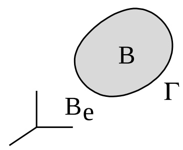
CAM 397: Exercises on the Stokes and Lamm equations

1. Consider the slow, diffusive motion of a body B in an incompressible, viscous fluid of viscosity μ. Let B_e and Γ denote the region exterior to and bounding surface of B. For a given boundary velocity $v : \Gamma \to \mathbb{R}^3$, the fluid velocity $u : B_e \to \mathbb{R}^3$ and pressure $p : B_e \to \mathbb{R}$ satisfy the Stokes flow equations



$$\begin{aligned} \mu\Delta u &= \nabla p & x \in B_e \\ \nabla \cdot u &= 0 & x \in B_e \\ u &= v & x \in \Gamma \\ u, p &\to 0 & |x| \to \infty. \end{aligned} \tag{1}$$

Under mild assumptions on Γ and v, these equations have a unique solution (u, p) with the decay properties

$$u = O(|x|^{-1}), \quad \nabla u = O(|x|^{-2}), \quad p = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty. \tag{2}$$

When the body B is rigid, the boundary velocity v in $(1)_3$ takes the general form

$$v = V + \Omega \times (x - c), \tag{3}$$

where V is the linear velocity of a given reference point c, and Ω is the angular velocity of the body. The fluid stress $\sigma : B_e \to \mathbb{R}^{3\times3}$ associated with (u, p) is defined by

$$\sigma = -pI + \mu(\nabla u + \nabla u^T), \tag{4}$$

where $I \in \mathbb{R}^{3\times3}$ is the identity matrix, and the traction $f : \Gamma \to \mathbb{R}^3$ exerted by the fluid on the body surface (force per unit area) is defined by

$$f = \sigma\nu, \tag{5}$$

where ν is the outward unit normal on Γ. The resultant force F and torque T, about the reference point c, of the fluid on the body are then given by

$$F = \int_\Gamma f(x)\, dA_x, \qquad T = \int_\Gamma (x - c) \times f(x)\, dA_x. \tag{6}$$

Here we study the relation between the loads (F, T) and the velocity data (V, Ω).

(a) Show there exists a matrix $L \in \mathbb{R}^{6\times6}$, depending only on B, such that $\begin{bmatrix} F \\ T \end{bmatrix} = -L \begin{bmatrix} V \\ \Omega \end{bmatrix}$.

(b) Using the usual summation convention, show that $(1)_1$ is equivalent to $\sigma_{ij,j} = 0$ in B_e.

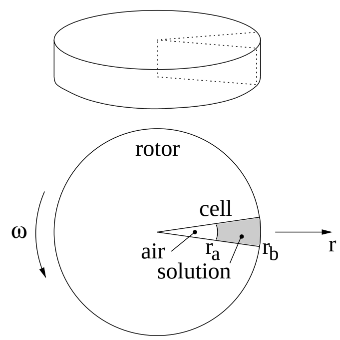
(c) Let (u, p) be the flow associated with (V, Ω), let $(\hat{u}, \hat{p})$ be associated with $(\hat{V}, \hat{\Omega})$, and for any square matrix A let $\operatorname{sym} A = \frac{1}{2}(A + A^T)$. Use (1)–(6) and the Divergence Theorem to establish the identity

$$\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot \begin{bmatrix} \hat{F} \\ \hat{T} \end{bmatrix} = -\frac{\mu}{2} \int_{B_e} [\operatorname{sym} \nabla u]_{ij} [\operatorname{sym} \nabla \hat{u}]_{ij}\, dV.$$

(d) Show that $L \in \mathbb{R}^{6\times6}$ is symmetric, that is, $\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot L \begin{bmatrix} \hat{V} \\ \hat{\Omega} \end{bmatrix} = \begin{bmatrix} \hat{V} \\ \hat{\Omega} \end{bmatrix} \cdot L \begin{bmatrix} V \\ \Omega \end{bmatrix}, \quad \forall \begin{bmatrix} V \\ \Omega \end{bmatrix}, \begin{bmatrix} \hat{V} \\ \hat{\Omega} \end{bmatrix} \in \mathbb{R}^6.$

(e) Show that $L \in \mathbb{R}^{6\times6}$ is positive-definite, that is, $\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot L \begin{bmatrix} V \\ \Omega \end{bmatrix} > 0, \quad \forall \begin{bmatrix} V \\ \Omega \end{bmatrix} \in \mathbb{R}^6, \quad \begin{bmatrix} V \\ \Omega \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

2. A model for the concentration $\rho(r,t)$ of macromolecules in solution inside a spinning ultracentrifuge cell is given by the Lamm equations



$$\begin{aligned}
&\frac{\partial \rho}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left[rD\frac{\partial \rho}{\partial r} - S\omega^2 r^2 \rho \right], && r_a < r < r_b, \quad t > 0 \\
&D\frac{\partial \rho}{\partial r} = S\omega^2 r\rho, && r = r_a, \quad t > 0 \\
&D\frac{\partial \rho}{\partial r} = S\omega^2 r\rho, && r = r_b, \quad t > 0 \\
&\rho = \rho_0, && r_a \le r \le r_b, \quad t = 0.
\end{aligned} \tag{7}$$

In this system, r is the radial coordinate from the axis of rotation and t is time. The positive constants ω, ρ_0, D and S are the angular velocity of the rotor, the initial concentration, and the diffusion and sedimentation coefficients of the macromolecule. Here we study a leading-order (outer) approximation of (7) in the case when D is small compared to $S\omega^2 r_a$, and solve it by the method of characteristics.

(a) In the case when $D \ll S\omega^2 r_a$, a leading-order approximation of (7) is

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}\left[S\omega^2 r^2 \rho \right], && r > r_a, \quad t > 0 \\
&0 = S\omega^2 r\rho, && r = r_a, \quad t > 0 \\
&\rho = \rho_0, && r \ge r_a, \quad t = 0.
\end{aligned} \tag{8}$$

Briefly explain why solutions of (7) are expected to satisfy (8) when r is near r_a, equivalently, away from r_b.

(b) Rewrite $(8)_1$ in the first-order form

$$\alpha \frac{\partial \rho}{\partial r} + \beta \frac{\partial \rho}{\partial t} = \gamma \rho, \tag{9}$$

where α, β and γ are coefficients depending on r, and define a change of variables $r = r(\zeta, \eta)$, $t = t(\zeta, \eta)$ by

$$\frac{\partial r}{\partial \eta} = \alpha, \quad \frac{\partial t}{\partial \eta} = \beta, \quad r(\zeta, 0) = \zeta, \quad t(\zeta, 0) = 0. \tag{10}$$

Solve (10) for $r = r(\zeta, \eta)$ and $t = t(\zeta, \eta)$, and invert these relations to obtain $\zeta = \zeta(r,t)$ and $\eta = \eta(r,t)$.

(c) Show that (9) can be rewritten in the form

$$\frac{\partial \hat{\rho}}{\partial \eta} = \gamma \hat{\rho}, \tag{11}$$

where $\hat{\rho}(\zeta, \eta) = \rho(r,t)|_{r=r(\zeta,\eta), t=t(\zeta,\eta)}$. By integrating this equation, find the general solution $\rho(r,t)$ of (9), equivalently, $(8)_1$.

(d) Use the result from (c) to show that the solution of the initial-boundary value problem (8) is given by

$$\rho(r,t) = \begin{cases} 0, & r < r_a e^{S\omega^2 t} \\ \rho_0 e^{-2S\omega^2 t}, & r \ge r_a e^{S\omega^2 t}. \end{cases} \tag{12}$$

(e) Briefly explain how data of ρ versus r at various times t can be used to determine the sedimentation coefficient S.