Introduction to Aspects of Multiscale Modeling as Applied to Porous Media

Part III

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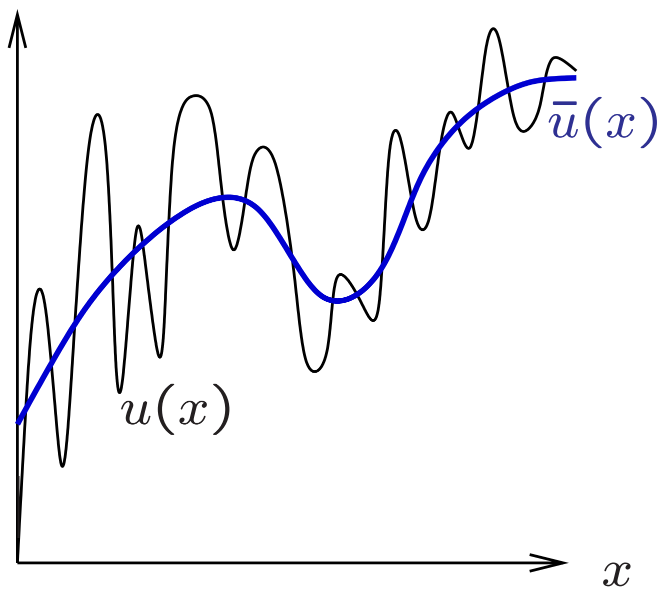
Mathematical Homogenization

Periodicity

The solution u has high frequency wiggles due to the heterogeneity of k.



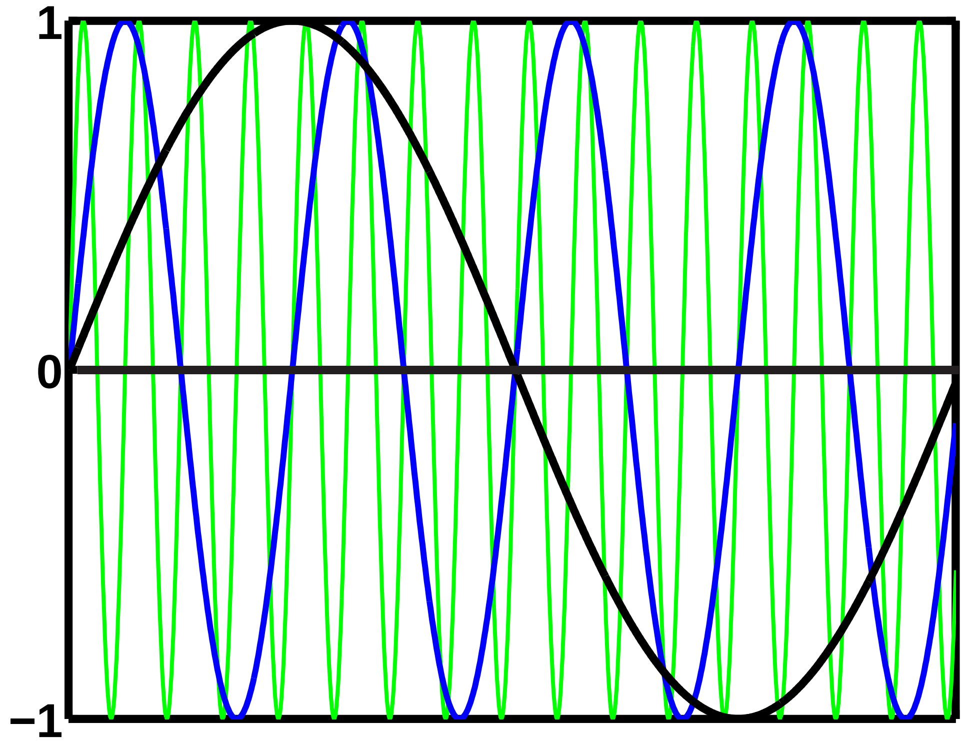
We want the average behavior! Can we find $\bar{u}(x)$ without knowing $u(x)$? The wiggles are irregular, so they are hard to deal with.

Idea 1: Assume that the heterogeneity is periodic, so that the wiggles are regular, and thus easily identified. This is basically our closure assumption.

Idea 2: Let the period of oscillation be ϵ, and let $\epsilon \to 0$. This should remove the wiggles.

Example: Let $u^\epsilon(x) = \sin(x/\epsilon)$. As $\epsilon \to 0$, $\sin(x/\epsilon)$ does not converge, it just oscillates more and more between -1 and 1.

But it becomes a blur, so is it not 0? (At least in *some* sense?)

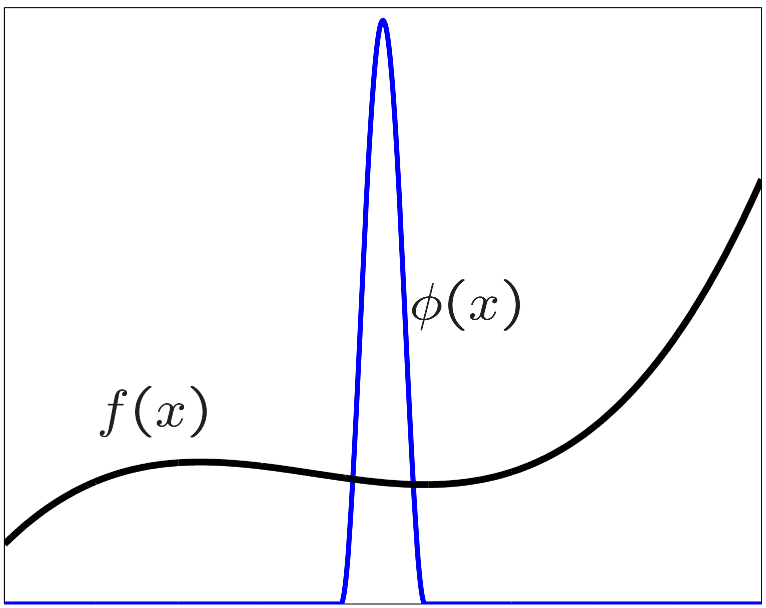


Idea 3: Consider a weaker form of convergence.

Theorem: If

$$\int_0^1 f(x)\,\phi(x)\,dx = \int_0^1 g(x)\,\phi(x)\,dx \qquad \forall \phi \in C_0^\infty$$

then $f = g$.



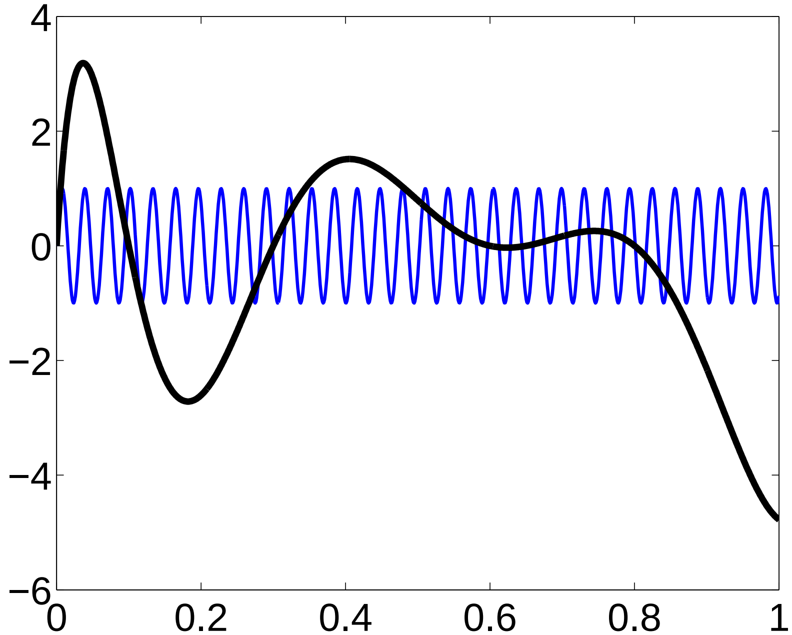
We "test" f by multiplying by ϕ and integrating. If we know all such tests, we know f.

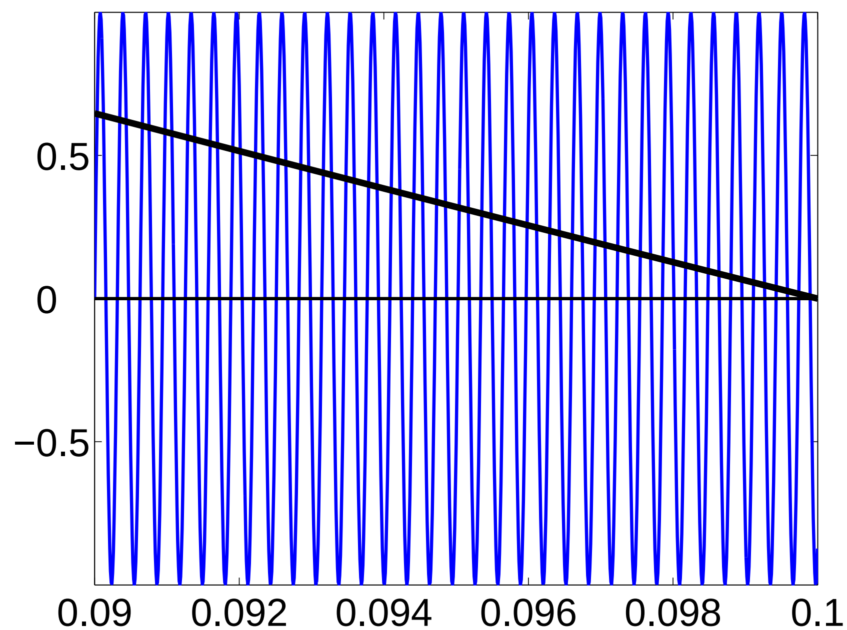
Convergence of Wiggles—2

Let $\phi(x) \in C_0^\infty$ and consider

$$\int_0^1 u^\epsilon(x)\,\phi(x)\,dx = \int_0^1 \sin(x/\epsilon)\,\phi(x)\,dx = \int_0^1 \epsilon\cos(x/\epsilon)\,\phi'(x)\,dx \longrightarrow 0$$

We call this *weak convergence*, and write $u^\epsilon \rightharpoonup 0$.





Weak Convergence

Definition: If

$$\lim_{\epsilon \longrightarrow 0} \int u^\epsilon(x)\, \phi(x)\, dx = \int u(x)\, \phi(x)\, dx \qquad \forall \phi \in C_0^\infty$$

then u^ϵ converges *weakly* to u. We write $u^\epsilon \rightharpoonup u$.

Definition: The L^2-norm of a function u is

$$\|u\| \equiv \left\{ \int |u(x)|^2\, dx \right\}^{1/2}$$

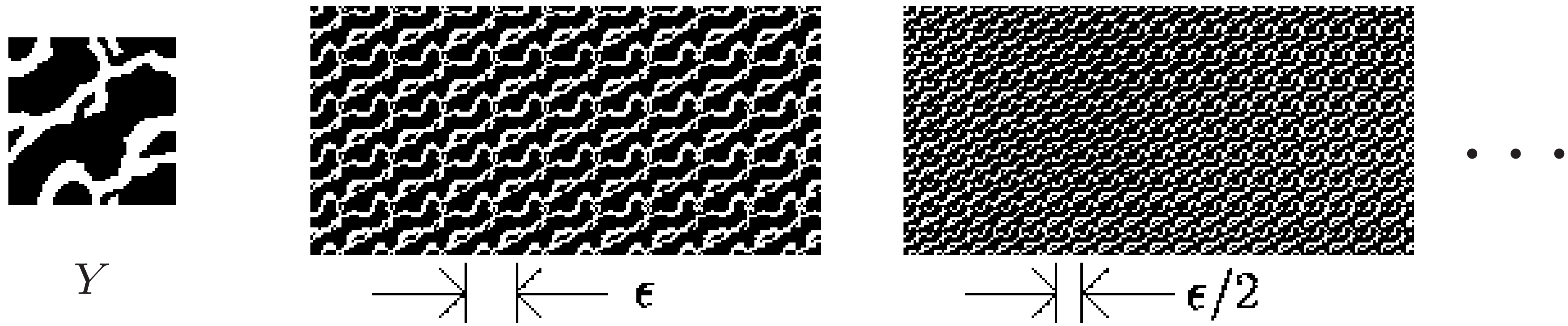
Theorem: If there is $C > 0$ *independent* of ϵ such that

$$\|u^\epsilon\| \leq C < \infty$$

then there exists u such that $u^\epsilon \rightharpoonup u$.

Obtaining Periodic Wiggles

Suppose that the domain Ω has a periodic structure with period ϵY. As $\epsilon \to 0$, we obtain our macro-scale model for the average flow.



Question: How do we proceed?

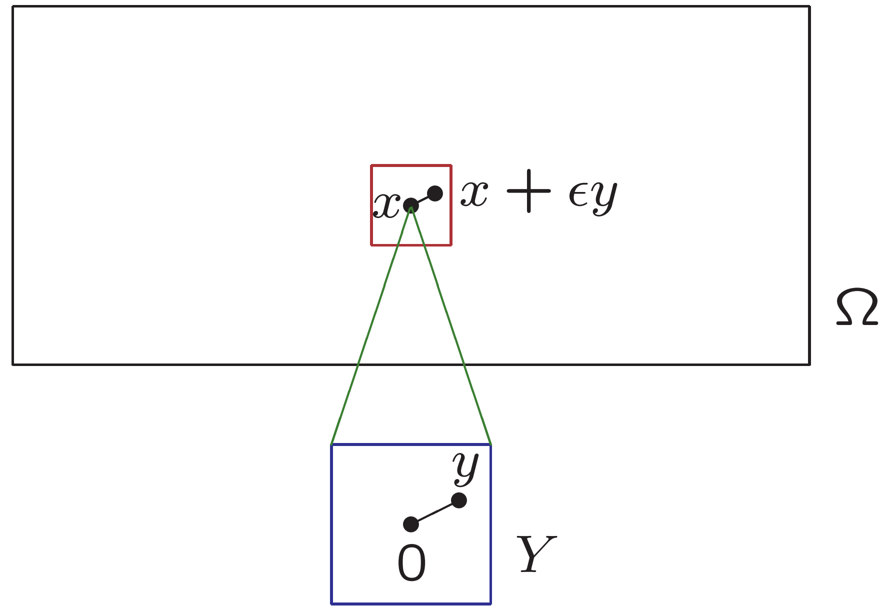
Homogenization is very mathematical, and involves deep analysis and partial differential equations.

Fortunately there is a simpler, more physical way to view homogenization.

Scaling: We assume that the space variable has both a slow (x) and fast (y) component.

$$x \sim x + \epsilon y$$

At any point x, y allows us to "see" the local details, which disappear as $\epsilon \to 0$.



Formal assumption: We *assume* without proof that we can expand the true solution $p(x)$ into a power series involving ϵ:

$$p(x) \sim p_0(x,y) + \epsilon p_1(x,y) + \epsilon^2 p_2(x,y) + \cdots$$

wherein $y = x/\epsilon$ and each p_k is periodic in y.

Gradient scaling: Then

$$\nabla \sim \nabla_x + \epsilon^{-1} \nabla_y$$

Recall the model of Darcy flow (ignore outer BCs):

$$-\nabla \cdot (k_\epsilon \nabla p_\epsilon) = f \quad \text{in } \Omega$$

Make the closure assumption that

$$k_\epsilon(x) = k(x, x/\epsilon),$$

where $k(x, y)$ is periodic of period 1 in y, so $k_\epsilon(x)$ has periodic oscillations of period ϵ. Note that $k_\epsilon(x)$ can vary slowly over the domain with only the local variability being periodic.

Formal expansion: Substitute

$$p_\epsilon(x) = p_0(x, y) + \epsilon\, p_1(x, y) + \epsilon^2 p_2(x, y) + \cdots \quad \text{and} \quad \nabla = \nabla_x + \epsilon^{-1}\nabla_y$$

to find

$$\begin{aligned} -\,(\epsilon^{-1}\nabla_y + \nabla_x) \cdot \Big[k(x, y)(\epsilon^{-1}\nabla_y + \nabla_x)\Big(p_0(x, y) \\ + \epsilon\, p_1(x, y) + \epsilon^2 p_2(x, y) + \cdots\Big)\Big] = f \quad \text{in } \Omega \times Y \end{aligned}$$

We rewrite our expansion

$$-(\epsilon^{-1}\nabla_y + \nabla_x)\cdot\Big[k(x,y)(\epsilon^{-1}\nabla_y + \nabla_x)\Big(p_0(x,y) + \epsilon\, p_1(x,y) + \epsilon^2 p_2(x,y) + \cdots\Big)\Big] = f \quad \text{in } \Omega\times Y$$

as a power series in ϵ

$$\begin{aligned}
&\epsilon^{-2}\Big\{-\nabla_y\cdot\Big[k(x,y)\nabla_y p_0(x,y)\Big]\Big\} \\
&+\epsilon^{-1}\Big\{-\nabla_y\cdot\Big[k(x,y)\Big(\nabla_y p_1(x,y)+\nabla_x p_0(x,y)\Big)\Big] - \nabla_x\cdot\Big[k(x,y)\nabla_y p_0(x,y)\Big]\Big\} \\
&+\epsilon^{0}\Big\{-\nabla_y\cdot\Big[k(x,y)\Big(\nabla_y p_2(x,y)+\nabla_x p_1(x,y)\Big)\Big] - \nabla_x\cdot\Big[k(x,y)\Big(\nabla_y p_1(x,y)+\nabla_x p_0(x,y)\Big)\Big]\Big\} \\
&+\cdots = f \quad \text{in } \Omega\times Y
\end{aligned}$$

This should hold for all ϵ as $\epsilon\to 0$, so it must hold for each term.

Step 1, ϵ^{-2}-terms:

$$\begin{cases} -\nabla_y \cdot \Big[k(x,y)\nabla_y p_0(x,y)\Big] = 0 \quad \text{in } \Omega \times Y \\ \quad p_0(x,y) \text{ is periodic in } y \end{cases}$$

Note that there are no derivatives in x, so x is just a parameter. We have a partial differential equation (PDE) in y only. It is not particularly difficult to see that $p_0(x,y)$ must be constant in y.

Conclusion:

$$\boxed{p_0 = p_0(x) \text{ only.}}$$

Question: Why is this result important? That is, why should the leading order of the solution not depend on y?

Step 2, ϵ^{-1}-terms:

$$-\nabla_y \cdot \Big[k(x,y)\Big(\nabla_y p_1(x,y) + \nabla_x p_0(x,y)\Big)\Big] \\ - \nabla_x \cdot \Big[k(x,y)\nabla_y p_0(x,y)\Big] \quad \text{in } \Omega \times Y$$

From Step 1, one term vanishes. Thus

$$\begin{cases} -\nabla_y \cdot \Big[k(x,y)\nabla_y p_1(x,y)\Big] = \nabla_y \cdot \Big[k(x,y)\nabla p_0(x)\Big] & \text{in } \Omega \times Y \\ p_1(x,y) \text{ is periodic in } y \end{cases}$$

Again x is basically a parameter, and this is a PDE in y for $p_1(x,y)$, if we are given $\nabla_x p_0(x)$. But $\nabla_x p_0(x)$ is a constant vector in y!

Trick: Use the linearity! Since

$$\nabla p_0 = \sum_j \mathbf{e}_j \partial_j p_0,$$

replace ∇p_0 by $\mathbf{e}_j$ and solve.

Homogenization of Darcy Flow—5

We solve (for each fixed x of interest)

$$\begin{cases} -\nabla_y \cdot \left[k(x,y)\nabla_y \omega_j(x,y)\right] = \nabla_y \cdot \left[k(x,y)\mathbf{e}_j\right] & \text{in } \Omega \times Y \\ \omega_j(x,y) \text{ is periodic in } y \end{cases}$$

Then, multiplying by $\partial_j p_0$ and summing,

$$-\sum_j \partial_j p_0(x) \nabla_y \cdot \left[k(x,y)\nabla_y \omega_j(x,y)\right] = \sum_j \partial_j p_0(x) \nabla_y \cdot \left[k(x,y)\mathbf{e}_j\right]$$

Linearity and constancy in y of $\partial_j p_0$ allow us to move things inside

$$-\nabla_y \cdot \left[k(x,y)\nabla_y \sum_j \omega_j(x,y)\partial_j p_0(x)\right] = \nabla_y \cdot \left[k(x,y)\nabla p_0(x)\right]$$

Conclusion:

$$\boxed{p_1(x,y) = \sum_j \omega_j(x,y)\partial_j p_0(x)}$$

solves our problem

$$\begin{cases} -\nabla_y \cdot \left[k(x,y)\nabla_y p_1(x,y)\right] = \nabla_y \cdot \left[k(x,y)\nabla p_0(x)\right] & \text{in } \Omega \times Y \\ p_1(x,y) \text{ is periodic in } y \end{cases}$$

Step 3 (Final Step), ϵ^0-terms:

$$-\nabla_y \cdot \Big[k(x,y)\Big(\nabla_y p_2(x,y) + \nabla_x p_1(x,y)\Big)\Big] \\ - \nabla_x \cdot \Big[k(x,y)\Big(\nabla_y p_1(x,y) + \nabla p_0(x)\Big)\Big] = f \quad \text{in } \Omega \times Y$$

Trick: Remove y by averaging (integrate over y and divide by the volume of Y, $|Y|$). For the first piece, we get

$$\frac{1}{|Y|}\int_Y \nabla_y \cdot \Big[k(x,y)\Big(\nabla_y p_2(x,y) + \nabla_x p_1(x,y)\Big)\Big]\,dy \\ = \frac{1}{|Y|}\int_{\partial Y} \Big[k(x,y)\Big(\nabla_y p_2(x,y) + \nabla_x p_1(x,y)\Big)\Big] \cdot \nu\,ds(y) = 0$$

due to periodicity!

Easily, the third piece is

$$\frac{1}{|Y|}\int_Y f(x)\,dy = f(x)$$

Note that
$$p_1(x,y) = \sum_j \omega_j(x,y)\partial_j p_0(x)$$
tells us that we know p_1 if we know p_0. Thus the second piece is

$$\begin{aligned}
&\frac{1}{|Y|}\int_Y \nabla_x \cdot \Big[k(x,y)\Big(\nabla_y p_1(x,y) + \nabla p_0(x)\Big)\Big]\, dy \\
&= \nabla \cdot \frac{1}{|Y|}\int_Y k(x,y)\Big(\sum_j \nabla_y \omega_j(x,y)\partial_j p_0(x) + \nabla p_0(x)\Big)\, dy \\
&= \sum_i \partial_i \frac{1}{|Y|}\int_Y k(x,y)\Big(\sum_j \partial_i^y \omega_j(x,y)\partial_j p_0(x) + \sum_j \delta_{ij}\partial_j p_0(x)\Big)\, dy \\
&= \sum_i \sum_j \partial_i \Big(\frac{1}{|Y|}\int_Y k(x,y)\Big(\partial_i^y \omega_j(x,y) + \delta_{ij}\Big)\, dy\Big)\partial_j p_0(x) \\
&= \sum_i \sum_j \partial_i \Big(\widehat{k}_{ij}\partial_j p_0(x)\Big) = \nabla \cdot (\widehat{k}\nabla p_0).
\end{aligned}$$

We have derived our homogenized coefficient $\widehat{k}$ from k. It is a tensor!

Conclusion: Collecting pieces, we have our desired result:

$$\boxed{\nabla \cdot \Big(\widehat{k}\nabla p_0\Big) = f \quad \text{in } \Omega}$$

Summary: Starting from

$$\nabla \cdot \left(k_\epsilon \nabla p_\epsilon\right) = f \quad \text{in } \Omega$$

we found that, as $\epsilon \to 0$, $p_\epsilon \to p_0$, where

$$\nabla \cdot \left(\widehat{k} \nabla p_0\right) = f \quad \text{in } \Omega$$

and $\widehat{k}(x)$ can be computed as the tensor

$$\widehat{k}_{ij}(x) = \frac{1}{|Y|} \int_Y k(x,y) \left(\partial_i^y \omega_j(x,y) + \delta_{ij}\right) dy$$

and $\omega_j(x,y)$ can be computed from the local cell problems:

$$\begin{cases} -\nabla_y \cdot \left[k(x,y) \nabla_y \omega_j(x,y)\right] = \nabla_y \cdot \left[k(x,y) \mathbf{e}_j\right] & \text{in } \Omega \times Y \\ \omega_j(x,y) \text{ is periodic in } y & \end{cases}$$

Lemma: $\widehat{k}$ is symmetric and positive definite:

$$\xi \cdot \widehat{k}\xi = \sum_{i,j} \xi_i \widehat{k}_{ij} \xi_j > 0 \quad \text{for all vectors } \xi.$$

Thus, $\widehat{k}$ has three principle eigenvectors and only positive eigenvalues.

Question: Why is this important?

Lemma (Voigt-Reiss Inequality): $\widehat{k}$ lies between the harmonic and arithmetic averages. More precisely, if

$$k_h = \left(\frac{1}{|Y|} \int_Y (k(x,y))^{-1}\, dy \right)^{-1} \quad \text{and} \quad k_a = \frac{1}{|Y|} \int_Y k(x,y)\, dy$$

then

$$\xi \cdot k_h \xi \le \xi \cdot \widehat{k}\xi \le \xi \cdot k_a \xi$$

Convergence

Theorem: As $\epsilon \to 0$, we have weak convergence

$$p_\epsilon \rightharpoonup p_0$$

In fact, $p_\epsilon \to p_0$ and

$$\|p_\epsilon - p_0\| \leq C\epsilon$$

Moreover, if

$$p_\epsilon^1 = p_0 + \epsilon \sum_j \omega_j(x, x/\epsilon)\partial_j p_0(x),$$

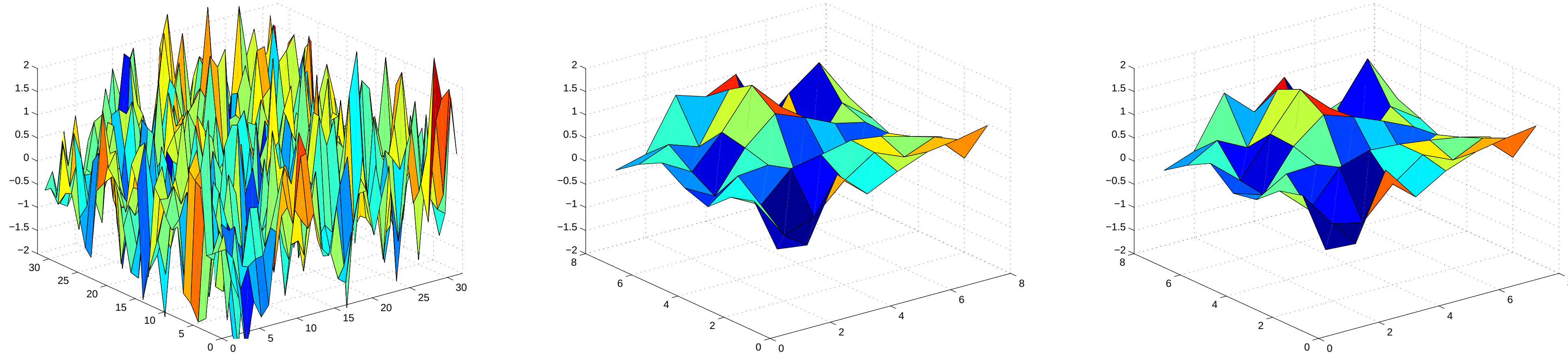
then

$$\|\nabla(p_\epsilon - p_\epsilon^1)\| \leq C\sqrt{\epsilon}$$

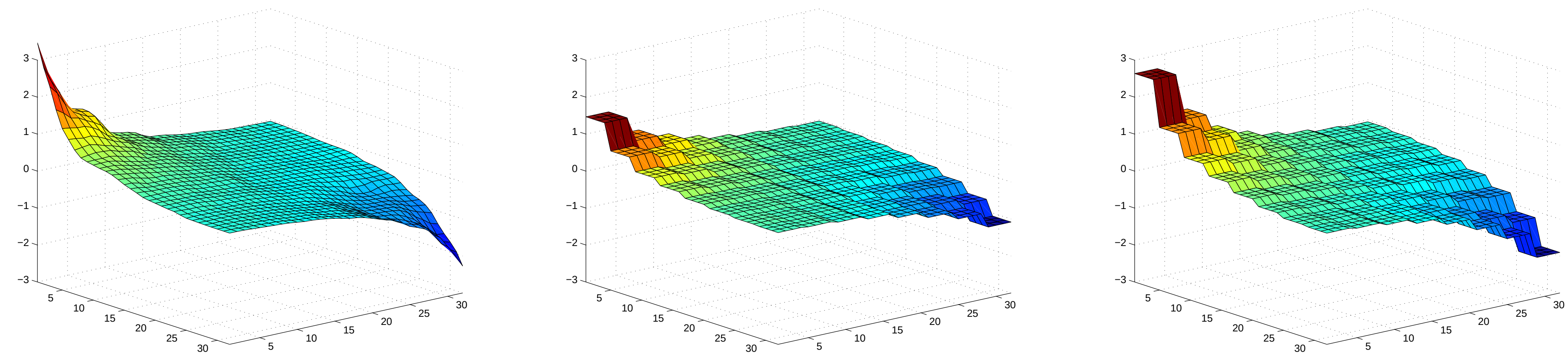
Computational Upscaling via Homogenization

In our small 2-D problem, we obtain the following.

Log-permeability and xx and yy local averages ($xy = yx$ set to 0):



Computed pressure:



32×32 $\quad$ 8×8 homogenized avg $\quad$ 8×8 computed average

Relative errors: Harmonic 0.40, Homog. 0.36, Computational 0.28.