional variance (from one period to the next) that will generate time–consistent portfolios?
- What is the continuous–time limit of this construction?
- Can it be addressed by mapping the MV problem to the time–consistent expected utility problem? No! because the expected utility approach produces a solution "backwards in time"!

Can the theoretically foundational approach of expected utility meet the investment practice?

MV and expected utility

Practice	**Theory**
	Criteria
Risk–return tradeoff practical but still does not capture much	Utility is an elusive concept Difficult to quantify, especially for longer horizons
	Evolution
time–inconsistent	time consistent (semigroup property)
sequential ad hoc implementation	backward construction (DPP)
captures info up to now, limited and ad hoc	requires forecasting of asset returns, major difficulties
single–period	inflexible investment horizon

Similar limitations and discrepancies arise between expected utility, and the Black-Litterman and Kelly criteria. Practical features of these industry criteria do not fit in the expected utility framework, which however has major deficiencies

Such issues and considerations prompted the development of the forward investment performance approach (Musiela and Z., 2002-)

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Market environment

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$; $W = (W^1, \ldots, W^d)$

- Traded securities

$$\begin{cases} dS_t^i = S_t^i\Big(\mu_t^i dt + \sigma_t^i \cdot dW_t\Big), & S_0^i > 0, \quad 1 \le i \le k \\ dB_t = r_t B_t dt \ , & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-measurable stochastic processes

- Wealth process $dX_t^\pi = \sigma_t \pi_t \cdot (\lambda_t dt + dW_t)$

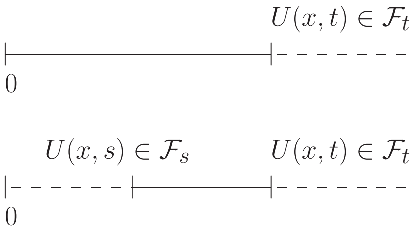
- Postulate existence of an $\mathcal{F}_t$-measurable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t \mathbb{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Forward investment performance process

optimality across trading times



$$U(x,s) = \sup_{\mathcal{A}} E_{\mathbb{P}}(U(X_t^{\pi}, t) | \mathcal{F}_s, X_s^{\pi} = x)$$

- Does such a process aways exist?
- Is it unique?
- Axiomatic construction?
- How does it relate to criteria in investment practice?

Forward investment performance process

$U(x,t)$ is an $\mathcal{F}_t$-adapted process, $t \geq 0$

- The mapping $x \rightarrow U(x,t)$ is strictly increasing and strictly concave
- For each self-financing strategy π and the associated (discounted) wealth X_t^π

$$E_{\mathbb{P}}(U(X_t^\pi, t) \mid \mathcal{F}_s) \leq U(X_s^\pi, s), \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy π^* and associated (discounted) wealth $X_t^{\pi^*}$ such that

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

The forward performance SPDE

The forward performance SPDE (MZ 2007)

Let $U(x,t)$ be an $\mathcal{F}_t-$measurable process such that the mapping $x \to U(x,t)$ is strictly increasing and concave. Let also $U(x,t)$ be the solution of the stochastic partial differential equation

$$dU(x,t) = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U(x,t)\lambda + a\right)\right|^2}{\mathcal{A}^2 U(x,t)}\, dt + a(x,t)\cdot dW_t$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$adapted process, and $\mathcal{A} = \frac{\partial}{\partial x}$. Then $U(x,t)$ is a forward performance process.

Once the volatility is chosen the drift is fully specified if we know (σ, λ)

The volatility of the investment performance process

This is the novel element in the new approach

- The volatility models how the current shape of the performance process is going to diffuse in the next trading period

- The volatility is up to the investor to choose, in contrast to the classical approach in which it is uniquely determined via the backward construction of the value function process

- In general, $a(x,t) = F(x, t, U, U_x, U_{xx})$ may depend on t, x, U, its spatial derivatives etc.

- When the volatility is not state-dependent, we are in the zero volatility case

Specifying the appropriate class of volatility processes is a central problem in the forward performance approach

Musiela and Z., Nadtochiy and Z., Nadtochyi and Tehranchi, Berrier et al., El Karoui and M'rad

The zero volatility case: $a(x,t) \equiv 0$

Time-monotone performance process

The forward performance SPDE simplifies to

$$dU(x,t) = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U(x,t)\lambda\right)\right|^{2}}{\mathcal{A}^{2}\,U(x,t)}\,dt$$

The process

$$U\left(x,t\right) = u\left(x,A_t\right) \quad \text{with} \quad A_t = \int_0^t \left|\lambda_s\right|^2 ds$$

and $u(x,t)$ a strictly increasing and concave w.r.t. x function solving

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}}$$

is a solution

MZ (2006, 2009)
Berrier, Rogers and Tehranchi (2009)

Optimal wealth and portfolio processes and a fast diffusion equation

Local risk tolerance function and a fast diffusion equation

$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

System of SDEs at the optimum

$$R_t^* \triangleq r(X_t^*, A_t) \quad \text{and} \quad A_t = \int_0^t |\lambda_s|^2 ds$$

Then

$$\begin{cases} dX_t^* = R_t^* \lambda_t \cdot (\lambda_t dt + dW_t) \\ dR_t^* = r_x(X_t^*, A_t) dX_t^* \end{cases}$$

and the optimal portfolio is $\pi_t^* = R_t^* \sigma_t^+ \lambda_t$

Complete construction

utility inputs, heat eqn and fast diffusion eqn

$$u_t = \frac{1}{2}\frac{u_x^2}{u_{xx}} \qquad \longleftrightarrow \qquad h_t + \frac{1}{2}h_{xx} = 0 \qquad \longleftrightarrow \qquad r_t + \frac{1}{2}r^2 r_{xx} = 0$$

positive solutions to heat eqn and Widder's thm

$$h_x(x,t) = \int_{\mathbb{R}} e^{xy-\frac{1}{2}y^2 t}\nu(dy)$$

optimal wealth process

$$X_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right) \qquad M = \int_0^t \lambda_s \cdot dW_s, \quad \langle M \rangle_t = A_t$$

optimal portfolio process

$$\pi_t^{*,x} = r(X_t^*, A_t)\sigma_t^+\lambda_t = h_x\left(h^{(-1)}\left(X_t^{*,x}, A_t\right), A_t\right)\sigma_t^+\lambda_t$$

Forward performance approach under Knightian uncertainty

Forward robust portfolio criterion

(Kallbald, Obłój, Z.)

- allow flexibility with respect to the investment horizon
- incorporate "learning"
- produce optimal investment strategies closer to the ones used in practice

Forward robust criterion

A pair $(U(x,t), \gamma_{t,T}(Q))$ of a utility process and a penalty criterion which satisfies, for all $0 \leq t \leq T$,

$$U(x,t) = ess \sup_{\alpha} ess \inf_{Q \in \mathcal{Q}_{t,T}} \left(E_Q \left(U(x + \int_t^T \alpha_s \cdot dS_s, T) \middle| \mathcal{F}_t \right) + \gamma_{t,T}(Q) \right)$$

with $\mathcal{Q}_T = \{Q \in \mathcal{Q} : Q|_{\mathcal{F}_T} \sim \mathbb{P}|_{\mathcal{F}_T}\}$

This criterion gives rise to an ill-posed SPDE corresponding to a zero-sum stochastic differential game

Connection with the Kelly criterion

Is there a pair $\left(U\left(x,t\right),\gamma_{t,T}\right)$ that yields the Kelly optimal-growth portfolio?

"True" model $dS_t = S_t\left(\lambda_t dt + \sigma_t dW_t^1\right)$, $\left(W^1, W^2\right)$ under $\mathbb{P}$
"Proxy" model: $dS_t = S_t\left(\hat{\lambda}_t dt + \sigma_t d\hat{W}_t^1\right)$, $\left(W^1, W^2\right)$ under $\hat{\mathbb{P}}$

For each $Q \sim \hat{\mathbb{P}}$ and each $T > 0$, let $\eta_t = (\eta_t^1, \eta_t^2)$, $0 \le t \le T$,

$$\left.\frac{dQ}{d\hat{\mathbb{P}}}\right|_{\mathcal{F}_T} = \mathcal{E}\left(\int_0^{\cdot} \eta_s^1 d\hat{W}_s^1 + \int_0^{\cdot} \eta_s^2 d\hat{W}_s^2\right)_T$$

Doléans-Dade exponential: $\mathcal{E}\left(Y\right)_t = \exp\left(Y_t - \frac{1}{2}\langle Y\rangle_t\right)$

Candidate penalty functionals

$$\gamma_{t,T}(Q) = \mathbb{E}_Q\left(\int_t^T g(\eta_s, s)\,ds \,\middle|\, \mathcal{F}_t\right)$$

Logarithmic risk preferences and quadratic penalty

$$U(x,t) = \ln x - \frac{1}{2}\int_0^t \frac{\delta_s}{1+\delta_s}\hat{\lambda}_s^2 ds, \quad t \geq 0,\ x > 0$$

$$\gamma_{t,T}(Q^\eta) = \mathrm{E}_{Q^\eta}\left(\int_t^T \frac{\delta_s}{2} \left|\eta_s\right|^2 ds \middle| \mathcal{F}_t \right)$$

- The process δ_t is adapted, non–negative and controls the strength of the penalization
- It models the confidence of the investor re the "true" model

(Fractional) Kelly strategies and forward optimal controls

Investor chooses proxy model $(\hat{\lambda}_t)$ and confidence level (δ_t)

Optimal measure Q^{η^*}

$$\eta_t^* = \left(-\frac{\hat{\lambda}_t}{(1+\delta_t)}, 0 \right) \quad \text{and} \quad \frac{dQ^{\eta^*}}{d\hat{\mathbb{P}}} = \mathcal{E}\left(-\int_0^\cdot \frac{\hat{\lambda}_t}{1+\delta_t} d\hat{W}_t^1 \right)_T$$

Optimal forward Kelly portfolio

$$\bar{\pi}_t = \frac{\delta_t}{1+\delta_t} \frac{\hat{\lambda}_t}{\sigma_t}$$

- If $\delta_t \uparrow \infty$ (infinite trust in the estimation), then $\bar{\pi}_t \downarrow \frac{\hat{\lambda}_t}{\sigma_t}$, which is the Kelly strategy associated with the most likely model $\hat{\mathbb{P}}$
- If $\delta_t \downarrow 0$ (no trust in the estimation), then $\bar{\pi}_t \downarrow 0$ and the optimal behavior is to invest nothing in the stock

Open problems

Academic research

study of forward SPDE

existence and uniqueness
for concave, increasing slns

characterize admissible
volatility processes

approx. of slns via finite-dim
Markovian multi-factor processes

ergodic properties of
portfolios and wealth processes

Forward performance approach

Investment practice

reconcile with MV
forward, dynamic construction

reconcile with Black-Litterman
inject manager's views in
the forward volatility and drift

reconcile with Kelly criterion
model ambiguity and
and fractional Kelly

provide a normative platform to
connect these three criteria