CAM 397: Exercises on the Stokes and Lamm equations

1. Consider the slow, diffusive motion of a body $B$ in an incompressible, viscous fluid of viscosity $\mu$. Let $B_e$ and $\Gamma$ denote the region exterior to and bounding surface of $B$. For a given boundary velocity $v : \Gamma \rightarrow \mathbb{R}^3$, the fluid velocity $u : B_e \rightarrow \mathbb{R}^3$ and pressure $p : B_e \rightarrow \mathbb{R}$ satisfy the Stokes flow equations

\[
\begin{align*}
\mu \Delta u &= \nabla p \quad x \in B_e \\
\nabla \cdot u &= 0 \quad x \in B_e \\
u &= v \quad x \in \Gamma \\
\nabla u &= 0 \quad x \in \Gamma \\
u, p &\rightarrow 0 \quad |x| \rightarrow \infty.
\end{align*}
\]

(1)

Under mild assumptions on $\Gamma$ and $v$, these equations have a unique solution $(u, p)$ with the decay properties

\[
u = O(|x|^{-1}), \quad \nabla u = O(|x|^{-2}), \quad p = O(|x|^{-2}) \quad \text{as} \quad |x| \rightarrow \infty.
\]

(2)

When the body $B$ is rigid, the boundary velocity $v$ in (1)_3 takes the general form

\[
v = V + \Omega \times (x - c),
\]

(3)

where $V$ is the linear velocity of a given reference point $c$, and $\Omega$ is the angular velocity of the body. The fluid stress $\sigma : B_e \rightarrow \mathbb{R}^{3 \times 3}$ associated with $(u, p)$ is defined by

\[
\sigma = -pI + \mu (\nabla u + \nabla u^T),
\]

(4)

where $I \in \mathbb{R}^{3 \times 3}$ is the identity matrix, and the traction $f : \Gamma \rightarrow \mathbb{R}^3$ exerted by the fluid on the body surface (force per unit area) is defined by

\[
f = \sigma \nu,
\]

(5)

where $\nu$ is the outward unit normal on $\Gamma$. The resultant force $F$ and torque $T$, about the reference point $c$, of the fluid on the body are then given by

\[
F = \int_{\Gamma} f(x) \, dA_x, \quad T = \int_{\Gamma} (x - c) \times f(x) \, dA_x.
\]

(6)

Here we study the relation between the loads $(F, T)$ and the velocity data $(V, \Omega)$.

(a) Show there exists a matrix $L \in \mathbb{R}^{6 \times 6}$, depending only on $B$, such that $\begin{bmatrix} F \\ T \end{bmatrix} = -L \begin{bmatrix} V \\ \Omega \end{bmatrix}$.

(b) Using the usual summation convention, show that (1)_1 is equivalent to $\sigma_{ij,j} = 0$ in $B_e$.

(c) Let $(u, p)$ be the flow associated with $(V, \Omega)$, let $(\tilde{u}, \tilde{p})$ be associated with $(\tilde{V}, \tilde{\Omega})$, and for any square matrix $A$ let $\text{sym} A = \frac{1}{2}(A + A^T)$. Use (1)–(6) and the Divergence Theorem to establish the identity

\[
\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot \begin{bmatrix} \tilde{F} \\ \tilde{T} \end{bmatrix} = -\frac{\mu}{2} \int_{B_e} \text{sym} \nabla u_{ij}[\text{sym} \nabla \tilde{u}]_{ij} \, dV.
\]

(d) Show that $L \in \mathbb{R}^{6 \times 6}$ is symmetric, that is, $\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot L \begin{bmatrix} V \\ \Omega \end{bmatrix} = \begin{bmatrix} \tilde{V} \\ \tilde{\Omega} \end{bmatrix} \cdot L \begin{bmatrix} \tilde{V} \\ \tilde{\Omega} \end{bmatrix}, \quad \forall \begin{bmatrix} V \\ \Omega \end{bmatrix}, \begin{bmatrix} \tilde{V} \\ \tilde{\Omega} \end{bmatrix} \in \mathbb{R}^6$.

(e) Show that $L \in \mathbb{R}^{6 \times 6}$ is positive-definite, that is, $\begin{bmatrix} V \\ \Omega \end{bmatrix} \cdot L \begin{bmatrix} V \\ \Omega \end{bmatrix} > 0, \quad \forall \begin{bmatrix} V \\ \Omega \end{bmatrix} \in \mathbb{R}^6, \quad \begin{bmatrix} V \\ \Omega \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
2. A model for the concentration $\rho(r, t)$ of macromolecules in solution inside a spinning ultracentrifuge cell is given by the Lamm equations

$$\frac{\partial \rho}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left[ rD \frac{\partial \rho}{\partial r} - S\omega^2 r^2 \rho \right], \quad r_a < r < r_b, \quad t > 0$$

$$D \frac{\partial \rho}{\partial r} = S\omega^2 r \rho, \quad r = r_a, \quad t > 0$$

$$D \frac{\partial \rho}{\partial r} = S\omega^2 r \rho, \quad r = r_b, \quad t > 0$$

$$\rho = \rho_0, \quad r_a \leq r \leq r_b, \quad t = 0.$$  \hspace{1cm} (7)

In this system, $r$ is the radial coordinate from the axis of rotation and $t$ is time. The positive constants $\omega$, $\rho_0$, $D$ and $S$ are the angular velocity of the rotor, the initial concentration, and the diffusion and sedimentation coefficients of the macromolecule. Here we study a leading-order (outer) approximation of (7) in the case when $D$ is small compared to $S\omega^2 r_a$, and solve it by the method of characteristics.

(a) In the case when $D \ll S\omega^2 r_a$, a leading-order approximation of (7) is

$$\frac{\partial \rho}{\partial t} = -\frac{1}{r} \frac{\partial}{\partial r} \left[ S\omega^2 r^2 \rho \right], \quad r > r_a, \quad t > 0$$

$$0 = S\omega^2 r \rho, \quad r = r_a, \quad t > 0$$

$$\rho = \rho_0, \quad r_a \leq r \leq r_b, \quad t = 0.$$  \hspace{1cm} (8)

Briefly explain why solutions of (7) are expected to satisfy (8) when $r$ is near $r_a$, equivalently, away from $r_b$.

(b) Rewrite (8) in the first-order form

$$\alpha \frac{\partial \rho}{\partial r} + \beta \frac{\partial \rho}{\partial t} = \gamma \rho,$$  \hspace{1cm} (9)

where $\alpha$, $\beta$ and $\gamma$ are coefficients depending on $r$, and define a change of variables $r = r(\zeta, \eta)$, $t = t(\zeta, \eta)$ by

$$\frac{\partial r}{\partial \eta} = \alpha, \quad \frac{\partial t}{\partial \eta} = \beta, \quad r(\zeta, 0) = \zeta, \quad t(\zeta, 0) = 0.$$  \hspace{1cm} (10)

Solve (10) for $r = r(\zeta, \eta)$ and $t = t(\zeta, \eta)$, and invert these relations to obtain $\zeta = \zeta(r, t)$ and $\eta = \eta(r, t)$.

(c) Show that (9) can be rewritten in the form

$$\frac{\partial \hat{\rho}}{\partial \eta} = \gamma \hat{\rho},$$  \hspace{1cm} (11)

where $\hat{\rho}(\zeta, \eta) = \rho(r, t)|_{r=r(\zeta, \eta), t=t(\zeta, \eta)}$. By integrating this equation, find the general solution $\rho(r, t)$ of (9), equivalently, (8).

(d) Use the result from (c) to show that the solution of the initial-boundary value problem (8) is given by

$$\rho(r, t) = \begin{cases} 
0, & r < r_a e^{S\omega^2 t} \\
\rho_0 e^{-2S\omega^2 t}, & r \geq r_a e^{S\omega^2 t}.
\end{cases}$$  \hspace{1cm} (12)

(e) Briefly explain how data of $\rho$ versus $r$ at various times $t$ can be used to determine the sedimentation coefficient $S$. 

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