

CAM 397 AREA C EXERCISES

September 3–10, 2008

1. Consider the 1-D problem

$$\begin{aligned} -(ap')' &= 0 \quad 0 < x < 1 \\ p(0) &= 1 \text{ and } p(1) = 0 \end{aligned}$$

(a) Integrate twice to find the solution.

(b) Find a constant \bar{a} such that the velocity $u = -ap'$ is close to the velocity $\bar{u} = -\bar{a}\bar{p}'$, where \bar{p} solves the same type of equation with \bar{a} in place of a . How close did you get? Why?

2. Consider modeling a porous medium with a fracture. Discuss the meaning of representative elementary volume (REV) for this case. Does it make sense to have an REV?

3. Consider modeling oil and water in a porous medium. Assume both fluids are incompressible (i.e., their densities are constant) and they “mix” without changing volume. Let $s = s_w$ be the ratio of the volume of water to the total fluid volume, called the water saturation (so then the oil saturation is $s_o = 1 - s$).

(a) The two fluids form distinct phases, with different pressures. As a first approximation, we assume that the pressure difference

$$p_o - p_w = p_c(s)$$

is a function p_c , called the capillary pressure of the saturation, which must be measured.

(b) The two fluids interfere with each other as they try to squeeze between pores. We model this by assuming that

$$k_w = kk_{rw}(s) \quad \text{and} \quad k_o = kk_{ro}(s)$$

where k_{rw} and k_{ro} must be measured. What can you say about these two functions? Why should we assume that $k_{rw} + k_{ro} \leq 1$?

4. Consider the 1-D problem

$$\begin{aligned} -(a_\epsilon p'_\epsilon)' &= 0 \quad 0 < x < 1 \\ p_\epsilon(0) &= 1 \text{ and } p_\epsilon(1) = 0 \end{aligned}$$

where a_ϵ is periodic of period 1.

(a) Find the homogenized coefficient \hat{a} (Hint: it is the harmonic average).

(b) Show that the solution is

$$p_\epsilon(x) = 1 - \hat{a} \int_0^x \frac{dy}{a_\epsilon(y)} \quad \text{and} \quad u_\epsilon(x) = -a_\epsilon p'_\epsilon = \hat{a}.$$

and the homogenized solution is

$$p_0(x) = 1 - \hat{a} \int_0^x \frac{dy}{a_0(y)} \quad \text{and} \quad u_0(x) = -\hat{a} p'_0 = \hat{a}.$$

Verify that

$$\left\{ \int_0^1 |p_\epsilon(x) - p_0(x)|^2 dx \right\}^{1/2} = \|p_\epsilon - p_0\|_{L^2(0,1)} = O(\epsilon).$$

5. Verify the following by direct calculation. [Hint: It is sufficient to test the expression with a function in C^∞ .]

(a) For $0 < x < 1$,

$$f_\epsilon(x) = f(x, x/\epsilon) = e^x[1 + \cos(x/\epsilon)] \rightarrow e^x .$$

(b) For $0 < x < 1$ and $0 < y < 1$,

$$f_\epsilon(x, y) = \cos^2(2\pi(x + y)/\epsilon) \rightarrow 1/2 .$$

6. Use formal homogenization to find the limit function u_0 and the homogenized problem it satisfies for the problem

$$\begin{aligned} -\nabla \cdot (a(x, x/\epsilon)\nabla u_\epsilon) + b(x, x/\epsilon) \cdot \nabla u_\epsilon + c(x, x/\epsilon)u_\epsilon &= q(x, x/\epsilon) & \text{in } \Omega , \\ u_\epsilon &= 0 & \text{on } \partial\Omega . \end{aligned}$$

Assume that there exists a unique solution and that $a(x, y)$, $b(x, y)$, $c(x, y)$, and $q(x, y)$ are smooth and periodic in y on the unit cube Y .

7. Suppose that $a(y) : [0, 1]^2 \rightarrow \mathbb{R}$ is defined as

$$a(y) = \begin{cases} a_1, & y_2 \geq 1/2, \\ a_2, & y_2 < 1/2, \end{cases}$$

and extended periodically. Note that this is a layered medium. Show that the homogenized coefficient is

$$\hat{a} = \begin{pmatrix} \frac{1}{2}(a_1 + a_2) & 0 \\ 0 & 2(a_1^{-1} + a_2^{-1})^{-1} \end{pmatrix},$$

which are the extremes of the Voigt-Reiss inequality (arithmetic versus harmonic averaging). [Hint: $\nabla \cdot (ae_j) = \partial a / \partial y_j$. But $a(y) = a(y_1, y_2) = a(y_2)$ only, so ω_1 is constant and $\omega_2(y) = \omega_2(y_2)$ only. Find ω_2 by integrating the differential equation that defines it.]

8. In this exercise, you give a formal derivation of Darcy's Law from homogenization theory. Begin with an elementary representative cell Y , consisting of the solid part Y_s and the fluid region Y_f . Let $\epsilon > 0$ and assume that the domain Ω is a union of translates of ϵY , with fluid part Ω_f being the union of translates of ϵY_f . The fluid obeys the Stokes equation

$$\begin{aligned} \epsilon^2 \mu \Delta v^\epsilon &= \nabla p^\epsilon & x \in \Omega_f, \\ \nabla \cdot v^\epsilon &= 0 & x \in \Omega_f, \\ v^\epsilon &= 0 & x \in \partial\Omega_f, \end{aligned}$$

where μ is the constant fluid viscosity. Formally expand both p^ϵ and v^ϵ and let $\epsilon \rightarrow 0$. You should obtain that the leading terms in the formal expansion, $u_0(x, y)$ and $p_0(x)$, satisfy a Darcy law

$$\int_{Y_f} u_0(x, y) dy = -\frac{1}{\mu} K \nabla_x p_0(x),$$

and also that $\nabla \cdot \int_{Y_f} u_0(x, y) dy = 0$. Show these facts, and also show how we can determine the permeability constant K .