Periodic macroscopic bodies

1 Boundary conditions

Suppose that the macroscopic body Ω is formed via periodic replication of the RVE ω. The class of problems in which Ω is subjected to general loading is central to mathematical homogenization theory. Here we are concerned with the problem in which the macroscopic strain and stress fields are constant. That is the value of $\bar{\epsilon}_{ij}$ and $\bar{\sigma}_{ij}$ are same for all replicas of ω. Further, we suppose that the macroscopic strain is prescribed, $\bar{\epsilon}_{ij} = \epsilon^0_{ij}$, while the macroscopic stress is to be determined as part of the solution.

For problems involving periodicity, it is natural to decompose the microscopic fields as

$$\epsilon_{ij}(\boldsymbol{x}) = \bar{\epsilon}_{ij} + \tilde{\epsilon}_{ij}(\boldsymbol{x}) = \epsilon^0_{ij} + \tilde{\epsilon}_{ij}(\boldsymbol{x}) \quad \text{and} \quad \sigma_{ij}(\boldsymbol{x}) = \bar{\sigma}_{ij} + \tilde{\sigma}_{ij}(\boldsymbol{x}) \,. \tag{1}$$

Here the tilde sign denotes the periodic "correction" fields with zero volume averages. Since $\bar{\epsilon}_{ij}$ is constant, the corresponding macroscopic displacement field is linear, and the microscopic displacement field can be expressed as

$$u_i(\boldsymbol{x}) = \bar{\epsilon}_{ij}(x_j - r_j) + \bar{\theta}_{ij}(x_j - r_j) + \tilde{u}_i(\boldsymbol{x}) \,. \tag{2}$$

Here r_i is an arbitrarily chosen reference point and $\bar{\theta}_{ij}$ is a skew-symmetric constant tensor representing rigid-body rotation. Both r_i and $\bar{\theta}_{ij}$ have to be specified independently of $\bar{\epsilon}_{ij}$. However, if r_i and $\bar{\theta}_{ij}$ are not specified, the ensuing indeterminacy is not too significant because these quantities do not affect the stress and strain fields. This indeterminacy parallels that encountered in traction-prescribed boundary-value problems. From now on, we set $r_i = 0$ and $\bar{\theta}_{ij} = 0$.

The principal objective of this section is to derive displacement and traction boundary conditions on $\partial\omega$ corresponding to the prescription $\bar{\epsilon}_{ij} = \epsilon^0_{ij}$, so that it becomes possible to formulate periodic boundary-value problems on the domain ω. Toward this objective we consider ω and its replica ω', such that ω and ω' have a common boundary γ (Fig. 1).

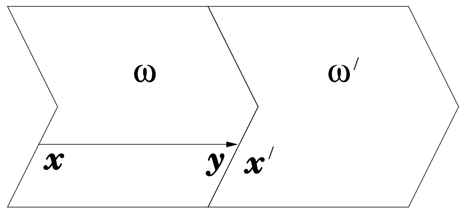

Figure 1: Periodic structure

Next, consider two points $\boldsymbol{y} \in \omega$ and $\boldsymbol{x}' \in \omega'$ that are very close to each other and the surface γ, so that the displacement and traction vectors at these points can be related via the continuity conditions:

$$u_i(\boldsymbol{x}') = u_i(\boldsymbol{y}) \quad \text{and} \quad \sigma_{ij}(\boldsymbol{x}')n_j(\boldsymbol{x}') = -\sigma_{ij}(\boldsymbol{y})n_j(\boldsymbol{y}). \tag{3}$$

The minus sign in the traction continuity condition appears because we use two (opposite) unit normal vectors on γ. Now consider a point $\boldsymbol{x} \in \omega$ such that $\boldsymbol{x}$ and $\boldsymbol{x}'$ are connected by periodicity,

$$x_i' = x_i + l_i, \tag{4}$$

where $\boldsymbol{l}$ is the period-vector. Periodicity implies that $\boldsymbol{x}$ is very close to $\partial\omega$, and the correction fields at $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the same. Thus we can combine (1), (2), and (4) to obtain

$$u_i(\boldsymbol{x}') = u_i(\boldsymbol{x}) + \bar{\epsilon}_{ij}l_j \quad \text{and} \quad \sigma_{ij}(\boldsymbol{x}')n_j(\boldsymbol{x}') = \sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x}). \tag{5}$$

Now the boundary conditions on $\partial\omega$, defined at the points $\boldsymbol{x}$ and $\boldsymbol{y}$, follow from (3)-(5):

$$u_i(\boldsymbol{y}) = u_i(\boldsymbol{x}) + \bar{\epsilon}_{ij}l_j \quad \text{and} \quad \sigma_{ij}(\boldsymbol{y})n_j(\boldsymbol{y}) = -\sigma_{ij}(\boldsymbol{x})n_j(\boldsymbol{x})\,. \tag{6}$$

Let us make two remarks about these boundary conditions. First, in general, periodic structures have several period-vectors. For example, if ω is a cube, then there are three period-vectors. Second, for periodic boundary conditions, the total count of specified versus unspecified boundary data conforms with the usual requirements. In particular, in three dimensions, the boundary conditions impose six equations on six displacement and six traction components, so that, in effect, only one half of the boundary data are specified by boundary conditions while the remaining half must be determined as part of the solution.

2 Cellular solid

As an example, consider a two-dimensional cellular solid formed by identical links arranged into regular triangles. Each link has length $2l$, in-plane thickness t, unit out-of-plane thickness, and Young's modulus E. We assume that the links are slender to the extent that the bending deformation is negligible in comparison to the axial deformation. Also we assume that that the out-of-plane thickness is small, so that the state of plane stress prevails. Our objective is to determine the relationship between the macroscopic stress $\bar{\sigma}_{ij}$ and the macroscopic strain $\bar{\epsilon}_{ij}$. In our development, we consider the macroscopic strain as prescribed, $\bar{\epsilon}_{ij} = \epsilon^0_{ij}$, so that the objective is to determine the macroscopic stress $\bar{\sigma}_{ij}$.

Let us construct the RVE as follows (Fig. 2):

1. Identify a reference node O and its six nearest neighbors.

2. Bisect each of the six links connecting O to its nearest neighbors.

3. The RVE is the six-point star formed as a result of bisection.

We carry out calculations using Cartesian coordinates with the origin at O and the x_1 axis aligned with a pair of links. The lattice vectors which identify directions of the links are

$$\boldsymbol{e}^1 = \{1, 0\}\,, \quad \boldsymbol{e}^2 = \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}\right\}, \quad \text{and} \quad \boldsymbol{e}^3 = \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}\right\}.$$

For a pair of links, (6) yields

$$e_i u_i(l\boldsymbol{e}) - e_i u_i(-l\boldsymbol{e}) = 2l\epsilon^0_{ij} e_i e_j\,, \tag{7}$$

where $\boldsymbol{e}$ can be any of the three lattice vectors. Since rigid body motion is chosen such that $u_\alpha(O) = 0$, (7) can be interpreted as a relationship for the elongations of the two links. Further, since (6) implies that the internal forces in the two links are equal, we obtain

$$\Delta(\boldsymbol{e}) = \Delta(-\boldsymbol{e}) = a\epsilon^0_{ij} e_i e_j\,,$$

where Δ denotes the elongation. From Hooke's law we obtain

$$\sigma(\boldsymbol{e}) = \sigma(-\boldsymbol{e}) = E\epsilon^0_{\alpha\beta} e_\alpha e_\beta\,.$$

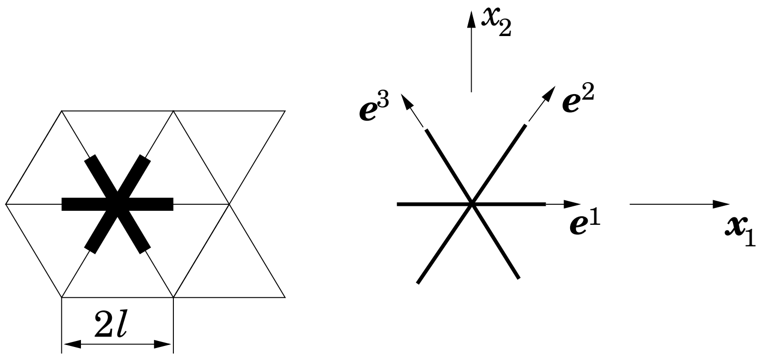

Figure 2: Triangular lattice and its representative volume element.

Further, since the stress field in the links is uniaxial tension, the stress tenor in the links is

$$\sigma_{ij}(\boldsymbol{e}) = \sigma_{ij}(-\boldsymbol{e}) = E\epsilon^0_{kl} e_i e_j e_k e_l \, .$$

Now the macroscopic stress is straightforward to calculate as

$$\bar{\sigma}_{ij} = \frac{1}{A} \sum_{m=1}^{3} \sigma_{ij}(\boldsymbol{e}^m)(2tl) = \left(\frac{2tlE}{A} \sum_{i=m}^{3} e^m_i e^m_j e^m_k e^m_l \right) \epsilon^0_{kl} = C^*_{ijkl} \epsilon^0_{kl} \, .$$

Here $A = 3\sqrt{3}l^2/2$ is the RVE area rather than volume; the vacuous phase is included. The fourth-rank tensor C^*_{ijkl} is the overall stiffness of the cellular solid, which can be expressed as

$$C^*_{ijkl} = \frac{tE}{4\sqrt{3}l}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) = \frac{E\phi}{4}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where ϕ is the volume fraction occupied by the solid phase. This stiffness tensor represents an isotropic elastic response with the elastic constants

$$E^* = \frac{2E\phi}{3} \quad G^* = \frac{E\phi}{4} \quad \text{and} \quad \nu^* = \frac{1}{3}.$$

3 Perforated plate

This problem involves a large (infinite) plate containing circular holes of the same size. The holes are arranged in a square array, so that their centers

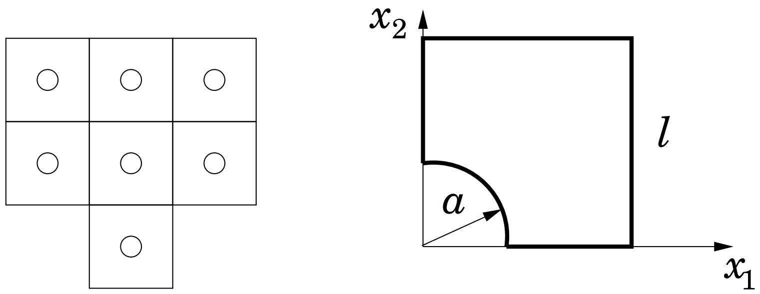

Figure 3: Perforated plate and its representative volume element

are at the locations $(2li_1, 2li_2)$, where i_1 and i_2 are integers and l is a given length. The plate is subjected to biaxial straining, so that $\epsilon^0_{12} = 0$ (Fig. 3).. Our objective is to derive the periodic boundary conditions. Furthermore, this has to be done by exploiting problem symmetry.

Let us begin by observing that the given problem is symmetric with respect to the coordinate axis. Formally, the symmetry conditions can be imposed using a projection operator

$$\pi_{ij} = \delta_{ij} - 2n_i n_j \ ,$$

where n_i is a unit normal to the symmetry line. Then points $\boldsymbol{x}$ and $\boldsymbol{y}$ are symmetric if $x_i = \pi_{ij} y_j$ or $y_i = \pi_{ij} x_j$. Further, this structure extends to all vector-valued fields. In particular,

$$u_i(\boldsymbol{x}) = \pi_{ij} u_j(\boldsymbol{y}) \quad u_i(\boldsymbol{y}) = \pi_{ij} u_j(\boldsymbol{x}),$$

and

$$t_i(\boldsymbol{x}) = \pi_{ij} t_j(\boldsymbol{y}) \quad t_i(\boldsymbol{y}) = \pi_{ij} t_j(\boldsymbol{x}).$$

Under general loading, the RVE ω is a $2l \times 2l$ square containing a hole of radius a at its center. However, this RVE does not take into account symmetry of the macroscopic stress tensor, and therefore it can be regarded as sub-optimal. To determine how symmetry affects periodic boundary conditions, consider two points $\boldsymbol{y} \in \partial\omega$ separated by the period-vector $2l\delta_{i1}$. These points are symmetric with respect to the x_2 axis, so that $\pi_{ij} = \delta_{ij} - 2\delta_{i1}\delta_{j1}$. Now we can combine the periodicity condition,

$$u_i(\boldsymbol{x}) = u_i(\boldsymbol{y}) + 2l\epsilon^0_{i1} \quad t_i(\boldsymbol{x}) = -t_i(\boldsymbol{y}) \ ,$$

with the symmetry condition

$$u_i(\boldsymbol{x}) = \pi_{ij} u_j(\boldsymbol{y}) \quad t_i(\boldsymbol{x}) = \pi_{ij} t_j(\boldsymbol{y}) \ ,$$

to obtain

$$u_1(\boldsymbol{x}) = l\bar{\epsilon}_{11} \text{ and } t_2(\boldsymbol{x}) = 0.$$

These boundary conditions give represents classical "roller" boundary conditions. Analogously, one can consider two points separated by the period-vector $l\delta_{i2}$.

In summary, the boundary-value problem can be stated on the first quadrant ($x_1 \geq 0,\ x_2 \geq 0$) only, and the boundary conditions are

$$\begin{aligned}
u_2 = 0 \quad t_1 &= 0 \quad \text{on } x_2 = 0 \\
u_1 = 0 \quad t_2 &= 0 \quad \text{on } x_1 = 0 \\
u_2 = l\bar{\epsilon}^0_{22} \quad t_1 &= 0 \quad \text{on } x_2 = l \\
u_1 = l\bar{\epsilon}^0_{11} \quad t_2 &= 0 \quad \text{on } x_1 = l \\
t_1 = 0 \quad t_2 &= 0 \quad \text{on } x_1^2 + x_2^2 = a^2
\end{aligned}$$

Note that all boundary conditions are classical in the sense that all of them are imposed at one point at a time.